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CHAIN TRANSITIVITY FOR MAPS ON G-SPACES

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Abstract. We define and study the notion of chain transitivity for maps on G-spaces. Through examples we justify that the notion of G-chain transitivity depends on the action of G. Further, we obtain characterization of G-chain transitivity in terms of chain transitivity. A relation between G-chain transitivity and G-chain recurrent points of a map is also obtained.

1. Introduction

By a discrete dynamical system we mean a pair (X, f), where X is a topological space and $f : X \to X$ is a continuous map. The primary aim of the theory of discrete dynamical systems is the study of behavior of the orbit, $O_f(x)$, of a point $x \in X$ given by $\{x, f(x), f^2(x), \ldots, f^n(x), \ldots\}$. In many situations, it is not always possible to find this exact trajectory. For instance, if the initial value of x is an approximate value, then the corresponding value of $f(x)$ will also be rough value, which further gives us an approximate value of $f^2(x)$ and so on. In this process we obtaine a new sequence of nearby values, say $\{x_0, x_1, x_2, \ldots, x_n, \ldots\}$, known as a *pseudo-orbit* or ϵ-*chain* of a map f. Applications of pseudo-orbits are much more diverse within and outside mathematics. For instance, Botelho [5] used it to study finite discrete neural networks, whereas recently Izhikevich used it in computational neuroscience [16].

Pseudo-orbits also play a key role in the study of different properties of a discrete dynamical system. For instance, one can study the theory of shadowing property if the pseudo-orbits are close to the actual orbits. Using the notion of pseudo-orbits of a map, it is possible to study various kinds of recurrence. One of such notions of recurrence, namely chain recurrence, was introduced by Conely [8] in 1978. Since its inception it has been extensively studied both for discrete dynamical systems and flows. Osipenko et al. used chain recurrence for the study of symbolic images [3]. Wiseman and Richeson [17] studied chain transitivity and chain mixing whereas Brian et al. used it to study the equivalence of various kinds of shadowing property [7].

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Good et al. studied chain transitivity on hyperspace [14]. In this paper we study chain transitivity for maps on G-spaces.

Let X be a metric G-space and $f : X \to X$ be a continuous map. Shah and Das introduced in [21] the notion of G-shadowing property for map f and through examples they observed that G-shadowing depends on the action of a group G acting on X. In [19] G-shadowing for the shift map on the inverse limit space generated by map f was studied. Choi and Kim [9] proved Spectral Decomposition type Theorem for weakly G-expansive homeomorphisms having G-shadowing property. Recently, Garg and Das [15] studied stronger forms of G-transitive maps, whereas Shah studied Devaney's chaos for maps on G-space [18]. The aim of current paper is to define and study chain transitivity for maps on G-spaces.

In Section 2 we discuss preliminaries required for the content of the paper. The notion of chain transitivity for maps on G-space is defined and studied in Section 3. Through examples it is observed that the notion of G-chain transitive depends on the action of G. We also obtain necessary and sufficient condition for the map to be G-chain transitive. Further, it is shown that the map f on a metric G-space X is G-chain transitive if and only if the corresponding induced map $\widehat{f}$ on the quotient space X/G is chain transitive. The notion of chain recurrent points for map f defined on a metric G-space X is defined in [19]. In Section 4, through examples we show that the notion of G-chain recurrent points for map depends on the action of G. Also, it is observed that the set of G-chain recurrent points, $CR_G(f)$ is a non-empty closed (G, f)-invariant subset of a compact G-space X. Further, it is shown that every G-non wandering point is a G-chain recurrent point but the converse is not true. Also, a condition is obtained for this converse to be true. In the last section of the paper we study relations between G-chain transitivity and G-chain recurrent points of maps.

