MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
72, 2 (2020), 117–123
June 2020

research paper
оригинални научни рад

HYPERBOLIC SETS FOR THE FLOWS ON PSEUDO-RIEMANNIAN MANIFOLDS

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Abstract. In this paper we introduce and consider the hyperbolic sets for the flows on pseudo-Riemannian manifolds. If Λ is a hyperbolic set for a flow Φ, then we show that at each point of Λ we have a unique decomposition for its tangent space up to a distribution on the ambient pseudo-Riemannian manifold. We prove that we have such decomposition for many points arbitrarily close to a given member of Λ.

1. Introduction

Hyperbolic sets for vector fields and discrete dynamical systems on Riemannian manifolds have been considered deeply by many mathematicians and physicists [1, 3, 5–8, 11–13], and nowadays it is one of the main tools for considering qualitative behavior of dynamical systems [3, 6]. We have extended this notion for discrete dynamical systems created by a diffeomorphism from a finite dimensional pseudo-Riemannian manifold to itself in [10], and here we present an extension of this notion for the flows on finite dimensional pseudo-Riemannian manifolds. We prove that the hyperbolic behavior creates a unique decomposition for the tangent space at each point of a hyperbolic set (see Theorem 2.2) with the exponential behavior on two components of this decomposition. By using a connection which preserves the pseudo-metric on parallel transition we find a kind of convergence of suitable bases of the decomposition of a sequence of points to suitable bases of their limit point (see Theorem 3.1).

2. Hyperbolic behavior on a set

We assume that M is a finite dimensional smooth manifold with a smooth pseudo-Riemannian metric g. If $p \in M$, then the vectors in the tangent space T_pM are divided

2010 Mathematics Subject Classification: 37D20, 37D30
Keywords and phrases: Hyperbolic set; pseudo-Riemannian manifold; flow; distribution.

into three classes named timelike, spacelike, and null classes. A vector $v \in T_pM$ belongs to timelike class, spacelike class or null class if $g_p(v, v) < 0$, $g_p(v, v) > 0$, or $g_p(v, v) = 0$ respectively. The nondegeneracy of g implies that its matrix in a local coordinate has no zero eigenvalues. The number of positive eigenvalues minus the number of negative eigenvalues of the matrix g at $p \in M$ is called the signature of g at p. Since g is continuous on M then its eigenvalues vary continuously, so the nondegeneracy of g implies that they are nonzero continuous functions on M. Hence if M is a connected manifold then the signature of g is constant at each point of M.

We assume that $\Phi = \{\phi^t : t \in R\}$ is a C^1-flow on M, i.e., the map $(t, p) \mapsto \phi^t(p)$ is a C^1-map, ϕ^0 is the identity map, and $\phi^t \circ \phi^s = \phi^{t+s}$ for all $t, s \in R$. A subset Λ of M is called an invariant set for Φ if $\phi^t(\Lambda) = \Lambda$ for all $t \in R$.

Definition 2.1. An invariant set Λ for Φ is called a hyperbolic set for Φ up to a distribution $p \mapsto E^n(p)$, if there exist positive constants a and b with $b < 1$ and a decomposition $T_pM = E^0(p) \oplus E^s(p) \oplus E^u(p) \oplus E^n(p)$ for each $p \in C$ such that:
(i) Each non-zero vector in the subspace $E^s(p)$ or the subspace $E^u(p)$ is timelike or spacelike, each vector of $E^n(p)$ is a null vector, and $E^0(p)$ is the subspace generated by the vector $X(p) = \frac{d}{dt}\phi^t(p)|_{t=0}$;

(ii) $D\phi^t(p)E^s(p) = E^s(\phi^t(p))$ and $D\phi^t(p)E^u(p) = E^u(\phi^t(p))$ for all $t \in R$;

(iii) if $v \in E^s(p)$ and $t > 0$ then $|g_{\phi^t(p)}(D\phi^t(p)(v), D\phi^t(p)(v))| \leq ab^t|g_p(v, v)|$ and $\lim_{t\to\infty} g_{\phi^t(p)}(D\phi^t(p)(v), D\phi^t(p)(w)) = 0$ for each non-null vector $w \in T_pM$ with the following property: $|g_{\phi^t(p)}(D\phi^t(p)(w), D\phi^t(p)(w))| \leq ab^t|g_p(w, w)$ for all $t > 0$;

(iv) if $v \in E^u(p)$ and $t > 0$ then $|g_{\phi^t(p)}(D\phi^t(p)(v), D\phi^t(p)(v))| \geq a^{-1}b^{-t}|g_p(v, v)|$.

