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ON SOMOS'S IDENTITIES OF LEVEL TWENTY ONE AND THEIR PARTITION INTERPRETATIONS

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Abstract. In this paper, proofs of Somos's theta function identities of level 21 will be given. Further, we deduce certain interesting partition identities from them.

1. Introduction

Let τ be a complex number satisfying $\mathrm{Im}(\tau) > 0$ and let $q = e^{2\pi i\tau}$. The Dedekind eta-function is defined by

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

Let $f(-q) = q^{-1/24}\eta(\tau)$ and $f_n = f(-q^n)$. The theta function identity which relates f_1, f_n, f_m and f_{mn}, is called the theta function identity of level mn. M. Somos [4] discovered around 6200 theta function identities of different levels using computer. Recently B. Yuttanan [7], K. R. Vasuki, R. G. Veeresha [6] and B. R. Srivatsa Kumar and D. AnuRadha [5] have obtained proofs for levels 8, 10, 12, 14 and 16. Somos discovered 13 identities of level 21, where three are equivalent to (1)–(3) below. In this paper, we provide a proof of identities of level 21.

Our proofs depend on the following three P-Q identities of Ramanujan:

Theorem 1.1 ([1, p. 236], [3, p.323]). *(i) Let* $P = \frac{f_1}{q^{1/4} f_7}$ *and* $Q = \frac{f_3}{q^{3/4} f_{21}}$. *Then*

$$PQ + \frac{7}{PQ} = \left(\frac{Q}{P}\right)^2 + \left(\frac{P}{Q}\right)^2 - 3. \tag{1}$$

(ii) Let $A = \frac{f_1}{q^{1/12} f_3}$ *and* $B = \frac{f_7}{q^{7/12} f_{21}}$. *Then*

$$(AB)^3 + \frac{27}{(AB)^3} = \left(\frac{B}{A}\right)^4 - 7\left(\frac{B}{A}\right)^2 + 7\left(\frac{A}{B}\right)^2 - \left(\frac{A}{B}\right)^4. \tag{2}$$

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(iii) Let $L = \frac{f_3}{q^{1/6} f_7}$ *and* $M = \frac{f_1}{q^{5/6} f_{21}}$. *Then*

$$\left(\frac{M}{L}\right)^3 - 27\left(\frac{L}{M}\right)^3 = (LM)^2 - LM + \frac{7}{LM} - \frac{49}{(LM)^2}. \tag{3}$$

2. Somos's identities

First let us list Somos's identities of level 21.

THEOREM 2.1.

$$\begin{aligned} q^2 f_1^4 f_{21}^4 + f_3^4 f_7^4 - f_1^3 f_3^3 f_7 f_{21} - 3q f_1^2 f_3^2 f_7^2 f_{21}^2 \\ -7q^2 f_1 f_3 f_7^3 f_{21}^3 = 0 \end{aligned} \tag{4}$$

$$\begin{aligned} q^4 f_1^8 f_{21}^8 + f_1^7 f_3 f_7^7 f_{21} + 27q^4 f_1 f_3^7 f_7 f_{21}^7 + 7q f_1^2 f_3^6 f_7^6 f_{21}^2 \\ -f_3^8 f_7^8 - 7q^3 f_1^6 f_3^2 f_7^2 f_{21}^6 = 0 \end{aligned}$$

$$\begin{aligned} q f_1^4 f_3^4 f_7^2 f_{21}^2 + f_1^6 f_7^6 + 49q^4 f_1 f_3 f_7^5 f_{21}^5 - f_1^5 f_3^5 f_7 f_{21} \\ -27q^4 f_3^6 f_{21}^6 - 7q^3 f_1^2 f_3^2 f_7^4 f_{21}^4 = 0 \end{aligned}$$

$$f_1^6 f_7 f_{21}^2 + f_3^2 f_7^7 + 36q^4 f_1 f_3 f_{21}^7 - 2f_1^3 f_3 f_7^4 f_{21} - 9q^2 f_3^4 f_7 f_{21}^4 = 0. \tag{5}$$

$$\begin{aligned} 10 f_1^7 f_{21} + 1323q^3 f_3^3 f_7 f_{21}^4 + 189q^2 f_1 f_3^4 f_{21}^3 + 49q f_1^3 f_7^4 f_{21} \\ +81 f_3^7 f_7 - 343q f_3 f_7^7 - 91 f_1^4 f_3 f_7^3 = 0 \end{aligned}$$