2. Preliminaries

By a metric G-space X, we mean a metric space X on which a topological group G acts continuously by an action ϑ. For $g \in G$ and $x \in X$ we denote $\vartheta(g, x)$ by gx. The *G-orbit* of a point x, denoted by $G(x)$, is the set $\{gx : g \in G\}$. The set X/G of all G-orbits in X with the quotient topology induced by the quotient map $\pi : X \to X/G$ defined by $\pi(x) = G(x)$, is called the *orbit space* of X and the map π is called the *orbit map*. Note that the map π is an open continuous map. A metric d on a metric G-space X is called an *invariant metric* if $d(x, y) = d(gx, gy)$, for each $g \in G$. If X is a metric G-space with G compact then there exists an invariant metric d on X which induces a metric d_G on X/G [6], given by $d_G(G(x), G(y)) = \inf\{d(gx, ky) | g, k \in G\}$. A continuous map $f : X \to X$ is said to be a *pseudoequivariant map* if $f(G(x)) = G(f(x))$, for all $x \in X$ [10]. For details on G-space one can refer to [6, 20]. It is known that if f is a pseudoequivariant continuous map, then it induces a continuous map $\widehat{f} : X/G \to X/G$ given by $\widehat{f}(G(x)) = G(f(x))$ [10]. A map f is said to be an *equivariant map* if $gf(x) = f(gx)$ for each $x \in X$ and each $g \in G$. A subset B of X is said to be *f-invariant* if $f(B) = B$ and a subset A of X is said to be *G-invariant* if

$G(A) = A$. Note that here $G(A) = \{ga : g \in G,\ a \in A\}$. Further, a subset A of X is said to be (G, f)-*invariant* if it is both f-invariant and G-invariant. Observe that A is (G, f)-invariant if and only if $G(f(A)) = A$. For $x \in X$, the G_f*-orbit of* x, denoted by $G_f(x)$, is given as the set $\{gf^k(x) : g \in G,\ k \geq 0\}$.

Let (X, f) be a dynamical system and let $x, y \in X$. For a $\delta > 0$, δ*-chain* from x to y is a finite sequence $\{x = x_0, x_1, \ldots, x_n = y\}$ in X such that $d(f(x_i), x_{i+1}) < \delta$ for all $i = 0, 1, \ldots, n-1$. If for each $\delta > 0$, there exists a δ-chain from x to y and y to x, then the points x and y are said to be *chained.* A map f is said to be *chain transitive* if any two points of X are chained [8]. A point $x \in X$ is said to be a *chain recurrent point* if x can be chained to itself. The set of all chain recurrent points is denoted by $CR(f)$. It is known that for the compact metric space X, $CR(f)$ is a non-empty f-invariant subset of X [4]. Much literature now exists for chain transitive maps and chain recurrent points of a map. For instance, see [1, 2, 7, 8, 11–13].

Let X be a metric G-space and $f : X \to X$ be a continuous map. The notion (ϵ, G)-pseudo orbits was first introduced in [21]. We recall the definition.

Definition 2.1. Let $f : X \to X$ be a continuous map defined on a metric G-space X. For a given $\delta > 0$, a sequence of points $\{x_n : n \geq 0\}$ in X is said to be a (δ, G)-*pseudo orbit* for f if for each n there is a $g_n \in G$ satisfying $d(g_n f(x_n), x_{n+1}) < \delta$.

Obviously every ϵ-pseudo orbit is an (ϵ, G)-pseudo orbit. But the converse need not be true (for example, see [21, Example 2.3(3)]). The notion of shadowing property for maps on G-spaces was defined and studied in [21]. We recall the definition.

Definition 2.2. Let $f : X \to X$ be a continuous map defined on a metric G-space X. Then f is said to have the G-*shadowing property* if for each $\epsilon > 0$ there is a $\delta > 0$ such that for every (δ, G)-chain $\{x_n : n \geq 0\}$ for f, there is a point x in X satisfying for each $n \geq 0$, $d(g_n x_n, f^n(x)) < \epsilon$, for some $g_n \in G$.

Through examples it was observed in [21], that the notion G-shadowing property depends on the action of G. For more details on G-shadowing property and other dynamical properties of maps defined on G-space see [9, 10, 15, 18, 20].