In the case of Riemannian manifolds we put the compactness condition in the definition of a hyperbolic set, but here we remove this condition. Since the spheres in pseudo-Riemannian manifolds may not be compact, we cannot use this tool here.

Theorem 2.2. *If Λ is a hyperbolic set for Φ up to a distribution $p \mapsto E^n(p)$, then for each $p \in \Lambda$, the tangent space of M at p has a unique decomposition with the properties described in Definition 2.1.*

Proof. Suppose that for a given $p \in \Lambda$ we have

$$T_pM = E^0(p) \oplus E^s_1(p) \oplus E^u_1(p) \oplus E^n(p) = E^0(p) \oplus E^s_2(p) \oplus E^u_2(p) \oplus E^n(p),$$

where $E^s_i(\cdot)$, and $E^u_i(\cdot)$ satisfy the axioms of Definition 2.1. Then $E^s_1(p) \oplus E^u_1(p) = E^s_2(p) \oplus E^u_2(p)$. Hence a given $u \in E^s_1(p)$ can be written as $u = v + w$, where $v \in E^s_2(p)$ and $w \in E^u_2(p)$. Since $w \in E^u_2(p)$ then for each $t > 0$ we have

$$\begin{aligned}
a^{-1}b^{-t}|g_p(w, w)| &\leq |g_{\phi^t(p)}(D\phi^t(p)(w), D\phi^t(p)(w))| \\
&= |g_{\phi^t(p)}(D\phi^t(p)(u - v), D\phi^t(p)(u - v))| \\
&= |g_{\phi^t(p)}(D\phi^t(p)(u), D\phi^t(p)(u)) + g_{\phi^t(p)}(D\phi^t(p)(v), D\phi^t(p)(v)) \\
&\quad - 2g_{\phi^t(p)}(D\phi^t(p)(u), D\phi^t(p)(v))| \\
&\leq |g_{\phi^t(p)}(D\phi^t(p)(u), D\phi^t(p)(u))| + |g_{\phi^t(p)}(D\phi^t(p)(v), D\phi^t(p)(v))|
\end{aligned}$$

$$+ 2|g_{\phi^t(p)}(D\phi^t(p)(u), D\phi^t(p)(v))|$$
$$\leq ab^t|g_p(u,u)| + ab^t|g_p(v,v)| + 2|g_{\phi^t(p)}(D\phi^t(p)(u), D\phi^t(p)(v))|.$$

Axiom (iii) of Definition 2.1 implies that the right-hand side of the former inequality tends to zero when t tends to infinity. Thus $|g_p(w,w)| = 0$. Hence $w \in E^n(p) \cap E_2^u(p) = \{0\}$. Therefore $E_1^s(p) \subseteq E_2^s(p)$. By replacing $E_1^s(p)$ with $E_2^s(p)$ we have $E_2^s(p) \subseteq E_1^s(p)$. Thus $E_2^s(p) = E_1^s(p)$, and this implies $E_2^u(p) = E_1^u(p)$. Hence we have a unique decomposition for T_pM. $\square$

We now give an example of a hyperbolic set up to a pseudo-Riemannian metric on $\mathbb{R}^2$ which is not a hyperbolic set with any Riemannian metric on $\mathbb{R}^2$.

Example 2.3. $\mathbb{R}^2$ with the metric $g((a,b),(c,d)) = ac - bd$ is a Lorentzian manifold. Let Φ be the flow of the smooth vector field $X(a,b) = (-ab+b^2, -ab+a^2)$. The set $\Lambda = \{(x,x) \ : \ x > 0\}$ is a hyperbolic set for Φ up to the distribution $E^n(\cdot) = \{(a,a) \ : \ a \in \mathbb{R}\}$. Since $X(x,x) = \{(0,0)\}$, then $E^0(x,x) = \{(0,0)\}$. For $x > 0$ we have $E^u(x,x) = \{(0,0)\}$, $E^s(x,x) = \{(-x,x) \ : \ x \in \mathbb{R}\}$ and $T_{(x,x)}\mathbb{R}^2 = E^0(x,x) \oplus E^s(x,x) \oplus E^u(x,x) \oplus E^n(x,x)$ (see Figure 1).

