HYPERBOLIC SETS FOR THE FLOWS ON PSEUDO-RIEMANNIAN MANIFOLDS

Mohammad Reza Molaei

Abstract. In this paper we introduce and consider the hyperbolic sets for the flows on pseudo-Riemannian manifolds. If $\Lambda$ is a hyperbolic set for a flow $\Phi$, then we show that at each point of $\Lambda$ we have a unique decomposition for its tangent space up to a distribution on the ambient pseudo-Riemannian manifold. We prove that we have such decomposition for many points arbitrarily close to a given member of $\Lambda$.

1. Introduction

Hyperbolic sets for vector fields and discrete dynamical systems on Riemannian manifolds have been considered deeply by many mathematicians and physicists [1, 3, 5–8, 11–13], and nowadays it is one of the main tools for considering qualitative behavior of dynamical systems [3, 6]. We have extended this notion for discrete dynamical systems created by a diffeomorphism from a finite dimensional pseudo-Riemannian manifold to itself in [10], and here we present an extension of this notion for the flows on finite dimensional pseudo-Riemannian manifolds. We prove that the hyperbolic behavior creates a unique decomposition for the tangent space at each point of a hyperbolic set (see Theorem 2.2) with the exponential behavior on two components of this decomposition. By using a connection which preserves the pseudo-metric on parallel transition we find a kind of convergence of suitable bases of the decomposition of a sequence of points to suitable bases of their limit point (see Theorem 3.1).

2. Hyperbolic behavior on a set

We assume that $M$ is a finite dimensional smooth manifold with a smooth pseudo-Riemannian metric $g$. If $p \in M$, then the vectors in the tangent space $T_p M$ are divided
into three classes named timelike, spacelike, and null classes. A vector \( v \in T_p M \) belongs to timelike class, spacelike class or null class if \( g_p(v, v) < 0 \), \( g_p(v, v) > 0 \), or \( g_p(v, v) = 0 \) respectively. The nondegeneracy of \( g \) implies that its matrix in a local coordinate has no zero eigenvalues. The number of positive eigenvalues minus the number of negative eigenvalues of the matrix \( g \) at \( p \in M \) is called the signature of \( g \) at \( p \). Since \( g \) is continuous on \( M \) then its eigenvalues vary continuously, so the nondegeneracy of \( g \) implies that they are nonzero continuous functions on \( M \). Hence if \( M \) is a connected manifold then the signature of \( g \) is constant at each point of \( M \).

We assume that \( \Phi = \{ \phi^t : t \in \mathbb{R} \} \) is a \( C^1 \)-flow on \( M \), i.e., the map \( (t, p) \mapsto \phi^t(p) \) is a \( C^1 \)-map, \( \phi^0 \) is the identity map, and \( \phi^s \circ \phi^t = \phi^{t+s} \) for all \( t, s \in \mathbb{R} \). A subset \( \Lambda \) of \( M \) is called an invariant set for \( \Phi \) if \( \phi^t(\Lambda) = \Lambda \) for all \( t \in \mathbb{R} \).

**Definition 2.1.** An invariant set \( \Lambda \) for \( \Phi \) is called a hyperbolic set for \( \Phi \) up to a distribution \( p \mapsto E^n(p) \), if there exist positive constants \( a \) and \( b \) with \( b < 1 \) and a decomposition \( T_p M = E^0(p) \oplus E^s(p) \oplus E^u(p) \oplus E^n(p) \) for each \( p \in C \) such that:

(i) Each non-zero vector in the subspace \( E^s(p) \) or the subspace \( E^u(p) \) is timelike or spacelike, each vector of \( E^u(p) \) is a null vector, and \( E^0(p) \) is the subspace generated by the vector \( X(p) = \frac{1}{2} \phi^1(p)|_{t=0} \);

(ii) \( D\phi^t(p)E^s(p) = E^s(\phi^t(p)) \) and \( D\phi^t(p)E^u(p) = E^u(\phi^t(p)) \) for all \( t \in \mathbb{R} \);

(iii) if \( v \in E^s(p) \) and \( t > 0 \) then \( |g_{\phi^t(p)}(D\phi^t(p)(v), D\phi^t(p)(v))| \leq ab^t|g_p(v, v)| \) and \( \lim_{t \to \infty} g_{\phi^t(p)}(D\phi^t(p)(v), D\phi^t(p)(v)) = 0 \) for each non-null vector \( v \in T_p M \) with the following property: \( |g_{\phi^t(p)}(D\phi^t(p)(w), D\phi^t(p)(w))| \leq ab^t|g_p(w, w)| \) for all \( t > 0 \);

(iv) if \( v \in E^u(p) \) and \( t > 0 \) then \( |g_{\phi^t(p)}(D\phi^t(p)(v), D\phi^t(p)(v))| \geq a^{-1}b^{-t}|g_p(v, v)| \).