$$\begin{aligned} q f_1^3 f_7^4 f_{21} + 1q^2 f_1 f_3^4 f_{21}^3 + f_1^4 f_3 f_7^3 + 10q^5 f_1 f_{21}^7 + 3q f_3 f_7^7 \\ -f_3^7 f_7 - 13q^3 f_3^3 f_7 f_{21}^4 = 0 \end{aligned}$$

$$\begin{aligned} 13 f_1^7 f_{21} + 1323q^5 f_1 f_{21}^7 + 14 f_1^4 f_3 f_7^3 + 196q f_1^3 f_7^4 f_{21} \\ +378q^2 f_1 f_3^4 f_{21}^3 - 27 f_3^7 f_7 - 49q f_3 f_7^7 = 0 \end{aligned}$$

$$\begin{aligned} f_1^7 f_{21} + 14q f_1^3 f_7^4 f_{21} + 14q^3 f_3^3 f_7 f_{21}^4 + 28q^2 f_1 f_3^4 f_{21}^3 \\ +91q^5 f_1 f_{21}^7 - f_3^7 f_7 - 7q f_3 f_7^7 = 0 \end{aligned}$$

$$\begin{aligned} 147q^4 f_1^2 f_3 f_{21}^6 + 3 f_3^7 f_7^2 + 4 f_1^7 f_7 f_{21} - 42q^2 f_1 f_3^4 f_7 f_{21}^3 \\ -7 f_1^4 f_3 f_7^4 = 0 \end{aligned}$$

$$\begin{aligned} q f_1^5 f_7^2 f_{21}^2 + 2 f_1^3 f_3^2 f_7^4 + 9q^2 f_3^5 f_7 f_{21}^3 - 2 f_1^6 f_3 f_7 f_{21} \\ -63q^4 f_1 f_3^2 f_{21}^6 - 7q f_1^2 f_3 f_7^5 f_{21} = 0 \end{aligned}$$

$$\begin{aligned} f_1^3 f_3 f_7^5 + 3q^3 f_1^2 f_3^2 f_{21}^5 + 6q^2 f_3^4 f_7^2 f_{21}^3 - f_1^6 f_7^2 f_{21} \\ -3q f_1 f_3^5 f_7 f_{21}^2 - 42q^4 f_1 f_3 f_7 f_{21}^6 = 0 \end{aligned}$$

$$\begin{aligned} f_1^2 f_3^2 f_7^5 + 2q f_1^4 f_7^3 f_{21}^2 + 9q f_3^6 f_7 f_{21}^2 - f_1^5 f_3 f_7^2 f_{21} \\ -14q f_1 f_3 f_7^6 f_{21} - 9q^3 f_1 f_3^3 f_{21}^5 = 0 \end{aligned}$$

$$\begin{aligned} {}_1^5 f_7^3 f_{21} + 21q^3 f_1 f_3^2 f_7 f_{21}^5 + 6 f_1 f_3^6 f_7 f_{21} - 3q f_3^5 f_7^2 f_{21}^2 \\ -6q^2 f_1^2 f_3^3 f_{21}^4 - 7 f_1^2 f_3 f_7^6 = 0. \end{aligned}$$

Proof. The first three identities are equivalent to (1)–(3) respectively. Let us prove the identity (5).

We shall set $P = \frac{f_1}{q^{1/3}f_7}$, $Q = \frac{f_3}{qf_{21}}$, $A = \frac{f_1}{q^{1/12}f_3}$ and $B = \frac{f_7}{q^{7/12}f_{21}}$. Dividing (5) throughout by $f_3^2 f_7^7$, we find that $\frac{P^6}{Q^2} + 1 - 2\frac{P^3}{Q} + \frac{9}{(AB)^3}\left(4\frac{P^4}{Q^4} - \frac{P^3}{Q}\right) = 0$ or $(AB)^3 = \frac{9(Q^3P^3-4P^4)}{Q^2(P^3-Q)^2}$. Using the equation above in (2) and factoring, we obtain

$$C(P,Q)D(P,Q) = 0, \tag{6}$$

where $C(P,Q) = P^4 + Q^4 - P^3Q^3 - 3P^2Q^2 - 7PQ$ and $D(P,Q) = P^8Q^4 - 4P^7Q^3 - P^5Q^5 + P^4Q^4 + 28P^5Q + 9P^3Q^3 - 52P^4 - 16P^2Q^2 - 3Q^4 + 28PQ$. From Theorem 1.1 (i) we have, $C(P,Q) = 0$. This implies that (6) holds, which verifies (5).