3. G-chain transitive maps

Definition 3.1. Let X be a metric G-space and $f : X \to X$ be a continuous map. For $x, y \in X$ and $\epsilon > 0$, if there exists a finite (ϵ, G)-pseudo orbit, $\{x = x_0, x_1, \ldots, x_n = y\}$, then the (ϵ, G)-pseudo orbit is said to be an (ϵ, G)-*chain from* x *to* y. Point x is said to be G-*chained to* y if for every $\epsilon > 0$ there is an (ϵ, G)-chain from x to y. If for every $x, y \in X$, x can be G-chained to y and y can be G-chained to x, then the map f is said to be G-*chain transitive.*

Under the trivial action of G on X, the notions 'chain transitive' and 'G-chain transitive' are the same. Since every δ-pseudo orbit is a (δ, G)-pseudo orbit it follows that every chain transitive map is G-chain transitive. In general the converse is not true, which is justified by the following example.

EXAMPLE 3.2. Consider the subspace $X = \{\pm\frac{1}{n}, \pm(1-\frac{1}{n}) : n \in \mathbb{N}\}$ of $\mathbb{R}$. For $x \in X$, let x_+ denote the element of X which is immediately right to x and x_- that element of X which is immediately left to x. Let $h : X \to X$ be a homeomorphism given by

$$h(x) = \begin{cases} x, & \text{if } x \in \{-1, 0, 1\} \\ -x_+, & \text{if } 0 < x < 1, \\ -x_-, & \text{if } -1 < x < 0. \end{cases}$$

Suppose the group $G_1 = \{h^n : n \in \mathbb{Z}\}$ acts on X by the usual action. Define $f : X \to X$ by

$$f(x) = \begin{cases} x, & \text{if } x \in \{-1, 0, 1\} \\ x_-, & \text{if } x < 0 \\ x_+, & \text{if } x > 0. \end{cases}$$

Then f is G_1-chain transitive but not chain transitive. In fact if $x \in X$ is such that $x < 0$, then x can never be chained to any point y, where $y > 0$. Next, suppose $G_2 = \mathbb{Z}_2$ acts on X by the action $1x = x$ and $-1x = -x$, for each $x \in X$. Then f is not $\mathbb{Z}_2$-chain transitive.

From Example 3.2 it can also be observed that f is G-chain transitive with respect to one group but not with respect to another group. It therefore follows that the notion of G-chain transitivity depends on the action of group G on X.

DEFINITION 3.3. Let X and Y be two G-spaces and let $f : X \to X$, $g : Y \to Y$ be two continuous maps. Then f and g are said to be *topologically G-conjugate* if there is a pseudoequivariant homeomorphism $h : X \to Y$ such that $hf = gh$. The map h is then called a *G-conjugancy* between f and g.

In the following result we show that G-chain transitivity is preserved under G-conjugancy if the space is compact.

PROPOSITION 3.4. *Let (X, d) and (Y, ρ) be two compact metric G-spaces and let $f_1 : X \to X$, $f_2 : Y \to Y$ be two continuous maps. Suppose f_1 and f_2 are topologically G-conjugate by G-conjugancy h. If f is G-chain transitive then so is g.*

Proof. Sincef_1 and f_2 are topologically G-conjugate by G-conjugancy h, therefore $h : X \to Y$ is homeomorphism satisfying $hf_1 = f_2h$. Let $\epsilon > 0$ be given. Since h is uniformly continuous, it follows that for this $\epsilon > 0$, there is a $\delta > 0$ such that

$$d(x, y) < \delta \Longrightarrow \rho(h(x), h(y)) < \epsilon.$$

Using the above inequality it is easy to observe that if $\{t_0, t_1, \dots, t_n\}$ is a (δ, G)-chain for f_1 in X then $\{h(t_0), h(t_1), \dots, h(t_n)\}$ is an (ϵ, G)-chain for f_2 in Y.

Let $y_1, y_2 \in Y$. Then we show that there are (ϵ, G)-chains for f_2 in Y from y_1 to y_2 and y_2 to y_1. For this $y_1, y_2 \in Y$, there are $x_1, x_2 \in X$ such that $x_1 = h^{-1}(y_1)$ and $x_2 = h^{-1}(y_2)$. But f_1 is G-chain transitive. Therefore there are (δ, G)-chains for f in X from x_1 to x_2 and x_2 to x_1. Suppose these (δ, G)-chains are given by $\{x_1 = s_0, s_1, \dots, s_n = x_2\}$ and $\{x_2 = w_0, w_1, \dots, w_m = x_1\}$. Then $\{h(x_1) = y_1 =$

$h(s_0), h(s_1), \ldots, h(s_n) = h(x_2) = y_2\}$ and $\{h(x_2) = y_2 = h(w_0), h(w_1), \ldots, h(w_m) = h(x_1) = y_1\}$ are (ϵ, G)-chains for f_2 in Y. $\square$

In the following proposition we obtain a necessary and sufficient condition for a pseudoequivariant map f to be G-chain transitive. We first recall the following result proved in [20].