In the case of Riemannian manifolds we put the compactness condition in the definition of a hyperbolic set, but here we remove this condition. Since the spheres in pseudo-Riemannian manifolds may not be compact, we cannot use this tool here.

**Theorem 2.2.** If \( \Lambda \) is a hyperbolic set for \( \Phi \) up to a distribution \( p \mapsto E^n(p) \), then for each \( p \in \Lambda \), the tangent space of \( M \) at \( p \) has a unique decomposition with the properties described in Definition 2.1.

**Proof.** Suppose that for a given \( p \in \Lambda \) we have

\[
T_p M = E^0(p) \oplus E^s_1(p) \oplus E^u_1(p) \oplus E^n(p) = E^0(p) \oplus E^s_2(p) \oplus E^u_2(p) \oplus E^n(p),
\]

where \( E^s_1(\cdot) \), and \( E^u_1(\cdot) \) satisfy the axioms of Definition 2.1. Then \( E^s_1(p) \oplus E^u_1(p) = E^s_2(p) \oplus E^u_2(p) \). Hence a given \( u \in E^s_1(p) \) can be written as \( u = v + w \), where \( v \in E^s_2(p) \) and \( w \in E^u_2(p) \). Since \( w \in E^u_2(p) \) then for each \( t > 0 \) we have

\[
a^{-1}b^{-t}|g_p(w, w)| \leq |g_{\phi^t(p)}(D\phi^t(p)(w), D\phi^t(p)(w))| = |g_{\phi^t(p)}(D\phi^t(p)(u - v), D\phi^t(p)(u - v))| = |g_{\phi^t(p)}(D\phi^t(p)(u), D\phi^t(p)(u)) + g_{\phi^t(p)}(D\phi^t(p)(v), D\phi^t(p)(v)) - 2g_{\phi^t(p)}(D\phi^t(p)(u), D\phi^t(p)(v))| \leq |g_{\phi^t(p)}(D\phi^t(p)(u), D\phi^t(p)(u))| + |g_{\phi^t(p)}(D\phi^t(p)(v), D\phi^t(p)(v))|
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M.R. Molaei 119

\[ + 2|g_{\phi^t(p)}(D\phi^t(p)(u), D\phi^t(p)(v))] \leq abt|g_p(u,u)| + abt|g_p(v,v)| + 2|g_{\phi^t(p)}(D\phi^t(p)(u), D\phi^t(p)(v))| \]

Axiom (iii) of Definition 2.1 implies that the right-hand side of the former inequality tends to zero when \( t \) tends to infinity. Thus \( |g_p(w,w)| = 0 \). Hence \( w \in E^n(p) \cap E_s^1(p) = \{0\} \). Therefore \( E_s^1(p) \subseteq E^1_s(p) \). By replacing \( E_s^1(p) \) with \( E^1_s(p) \) we have \( E_s^2(p) \subseteq E^1_s(p) \). Thus \( E_s^2(p) = E^1_s(p) \), and this implies \( E_s^2(p) = E^1_s(p) \). Hence we have a unique decomposition for \( T_pM \). □

We now give an example of a hyperbolic set up to a pseudo-Riemannian metric on \( \mathbb{R}^2 \) which is not a hyperbolic set with any Riemannian metric on \( \mathbb{R}^2 \).

Example 2.3. \( \mathbb{R}^2 \) with the metric \( g((a,b),(c,d)) = ac - bd \) is a Lorentzian manifold. Let \( \Phi \) be the flow of the smooth vector field \( X(a,b) = (-ab + b^2, -ab + a^2) \). The set \( \Lambda = \{(x,x) : x > 0\} \) is a hyperbolic set for \( \Phi \) up to the distribution \( E^n(\cdot) = \{(a,a) : a \in \mathbb{R}\} \). Since \( X(x,x) = \{(0,0)\} \), then \( E^0(x,x) = \{(0,0)\} \). For \( x > 0 \) we have \( E^u(x,x) = \{(0,0)\} \), \( E^s(x,x) = \{(-x,x) : x \in \mathbb{R}\} \) and \( T_{(x,x)}\mathbb{R}^2 = E^0(x,x) \oplus E^s(x,x) \oplus E^u(x,x) \) (see Figure 1).