We shall omit the proof of the remaining identities as the proofs are similar to the previous one. □

3. Application to partitions

As usually, for complex numbers a and q set $(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n)$, $|q|<1$, and

$$\left(q_{a_1}^{x_1\pm1}, q_{a_2}^{x_2\pm1}, \cdots, q_{a_k}^{x_k\pm1}; q^y\right)_\infty = (q^{x_1};q^y)_\infty^{a_1} \times (q^{x_2};q^y)_\infty^{a_2} \times \cdots \times (q^{x_k};q^y)_\infty^{a_k}$$
$$\times (q^{y-x_1};q^y)_\infty^{a_1} \times (q^{y-x_2};q^y)_\infty^{a_2} \times \cdots \times (q^{y-x_k};q^y)_\infty^{a_k}.$$

Color partition was introduced by S.-S. Haung [2]. In the context of partitions, we say that a positive integer n has k colors if there are k copies of n available and all of them are viewed as distinct objects. Partitions of positive integers into parts with colors are called colored partitions. For example, if 1 is allowed to have 2 colors, then all the (colored) partitions of 2 are 2, $1_r + 1_r$, $1_g + 1_g$ and $1_r + 1_g$, where we use the indices's r (red) and g (green) to distinguish the two colors of 1. The generating function is given by $\frac{1}{(q^a;q^b)_\infty^k}$, where all the parts are congruent to $a(\bmod\ b)$. Through this section we shall set

$$q^m = q^{1\pm}, q^{2\pm}, q^{4\pm}, q^{5\pm}, q^{8\pm}, q^{10\pm}, \quad q^t = q^{3\pm}, q^{6\pm}, q^{9\pm} \quad \text{and} \quad q^s = q^{7\pm}, \tag{7}$$

where

$$(q^{1\pm};q^{21})_\infty = (q^1,q^{20};q^{21})_\infty, \qquad (q^{2\pm};q^{21})_\infty = (q^2,q^{19};q^{21})_\infty,$$
$$(q^{3\pm};q^{21})_\infty = (q^3,q^{18};q^{21})_\infty, \qquad (q^{4\pm};q^{21})_\infty = (q^4,q^{17};q^{21})_\infty,$$
$$(q^{5\pm};q^{21})_\infty = (q^5,q^{16};q^{21})_\infty, \qquad (q^{6\pm};q^{21})_\infty = (q^6,q^{15};q^{21})_\infty,$$
$$(q^{7\pm};q^{21})_\infty = (q^7,q^{14};q^{21})_\infty, \qquad (q^{8\pm};q^{21})_\infty = (q^8,q^{13};q^{21})_\infty,$$
$$(q^{9\pm};q^{21})_\infty = (q^9,q^{12};q^{21})_\infty \quad \text{and} \quad (q^{10\pm};q^{21})_\infty = (q^{10},q^{11};q^{21})_\infty.$$

Definition 3.1. Let $P(n,a,b,c)$ denote the number of partition of n into parts not congruent to 0(mod 21) , with parts congruent to 0(mod 3) having **a** colors and parts congruent to 0(mod 7) having **b** colors and parts not congruent to 0(mod 3) or 0(mod 7) having **c** colors.

THEOREM 3.2. *If* $q^2 f_1^4 f_{21}^4 + f_3^4 f_7^4 - f_1^3 f_3^3 f_7 f_{21} - 3q f_1^2 f_3^2 f_7^2 f_{21}^2 - 7q^2 f_1 f_3 f_7^3 f_{21}^3 = 0$, *then for* $n \geq 2$

$$P(n-2,2,0,0) + P(n,2,0,4) - P(n,0,0,1) - 3P(n-1,2,0,2) - 7P(n-2,4,0,3) = 0,$$

where $P(0) = 1$.

Proof. Dividing (4) throughout by f_1^8, we find that

$$q^2 \frac{f_{21}^4}{f_1^4} + \frac{f_3^4 f_7^4}{f_1^8} - \frac{f_3^3 f_7 f_{21}}{f_1^5} - 3q \frac{f_3^2 f_7^2 f_{21}^2}{f_1^6} - 7q^2 \frac{f_3 f_7^3 f_{21}^3}{f_1^7} = 0.$$