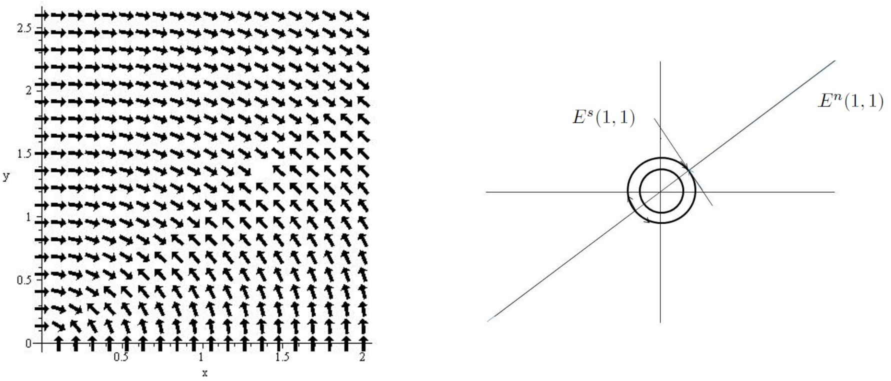

Figure 1: $\Lambda = \{(x,x) : x > 0\}$ is a hyperbolic set for the flow of $X(a,b) = (-ab+b^2, -ab+a^2)$.

3. Hyperbolic decomposition

Now we assume that ∇ is a Levi-Civita connection on a pseudo-Riemannian manifold M, i.e., it is a torsion free pseudo-Riemannian connection on M compatible with the metric g. This means that in a local coordinate of $p \in M$ we have $\nabla_{\partial_i}\partial_j = \Gamma_{ij}^k \partial_k$, where $\{\partial_i \ : \ i = 1, \ldots, m\}$ is a basis for T_pM, and the Christoffel symbols Γ_{ij}^k are determined by the following equations [9]: $\frac{1}{2}(\partial_j g_{li} + \partial_i g_{lj} - \partial_l g_{ij}) = g_{lk}\Gamma_{ij}^k$, where $g_{ij} = g(\partial_i, \partial_j)$.

The reader has to pay attention at this point that we use Einstein's summation convention.

If $\gamma : (-\epsilon, \epsilon) \to M$ is a smooth curve passing through p, then a smooth map $X : (-\epsilon, \epsilon) \to TM$ is called a smooth vector field along γ if $X(t) \in T_{\gamma(t)}M$. A vector field Y along γ is called a parallel vector field if $\frac{DY}{dt} = 0$, where $\frac{DY}{dt}$ is the covariant derivative of Y which is defined in a local chart by

$$\frac{DY}{dt}(t) = \frac{dY^j}{dt}(t)\partial_j + Y^j(t)\nabla^{\partial_j}_{\dot{\gamma}(t)} = \frac{dY^j}{dt}(t)\partial_j + Y^k(t)\dot{\gamma}^i(t)\Gamma^j_{ik}(\gamma(t))\partial_j, \qquad (1)$$

where $Y = Y^k\partial_k$. If we take $v \in T_pM$ then the existence and uniqueness theorem for ordinary differential equations implies that equation (1) with the initial condition $Y(0) = v$ has a unique solution $Y(t)$. We denote the parallel vector field $Y(t)$ deduced from the initial condition $Y(0) = v$ by $P_t(v)$ or $v(t)$. As in [10], if $v \in T_{\gamma(t)}M$ and E is a subspace of $T_{\gamma(s)}M$ with the basis B_E, where $t, s \in (-\epsilon, \epsilon)$, then $d(v, B_E)$ is defined by $d(v, B_E) = \inf\{|g_{\gamma(s)}(v(s-t) - w, v(s-t) - w)| : w \in B_E\}$. For $s, t \in (-\epsilon, \epsilon)$, if E and F are two subspaces of $T_{\gamma(s)}M$ and $T_{\gamma(t)}M$ with the basis B_E and B_F, respectively, then $d(B_E, B_F)$ is defined by $d(B_E, B_F) = \max\{a, b\}$, where $a = \max\{d(v, B_F) : v \in B_E\}$, and $b = \max\{d(u, B_E) : u \in B_F\}$. We now assume that Λ is a hyperbolic set for the flow Φ up to an r-dimensional distribution $q \mapsto E^n(q)$, and $\gamma : (-\epsilon, \epsilon) \to M$ is a smooth curve passing through $p \in \Lambda$. With these assumptions we have the next theorem.