![Figure 1](image)

Figure 1: \( \Lambda = \{(x,x) : x > 0\} \) is a hyperbolic set for the flow of \( X(a,b) = (-ab+b^2, -ab+a^2) \).

3. Hyperbolic decomposition

Now we assume that \( \nabla \) is a Levi-Civita connection on a pseudo-Riemannian manifold \( M \), i.e., it is a torsion free pseudo-Riemannian connection on \( M \) compatible with the metric \( g \). This means that in a local coordinate of \( p \in M \) we have \( \nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k \), where \( \{\partial_i : i = 1, \ldots, m\} \) is a basis for \( T_pM \), and the Christoffel symbols \( \Gamma^k_{ij} \) are determined by the following equations [9]: \( \frac{1}{2}(\partial_j g_{li} + \partial_l g_{ij} - \partial_i g_{lj}) = g_{lk}\Gamma^k_{ij} \), where \( g_{ij} = g(\partial_i, \partial_j) \).
The reader has to pay attention at this point that we use Einstein’s summation convention.

If $\gamma : (-\epsilon, \epsilon) \to M$ is a smooth curve passing through $p$, then a smooth map $X : (-\epsilon, \epsilon) \to TM$ is called a smooth vector field along $\gamma$ if $X(t) \in T_{\gamma(t)}M$. A vector field $Y$ along $\gamma$ is called a parallel vector field if $DY/dt = 0$, where $DY/dt$ is the covariant derivative of $Y$ which is defined in a local chart by

$$\frac{DY}{dt}(t) = \frac{dY^j}{dt}(t)\partial_j + Y^j(t)\frac{\partial \gamma^i}{\partial x^j}(t)\Gamma^j_{ik}(\gamma(t))\partial_i,$$

where $Y = Y^k\partial_k$. If we take $v \in T_pM$ then the existence and uniqueness theorem for ordinary differential equations implies that equation (1) with the initial condition $Y(0) = v$ has a unique solution $Y(t)$. We denote the parallel vector field $Y(t)$ deduced from the initial condition $Y(0) = v$ by $P_t(v)$ or $v(t)$. As in [10], if $v \in T_{\gamma(t)}M$ and $E$ is a subspace of $T_{\gamma(t)}M$ with the basis $B_E$, where $t \in (-\epsilon, \epsilon)$, then $d(v, B_E)$ is defined by $d(v, B_E) = \inf\{|g_{\gamma(t)}(v(s-t) - s, v(s-t) - w)| : w \in B_E\}$. For $s, t \in (-\epsilon, \epsilon)$, if $E$ and $F$ are two subspaces of $T_{\gamma(s)}M$ and $T_{\gamma(t)}M$ with the basis $B_E$ and $B_F$, respectively, then $d(B_E, B_F)$ is defined by $d(B_E, B_F) = \max\{|a, b\}$, where $a = \max\{d(v, B_F) : v \in B_E\}$, and $b = \max\{d(u, B_E) : u \in B_F\}$. We now assume that $\Lambda$ is a hyperbolic set for the flow $\Phi$ up to an $r$-dimensional distribution $q \rightarrow E^a(q)$, and $\gamma : (-\epsilon, \epsilon) \to M$ is a smooth curve passing through $p \in \Lambda$. With these assumptions we have the next theorem.

**Theorem 3.1.** Suppose $\{t_n\}$ is a sequence with $\gamma(t_n) \in \Lambda$ and $t_n \to 0$. If $P_t(E^0(p)) = E^0(\gamma(t))$, then for a subsequence $\{s_n\}$ of $\{t_n\}$, there exist bases $B_{E^s(\gamma(s_n))}$ and $B_{E^u(\gamma(s_n))}$ for $E^s(\gamma(s_n))$ and $E^u(\gamma(s_n))$, and bases $B_{E^c(p)}$ and $B_{E^u(p)}$ for $E^c(p)$ and $E^u(p)$ so that $d(B_{E^s(a(s_n))}, B_{E^c(p)}) \to 0$, and $d(B_{E^u(a(s_n))}, B_{E^c(p)}) \to 0$.