Lemma 3.5. *Let (X, d) be a compact metric G-space, where G is compact, then for $\epsilon > 0$ there are $\eta > 0$ and $\delta > 0$ such that for all g in G and x in X, $U_\eta(gx) \subset gU_\epsilon(x)$ and $gU_\delta(x) \subset U_\epsilon(gx)$.*

Proposition 3.6. *Let X be a compact metric G-space with G compact and let Y be a (G, f)-invariant dense subset of X. Suppose $f : X \to X$ is a pseudoequivariant continuous map. Then $f : X \to X$ is G-chain transitive if and only if $f_{|Y} : Y \to Y$ is G-chain transitive.*

Proof. Since Y is (G, f)-invariant subset of X, therefore $G(Y) = Y$ and $f(Y) = Y$. Also Y is dense in X implies that every point of x is either in Y or a limit point of Y.

Suppose $f : X \to X$ is G-chain transitive. Let $y_1, y_2 \in Y$ and let $\epsilon > 0$ be given. Then we show that there is an (ϵ, G)-chain from y_1 to y_2 in Y. By uniform continuity of f, for $\epsilon > 0$ there is δ, $0 < \delta < \frac{\epsilon}{2}$, such that $d(a, b) < \delta \implies d(f(a), f(b)) < \frac{\epsilon}{2}$. For $\delta > 0$, by Lemma 3.5, there is η, $0 < \eta < \frac{\delta}{2}$, such that

$$gU_\eta(x) \subset U_{\frac{\delta}{2}}(gx) \tag{1}$$

for all $g \in G$. Since $f : X \to X$ is G-chain transitive, there is an $\left(\frac{\eta}{2}, G\right)$-chain $\{y_1 = z_0, z_1, \ldots z_k = y_2\}$ for f in X. Therefore, for each $0 \leq n \leq k-1$, there exist $g_n \in G$ satisfying $d(f(g_n z_n), z_{n+1}) < \frac{\eta}{2}$. Further, Y is dense in X. Therefore for $z_n \in X$, there exists $t_n \in Y$ such that $t_n \in U_\eta(z_n)$. By using the equation (1), it follows that for each $0 \leq n \leq k-1$, $g_n t_n \in U_{\frac{\delta}{2}}(g_n z_n)$. Note that G-invariancy of Y implies that $g_n t_n$ is also in Y. Now for $n, 0 \leq n \leq k-1$, consider

$$d(f(g_n t_n), t_{n+1}) \leq d(f(g_n t_n), f(g_n z_n)) + d(f(g_n z_n), z_{n+1}) + d(z_{n+1}, t_{n+1}) < \epsilon$$

Thus, $\{y_1 = t_0, t_1, \ldots, t_k = y_2\}$ is an (ϵ, G)-chain for f in Y. Similarly we can obtain an (ϵ, G)-chain from y_2 to y_1.

Conversely, suppose that $f : Y \to Y$ is G-chain transitive. Let $x_1, x_2 \in X$ and let $\epsilon > 0$ be given. We show that there is an (ϵ, G)-chain from x_1 to x_2 for f in X. By uniform continuity of f, there is $\delta > 0$ such that $d(a, b) < \delta \implies d(f(a), f(b)) < \epsilon$. Now, let $w \in f^{-1}(x_2)$. Then there are $z_0, z_1 \in Y$ such that $d(z_0, f(x_1)) < \epsilon$ and $d(z_1, w) < \delta$. This further implies that $d(f(z_1), x_2) < \epsilon$. Using G-chain transitivity of $f : Y \to Y$ there is an (ϵ, G)-chain $\{z_0 = a_0, a_1, \ldots, a_k = z_1\}$ from z_0 to z_1. Since $d(f(x_1), z_0) < \epsilon$ and $d(f(z_1), x_2) = d(f(z_1), f(w)) < \epsilon$, it follows that $\{x_1, a_0, a_1, \ldots, a_k = z_1, x_2\}$ is an (ϵ, G)-chain for f from x_1 to x_2 in X. $\square$