Using (7) in the equation above gives

$$\frac{q^2}{(q_4^m, q_4^t, q_4^s; q^{21})_\infty} + \frac{1}{(q_8^m, q_4^t, q_4^s; q^{21})_\infty} - \frac{1}{(q_5^m, q_2^t, q_4^s; q^{21})_\infty} - \frac{3q}{(q_6^m, q_4^t, q_4^s; q^{21})_\infty} - \frac{7q^2}{(q_7^m, q_6^t, q_4^s; q^{21})_\infty} = 0.$$

Multiplying the equation above with $(q_4^m, q_2^t, q_4^s; q^{21})_\infty$ gives

$$\frac{q^2}{(q_2^t; q^{21})_\infty} + \frac{1}{(q_4^m, q_2^t; q^{21})_\infty} - \frac{1}{(q_1^m; q^{21})_\infty} - \frac{3q}{(q_2^m, q_2^t, ; q^{21})_\infty} - \frac{7q^2}{(q_3^m, q_4^t; q^{21})_\infty} = 0.$$

Using the definition of $P(n,a,b,c)$ in the equation above, we obtain

$$\sum_{n=0}^{\infty} P(n-2,2,0,0)q^n + \sum_{n=0}^{\infty} P(n,2,0,4)q^n - \sum_{n=0}^{\infty} P(n,0,0,1)q^n - 3\sum_{n=0}^{\infty} P(n-1,2,0,2)q^n - 7\sum_{n=0}^{\infty} P(n-2,4,0,3)q^n = 0.$$

Comparing the coefficients of q^n in the above equation, we obtain the required result. □

As the proof of the remaining identities is similar to the one proved, we shall only state the results. The following identities are equivalent to the identities from Theorem 2.1.

THEOREM 3.3. *The following identities hold.*

For $n \geq 4$ $\quad P(n-4,0,6,0) + P(n,0,0,1) + 27P(n-4,0,12,7) + 7P(n-1,0,6,6) - P(n,0,6,8) - 7P(n-3,0,6,2) = 0.$

For $n \geq 4$ $\quad P(n-1,2,0,2) + P(n,4,0,0) + 49P(n-4,8,6,5) - P(n,0,6,1) - 27P(n-4,4,12,6) - 7P(n-3,6,6,4) = 0.$

For $n \geq 4$ $\quad P(n,0,1,0) + P(n,4,1,6) + 36P(n-4,4,0,5) + 7P(n,2,1,3) - 9P(n,2,7,6) = 0.$

For $n \geq 3$ $\quad 10 + 1323P(n-3,4,6,7) + 189P(n-2,2,6,6) + 49P(n-1,4,0,4) + 81P(n,0,6,7) - 343P(n,6,0,7) - 91P(n,2,6,3) = 0.$

For $n \geq 5$ $\quad P(n-1,2,0,1) + P(n-2,1,6,3) + 1 + 3P(n-1,4,0,4)$

$$+10P(n-5,4,6,3)+P(n,4,0,4)-13P(n-3,2,6,4)=0.$$

For $n \geq 5$ $$13+P(n-5,6,6,6)+14P(n,2,0,3)+1+196P(n-1,4,0,4)$$
$$+378P(n-2,2,3,6)-27P(n,0,7,6)-49P(n-1,6,0,7)=0.$$

For $n \geq 5$ $$1+14P(n-1,4,0,4)+14P(n-3,4,6,7)+28P(n-2,3,7,7)$$
$$+91P(n-5,6,6,6)-P(n,0,6,7)-7P(n-1,6,0,7)=0.$$

For $n \geq 4$ $$147P(n-4,5,7,6)+3P(n,1,7,8)+4$$
$$-42P(n-2,3,7,7)-7P(n,3,1,6)=0.$$

For $n \geq 4$ $$P(n-1,2,0,1)+2P(n,2,0,3)+9P(n-2,2,6,6)$$
$$-63P(n-4,4,6,5)-7P(n-1,4,0,4)=0.$$

For $n \geq 4$ $$P(n,2,0,3)+3P(n-3,2,6,5)+6P(n-2,2,6,4)-1$$
$$-3P(n-1,0,1,5)-49P(n-4,4,6,5)=0.$$

For $n \geq 5$ $$P(n,1,0,2)+2P(n-1,1,0,0)+9P(n-1,2,6,4)$$
$$-P(n-1,0,0,1)+14P(n-1,3,0,3)-9P(n-5,1,5,3)=0.$$

For $n \geq 2$ $$P(n,2,0,0)+21P(n-3,4,6,4)+6P(n,0,6,4)-3P(n-1,2,6,3)$$
$$-6P(n-2,2,2,3)-7P(n,4,0,3)=0.$$

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