**Proof.** Since $0 \leq \dim(E^s(\gamma(t_n))) \leq m = \dim M$ for all $n \in N$, then there exist a subsequence $\{s_n \in [-\frac{\epsilon}{2}, \frac{\epsilon}{2}] : n \in N\}$ of $\{t_n\}$ and a constant $k \in N$ such that $\dim(E^s(\gamma(s_n))) = k$ for all $n \in N$. We take a pseudo-orthonormal basis $B_{E^s(\gamma(s_1))} = \{v_{n1}, v_{n2}, \ldots, v_{nk}\}$ for $E^s(\gamma(s_1))$. The pseudo-orthonormal basis is a basis with $g_{\gamma(s_1)}(v_{ni}, v_{nj}) = \delta_{ij}$. Clearly $B_{E^c(\gamma(s_n))} = \{v_{n1} = v_{11}(s_n - s_1), v_{n2} = v_{12}(s_n - s_1), \ldots, v_{nk} = v_{ik}(s_n - s_1)\}$ is a pseudo-orthonormal basis for $E^c(\gamma(s_n))$. If we fix $i$, then the sequence $\{v_{ni}\}$ is a convergence sequence in $TM$, and its limit is $v_i = \lim_{n \to \infty} v_{1i}(s_n - s_1) = v_{i1}^{-1}(s_1)$. Since $g$ is a smooth tensor, then its continuity implies that $v_i \notin E^u(p)$. Moreover, the condition $P_t(E^0(p)) = E^0(\gamma(t))$ implies $v_i \notin E^0(p)$, so $v_i \in E^c(p) \oplus E^u(p)$. Hence $v_i = w + u$ with $u \in E^c(p)$ and $w \in E^u(p)$. If $t > 0$, then $a^{-1/b^{-1}}|g_p(w, w)| \leq |g_{\phi_t(p)}(D\phi_t(p)(w), D\phi_t(p)(w))|$

$= |g_{\phi_t(p)}(D\phi_t(p)(v_i - u), D\phi_t(p)(v_i - u))|$

$\leq |g_{\phi_t(p)}(D\phi_t(p)(v_i), D\phi_t(p)(v_i))| + |g_{\phi_t(p)}(D\phi_t(p)(u), D\phi_t(p)(u))|$

$+ 2|g_{\phi_t(p)}(D\phi_t(p)(v_i), D\phi_t(p)(u))|$

$= \lim_{n \to \infty} |g_{\phi_t(\gamma(s_n))}(D\phi_t(\gamma(s_n))(v_{ni}), D\phi_t(\gamma(s_n))(v_{ni}))|$

$+ |g_{\phi_t(p)}(D\phi_t(p)(u), D\phi_t(p)(u))|$. 


\[ + 2 \lim_{n \to \infty} |g_{\varphi^t(\gamma(s_n))}(D\varphi^t(\gamma(s_n)))(v_{n1}), D\varphi^t(\gamma(s_n))(u(s_n))| \]
\[ \leq ( \lim_{n \to \infty} ab'g_{\varphi^t(\gamma(s_n))}(v_{n1}, v_{n1})) + ab'g_p(u, u) \]
\[ + 2 \lim_{n \to \infty} |g_{\varphi^t(\gamma(s_n))}(D\varphi^t(\gamma(s_n)))(v_{n1}), D\varphi^t(\gamma(s_n))(u(s_n))| \]
\[ = ab'g_p(v_1, v_1) + ab'g_p(u, u) \]
\[ + 2 \lim_{n \to \infty} |g_{\varphi^t(\gamma(s_n))}(D\varphi^t(\gamma(s_n)))(v_{n1}), D\varphi^t(\gamma(s_n))(u(s_n))| = 0. \]