Recall that a continuous group action $\theta : G \times X \to X$ acts equicontinuously on X, if for every $\epsilon > 0$ there is $\delta > 0$ such that for any $x, y \in X$ with $d(x, y) < \delta$ implies $d(\theta(g, x), \theta(g, y)) = d(gx, gy) < \epsilon$, for all $g \in G$. Equivalently, an action is equicontinuous, if the family of homeomorphisms given by $\{\theta_g : X \to X : g \in G\}$ is equicontinuous. It is known that every compact topological group acts equicontinuously on compact metric space X (for example, see [9, Lemma 2.3]).

If X contains a proper, clopen, nonempty, (G, f)-invariant set A, then f is not G-chain transitive on X. For, if $\epsilon > 0$ is smaller than the distance from A to its complement, then there is no (ϵ, G)-pseudo orbit between points of A and points of A complement. The following proposition shows that the conditions of clopen and (G, f)-invariancy for A is essential.

Proposition 3.7. *Suppose the action of G on a compact metric space X is equicontinuous and suppose $f : X \to X$ is a continuous map. Let A, B be two non-empty (G, f)-subsets of X such that $d(\overline{A}, \overline{B}) = 0$. If $f_{|A}$ and $f_{|B}$ are G-chain transitive then $f_{|(A\cup B)}$ is G-chain transitive.*

Proof. Let $p, q \in A \cup B$ and $\epsilon > 0$ be given. Then without loss of generality, we can assume that $p \in A$ and $q \in B$. Since $d(\overline{A}, \overline{B}) = 0$ and G acts equicontinuously on X, it follows that there exist $x \in A$ and $y \in B$ such that for all $g \in G$, $d(gf(x), gf(y)) < \frac{\epsilon}{2}$. It is now easy to verify that if $\{x_0 = p, x_1, \dots, x_n = x\}$ is an $(\frac{\epsilon}{2}, G)$-chain from p to x, and $\{y_0 = y, y_1, \dots, y_m = q\}$ is an $(\frac{\epsilon}{2}, G)$-chain from y to q, then $\{x_0 = p, x_1, \dots, x_n = x, y_1, \dots, y_m = q\}$ is an (ϵ, G)-chain from p to q. □

Let (X, d) be a compact metric G-space with G compact and let the corresponding orbit space be given by X/G with the induced metric d_G. Let $f : X \to X$ be a continuous pseudoequivariant map with the corresponding induced map $\widehat{f} : X/G \to X/G$ given by $\widehat{f}(G(x) = G(f(x))$. We now study the relation between G-chain transitivity of the map f and chain transitivity of the map $\widehat{f}$.

Theorem 3.8. *Let X be a compact metric G-space with G compact. Suppose that $f : X \to X$ is a pseudoequivariant continuous map. Then f is G-chain transitive if and only if the corresponding induced map $\widehat{f} : X/G \to X/G$ is chain transitive.*

Proof. Suppose $\widehat{f} : X/G \to X/G$ is chain transitive. Let $x, y \in X$ and let $\epsilon > 0$ be given. Then we show that there is an (ϵ, G)-chain from x to y. Since G is compact it follows that the action of G on X is an equicontinuous action. Therefore there is $\delta > 0$ such that for all $g \in G$

$$d(t, w) < \delta \Longrightarrow d(gt, gw) < \epsilon. \tag{2}$$

For $x, y \in X$ consider the corresponding points $G(x), G(y)$ in X/G. Since $\widehat{f}$ is chain transitive, it follows that there is a δ-chain for $\widehat{f}$ in X/G, say $\{G(x) = G(x_0), G(x_1), \dots, G(x_k) = G(y)\}$, from $G(x)$ to $G(y)$. Therefore for $0 \le n \le k-1$, $d_G(\widehat{f}(G(x_n)), G(x_{n+1})) = \inf \{d(gf(x_n), hx_{n+1}) \mid g, h \in G\} < \delta$. But G is compact, therefore for each n, there are $g_n, h_n \in G$ such that $d(g_n f(x_n), h_n x_{n+1}) < \delta$. Thus the equation (2) implies that for each n, there is $t_n = h_n^{-1} g_n \in G$ satisfying $d(t_n f(x_n), x_{n+1}) < \epsilon$. Hence $\{x = x_0, x_1, \dots, x_n = y\}$ is an (ϵ, G)-chain for f. Thus x is G-chained to y. But $x, y \in X$ are arbitrary and therefore f is G-chain transitive.