THEOREM 3.1. *Suppose $\{t_n\}$ is a sequence with $\gamma(t_n) \in \Lambda$ and $t_n \to 0$. If $P_t(E^0(p)) = E^0(\gamma(t))$, then for a subsequence $\{s_n\}$ of $\{t_n\}$, there exist bases $B_{E^s(\gamma(s_n))}$ and $B_{E^u(\gamma(s_n))}$ for $E^s(\gamma(s_n))$ and $E^u(\gamma(s_n))$, and bases $B_{E^s(p)}$ and $B_{E^u(p)}$ for $E^s(p)$ and $E^u(p)$ so that $d(B_{E^s(\alpha(s_n))}, B_{E^s(p)}) \to 0$, and $d(B_{E^u(\alpha(s_n))}, B_{E^u(p)}) \to 0$.*

Proof. Since $0 \le \dim(E^s(\gamma(t_n))) \le m = \dim M$ for all $n \in N$, then there exist a subsequence $\{s_n \in [-\frac{\epsilon}{2}, \frac{\epsilon}{2}] : n \in N\}$ of $\{t_n\}$ and a constant $k \in N$ such that $\dim(E^s(\gamma(s_n)) = k$ for all $n \in N$. We take a pseudo-orthonormal basis $B_{E^s(\gamma(s_1))} = \{v_{11}, v_{12}, \ldots, v_{1k}\}$ for $E^s(\gamma(s_1))$. The pseudo-orthonormal basis is a basis with $|g_{\gamma(s_1)}(v_{1i}, v_{1j})| = \delta_{ij}$. Clearly $B_{E^s(\gamma(s_n))} = \{v_{n1} = v_{11}(s_n - s_1), v_{n2} = v_{12}(s_n - s_1), \ldots, v_{nk} = v_{1k}(s_n - s_1)\}$ is a pseudo-orthonormal basis for $E^s(\gamma(s_n))$. If we fix i, then the sequence $\{v_{ni}\}$ is a convergence sequence in TM, and its limit is $v_i = \lim_{n\to\infty} v_{1i}(s_n - s_1) = v_{1i}(-s_1)$. Since g is a smooth tensor, then its continuity implies that $v_i \notin E^n(p)$. Moreover, the condition $P_t(E^0(p)) = E^0(\gamma(t))$ implies $v_i \notin E^0(p)$, so $v_i \in E^s(p) \oplus E^u(p)$. Hence $v_i = u + w$ with $u \in E^s(p)$ and $w \in E^u(p)$. If $t > 0$, then

$$\begin{aligned}
a^{-1}b^{-t}|g_p(w, w)| &\le |g_{\phi^t(p)}(D\phi^t(p)(w), D\phi^t(p)(w))| \\
&= |g_{\phi^t(p)}(D\phi^t(p)(v_i - u), D\phi^t(p)(v_i - u))| \\
&\le |g_{\phi^t(p)}(D\phi^t(p)(v_i), D\phi^t(p)(v_i))| + |g_{\phi^t(p)}(D\phi^t(p)(u), D\phi^t(p)(u))| \\
&\quad + 2|g_{\phi^t(p)}(D\phi^t(p)(v_i), D\phi^t(p)(u))| \\
&= \lim_{n\to\infty} |g_{\phi^t(\gamma(s_n))}(D\phi^t(\gamma(s_n))(v_{ni}), D\phi^t(\gamma(s_n))(v_{ni}))| \\
&\quad + |g_{\phi^t(p)}(D\phi^t(p)(u), D\phi^t(p)(u))|
\end{aligned}$$

$$
\begin{aligned}
&+2\lim_{n\to\infty}|g_{\phi^t(\gamma(s_n))}(D\phi^t(\gamma(s_n))(v_{ni}),D\phi^t(\gamma(s_n))(u(s_n)))| \\
&\le (\lim_{n\to\infty} ab^t g_{\phi^t(\gamma(s_n))}(v_{ni},v_{ni})) + ab^t g_p(u,u) \\
&\quad +2\lim_{n\to\infty}|g_{\phi^t(\gamma(s_n))}(D\phi^t(\gamma(s_n))(v_{ni}),D\phi^t(\gamma(s_n))(u(s_n))| \\
&= ab^t g_p(v_i,v_i) + ab^t g_p(u,u) \\
&\quad +2\lim_{n\to\infty}|g_{\phi^t(\gamma(s_n))}(D\phi^t(\gamma(s_n))(v_{ni}),D\phi^t(\gamma(s_n))(u(s_n))|.
\end{aligned}
$$

We have $\lim_{n\to\infty}|g_{\phi^t(\gamma(s_n))}(D\phi^t(\gamma(s_n))(v_{ni}),D\phi^t(\gamma(s_n))(u(s_n))| = 0$. Hence the above inequality is valid if $|g_p(w,w)| = 0$, and this implies that $w = 0$, and $v_i \in E^s(p)$. Therefore $\{v_1, v_2, \ldots, v_k\}$ is a pseudo-orthonormal subset of $E^s(p)$. Hence $\dim(E^s(p)) \ge k$. The similar calculations imply that $\dim(E^u(p)) \ge m-r-k$. Therefore $\dim(E^s(p)) = k$ and $\dim(E^u(p)) = m-r-k$. As a result $B_{E^s(p)} = \{v_1, v_2, \ldots, v_k\}$ is a basis for $E^s(p)$, and we have $d(B_{E^s(\gamma(s_n))}, B_{E^s(p)}) \to 0$, when $n \to \infty$. The similar calculations imply that $d(B_{E^u(\alpha(s_n))}, B_{E^u(p)}) \to 0$. □