Hence the above inequality is valid if \(|g_p(w, w)| = 0\), and this implies that \(w = 0\), and \(v_i \in E^s(p)\). Therefore \(\{v_1, v_2, \ldots, v_k\} \) is a pseudo-orthonormal subset of \(E^s(p)\). Hence \(\dim(E^s(p)) \geq k\). The similar calculations imply that \(\dim(E^u(p)) \geq m - r - k\). Therefore \(\dim(E^s(p)) = k\) and \(\dim(E^u(p)) = m - r - k\). As a result \(B_{E^s(p)} = \{v_1, v_2, \ldots, v_k\}\) is a basis for \(E^s(p)\), and we have \(d(B_{E^s(\gamma(s_n))}, B_{E^s(p)}) \to 0\), when \(n \to \infty\). The similar calculations imply that \(d(B_{E^u(\alpha(s_n))}, B_{E^u(p)}) \to 0\).

\[ \Box \]

Figure 2: Λ = \{(0, a) : a > 0\} is a partial hyperbolic set for the flow of \(X(a, b) = (-ab, a^2)\) on the Lorentzian manifold \(\mathbb{R}^2\).

4. Conclusion

We see that if we separate the null vectors via a null distribution then we can detect the hyperbolic dynamics on pseudo-Riemannian manifolds. In Example 2.3 we see that a set of stationary points of a vector field is a hyperbolic set by the given Lorentzian metric. This set is not a hyperbolic set in the case of Riemannian metrics.

The notion of partial hyperbolic set as another main object in smooth dynamical systems on Riemannian manifolds [2,4] can be extended for a \(C^1\) flow \(\Phi = \{\varphi^t : t \in \mathbb{R}\}\) on a pseudo-Riemannian manifolds via the results of this paper. In fact we say that an invariant set \(\Lambda\) is a partial hyperbolic set for \(\Phi\) if for each \(p \in \Lambda\) there exist a splitting \(T_p M = E_p \oplus F_p \oplus G_p\), and positive real numbers \(a, b < 1, c\) with the
Hyperbolic sets for the flows

following properties:
(i) $D\phi^t(p)E_p = E_{\phi^t(p)}$, $D\phi^t(p)F_p = F_{\phi^t(p)}$, and $D\phi^t(p)G_p = G_{\phi^t(p)}$ for all $p \in \Lambda$;
(ii) $E_p \neq \{0\}$, $F_p \neq \{0\}$ and there is no any non-zero null vector in $E_p \cup F_p$;
(iii) if $v \in E_p$ and $t > 0$ then $|g_{\phi^t(p)}(D\phi^t(p)(v), D\phi^t(p)(v))| \leq ab^t|g_p(v, v)|$ and
\[ \lim_{t \to \infty} g_{\phi^t(p)}(D\phi^t(p)(v), D\phi^t(p)(w)) = 0 \] for each non-null vector $w \in T_pM$ with
the following property $|g_{\phi^t(p)}(D\phi^t(p)(w), D\phi^t(p)(w))| \leq ab^t|g_p(w, w)|$ for all $t > 0$;
(iv) if $0 \neq v \in E_p$, $0 \neq w \in F_p$ and $t > 0$ then
\[ |g_{\phi^t(p)}(D\phi^t(p)(v), D\phi^t(p)(v))||g_{\phi^{-t}(p)}(D\phi^{-t}(p)(w), D\phi^{-t}(p)(w))| \leq cb^t|g_p(v, v)||g_p(w, w)|; \]
(v) each vector of $G_p$ is a null vector.

We see that any hyperbolic set is a partially hyperbolic set (in this case $c = a^2$),
but the converse is not true. For example with the space of Example 2.3 the set
$\Lambda = \{(0, a) : a \in \mathbb{R} \text{ and } a > 0\}$ is a partially hyperbolic set for the flow of the
vector field $X(a, b) = (\frac{-ab^2}{1+b^2}, \frac{a^2}{1+b^2})$ on $\mathbb{R}^2$, but it is not a hyperbolic set up to any null
distribution on $\mathbb{R}^2$ (see Figure 2).

The consideration of partially hyperbolic sets in pseudo-Riemannian manifolds
may be a topic for further research.

We conclude this paper by posing a problem on hyperbolic sets: Suppose $\Lambda$ is a
hyperbolic set for a flow $\Phi$ on $M$ with the metric $g$. Is there any other metric $\tilde{g}$ on $M$
such that $\Lambda$ is also a hyperbolic set with the metric $\tilde{g}$ and in Definition 2.1, $a$ takes
the value one?

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Mahani Mathematical Research Center and Department of Mathematics, Shahid Bahonar University of Kerman, Kerman, Iran

E-mail: mrmolaei@uk.ac.ir