Conversely, suppose $f : X \to X$ is G-chain transitive. Let $G(x), G(y) \in X/G$ and let $\epsilon > 0$ be given. We show that there is an ϵ-chain for $\widehat{f}$ from $G(x)$ to $G(y)$ in X/G.

Now, X is compact and the orbit map $\pi : X \to X/G$ is continuous. Therefore there is $\delta > 0$ such that

$$d(t,w) < \delta \Longrightarrow d_G(G(t), G(w)) < \epsilon. \tag{3}$$

For $G(x), G(y) \in X/G$, consider corresponding $x, y \in X$. Then, f is G-chain transitive implies that there is a (δ, G)-chain for f in X, say $\{x = x_0, x_1, \ldots, x_k = y\}$ from x to y. This implies that for each $0 \leq n \leq k-1$, there is $g_n \in G$ satisfying $d(g_n f(x_n), x_{n+1}) < \delta$. Therefore, using the equation (3), we obtain $d_G(\widehat{f}(G(x_n)), G(x_{n+1})) < \epsilon$. Hence $\{G(x) = G(x_0), G(x_1), \ldots, G(x_k) = G(y)\}$ is an ϵ-chain for $\widehat{f}$ from $G(x)$ to $G(y)$. Therefore $\widehat{f}$ is chain transitive. □

4. G-chain recurrent points

We recall the definition of G-chain recurrent points for a map defined in [19].

Definition 4.1. Let X be a metric G-space and let $f : X \to X$ be a continuous map. A point $x \in X$ is called a *G-chain recurrent* point if x can be G-chained to itself. The set of G-chain recurrent points is called the G-chain recurrent set of f and denoted by $\mathcal{CR}_G(f)$.

Under the trivial action of G on X, the notions of chain recurrent points and G-chain recurrent points are the same. Further, under non-trivial action of G it follows that $CR(f) \subset CR_G(f)$ and therefore $CR_G(f)$ is always non-empty for compact spaces. A G-chain recurrent point need not be chain recurrent point, as can be seen from Example 4.2.

Example 4.2. Consider the subspace $X = \left\{\pm\frac{1}{n}, \pm\left(1 - \frac{1}{n}\right) : n \in \mathbb{N}\right\}$ of $\mathbb{R}$ with the usual metric of $\mathbb{R}$. Suppose groups G_1 and G_2 act on X as in Example 3.2. If f is the left shift fixing $-1, 0, 1$ then $CR_{G_1}(f) = X$ but $CR(f) = \{-1, 0, 1\} = CR_{G_2}(f)$.

From Example 4.2, it can also be observed that a point can be G-chain recurrent with respect to one group, but need not be with respect to another group. Hence the notion depends on the action of G. It is known that $CR(f)$ is a closed f-invariant set [4]. In the following proposition we show that $CR_G(f)$ is a closed (G,f)-invariant set.

Proposition 4.3. *Let X be a compact metric G-space with G compact and let $f : X \to X$ be a continuous pseudoequivariant map. Then $CR_G(f)$ is a closed (G,f)-invariant subset of X.*

Proof. Let $\epsilon > 0$ be given. Then by uniform continuity of f there is a positive real number δ such that $d(a,b) < \delta \Longrightarrow d(f(x), f(y)) < \epsilon$.

We first show that $CR_G(f)$ is a closed subset of X. Let x be a limit point of $CR_G(f)$. Then there is a sequence $\{x_n\}$ in $CR_G(f)$ such that $\{x_n\}$ converges to x. Since x_n is a G-chain recurrent point of f, it follows that there is a (δ, G)-chain, $\{x_n =$

$y_0, y_1, \dots, y_k = x_n\}$, for f in X. It is now easy to verify that $\{x = y_0, y_1, \dots, y_k = x\}$ is an (ϵ, G)-chain for f from x to itself. Hence $x \in CR_G(f)$.