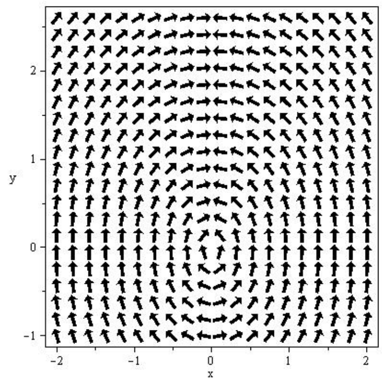

Figure 2: $\Lambda = \{(0,a) : a > 0\}$ is a partial hyperbolic set for the flow of $X(a,b) = (\frac{-ab}{3}, \frac{a^2}{2})$ on the Lorentzian manifold $\mathbb{R}^2$.

4. Conclusion

We see that if we separate the null vectors via a null distribution then we can detect the hyperbolic dynamics on pseudo-Riemannian manifolds. In Example 2.3 we see that a set of stationary points of a vector field is a hyperbolic set by the given Lorentzian metric. This set is not a hyperbolic set in the case of Riemannian metrics.

The notion of partial hyperbolic set as another main object in smooth dynamical systems on Riemannian manifolds [2,4] can be extended for a C^1 flow $\Phi = \{\phi^t : t \in \mathbb{R}\}$ on a pseudo-Riemannian manifolds via the results of this paper. In fact we say that an invariant set Λ is a partial hyperbolic set for Φ if for each $p \in \Lambda$ there exist a splitting $T_pM = E_p \oplus F_p \oplus G_p$, and positive real numbers $a, b < 1$, c with the

following properties:
(i) $D\phi^t(p)E_p = E_{\phi^t(p)}$, $D\phi^t(p)F_p = F_{\phi^t(p)}$, and $D\phi^t(p)G_p = G_{\phi^t(p)}$ for all $p \in \Lambda$;

(ii) $E_p \neq \{0\}$, $F_p \neq \{0\}$ and there is no any non-zero null vector in $E_p \cup F_p$;

(iii) if $v \in E_p$ and $t > 0$ then $|g_{\phi^t(p)}(D\phi^t(p)(v), D\phi^t(p)(v))| \leq ab^t|g_p(v,v)|$ and $\lim_{t\to\infty} g_{\phi^t(p)}(D\phi^t(p)(v), D\phi^t(p)(w)) = 0$ for each non-null vector $w \in T_pM$ with the following property $|g_{\phi^t(p)}(D\phi^t(p)(w), D\phi^t(p)(w))| \leq ab^t|g_p(w,w)|$ for all $t > 0$;

(iv) if $0 \neq v \in E_p$, $0 \neq w \in F_p$ and $t > 0$ then

$$|g_{\phi^t(p)}(D\phi^t(p)(v), D\phi^t(p)(v))||g_{\phi^{-t}(p)}(D\phi^{-t}(p)(w), D\phi^{-t}(p)(w))| \leq cb^t|g_p(v,v)||g_p(w,w)|;$$

(v) each vector of G_p is a null vector.

We see that any hyperbolic set is a partially hyperbolic set (in this case $c = a^2$), but the converse is not true. For example with the space of Example 2.3 the set $\Lambda = \{(0,a) : a \in \mathbb{R} \text{ and } a > 0\}$ is a partially hyperbolic set for the flow of the vector field $X(a,b) = (\frac{-ab}{3}, \frac{a^2}{2})$ on $\mathbb{R}^2$, but it is not a hyperbolic set up to any null distribution on $\mathbb{R}^2$ (see Figure 2).

The consideration of partially hyperbolic sets in pseudo-Riemannian manifolds can be a topic for further research.

We conclude this paper by posing a problem on hyperbolic sets: Suppose Λ is a hyperbolic set for a flow Φ on M with the metric g. Is there any other metric $\tilde{g}$ on M such that Λ is also a hyperbolic set with the metric $\tilde{g}$ and in Definition 2.1, a takes the value one?

Acknowledgement. Our special thanks to the referee for his/her valuable comments.

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(received 23.08.2018; in revised form 06.02.2018; available online 12.09.2019)

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