For $x \in CR_G(f)$ and $g \in G$ we show that $gx \in CR_G(f)$. Since G is compact, it follows that the action G on X is equicontinuous. Therefore, for $\epsilon > 0$ there is $0 < \eta < \epsilon$, such that $d(a,b) < \eta \Longrightarrow d(ta,tb) < \epsilon$, for all $t \in G$. Let $\{x = x_0, x_1, \dots\dots, x_k = x\}$ be an (η, G)-chain for f from x to itself. Then, there is $g_0 \in G$ such that $d(g_0 f(x_0), x_1) = d(k_0 f(gx_0), x_1) < \eta < \epsilon$, where $k_0 = g_0 l \in G$. Here l is obtained by using pseudoequivariancy of f. Next, there is $g_{n-1} \in G$ such that

$$d(g_{n-1}f(x_{n-1}), x_n) < \eta \Longrightarrow d(gg_{n-1}f(x_{n-1}), gx_n) = d(k_{n-1}f(x_{n-1}), gx) < \epsilon,$$

for $k_{n-1} = gg_{n-1} \in G$. Hence $\{gx = gx_0, x_1, \dots\dots, gx_k = gx\}$ is an (ϵ, G)-chain for f from gx to itself. Therefore $gx \in CR_G(f)$. But $g \in G$ is arbitrary. Therefore $CR_G(f)$ is a G-invariant set.

Next, we show that $f(CR_G(f)) \subset CR_G(f)$. For $y \in f(CR_G(f))$ then there is $x \in CR_G(f)$ such that $f(x) = y$. If $\{x = x_0, x_1, \dots, x_k = x\}$ is a finite (δ, G)-chain from x to itself then $\{y = f(x_0), f(x_1), \dots, f(x_k) = y\}$ is a finite (ϵ, G)-chain from y to itself and hence $y \in CR_G(f)$.

Conversely, we show that $\mathcal{CR}_G(f) \subseteq f(\mathcal{CR}_G(f))$. Let $x \in \mathcal{CR}_G(f)$. Then for every $m \in \mathbb{N}$, there is a $(\frac{1}{m}, G)$-chain, $\{x_i^m : 0 \leq i \leq n_m + 1\}$, from x to itself. Therefore for each $0 \leq i \leq n_m + 1$, there is $g_i \in G$ such that $d\left(f(g_i x_i^m), x_{i+1}^m\right) < \frac{1}{m}$. In particular, for each $m \in \mathbb{N}$, there is $g_{n_m} \in G$ such that

$$d(g_{n_m} f(x_{n_m}), x) < \frac{1}{m}. \tag{4}$$

Let y be the limit point of convergent sequence $\{g_{n_m} x_{n_m}\}$ in the compact metric space X. Note that we are denoting the convergent subsequence as the same sequence. Also, the inequality (4) implies that $f(y) = x$. We complete the proof by showing that $y \in \mathcal{CR}_G(f)$.

Let $\epsilon_1 > 0$ be given. Since G is a compact space it follows that the action G on X is equicontinuous. Therefore there is δ_1, $0 < \delta_1 < \frac{\epsilon_1}{6}$ such that for all $g \in G$, $d(a,b) < \delta_1 \Longrightarrow d(gf(a), gf(b)) < \frac{\epsilon_1}{6}$. Choose $m \in \mathbb{N}$ such that $0 < \frac{1}{m} < \delta_1$. Let the corresponding $(\frac{1}{m}, G)$-chain from x to itself be given by $\{x_i^m : 0 \leq i \leq n_m + 1\}$. Consider the sequence $\left\{y, x = x_0^m, x_1^m, \dots, x_{n_m-1}^m, y\right\}$. Then this is an (ϵ, G)-chain from y to itself as there is $e \in G$ such that $d(ef(y), x_0^m) = d(ex, x) = 0 < \epsilon_1$ and there is $h = g_{n_m} g_{n_m - 1} \in G$ satisfying

$$\begin{aligned} d(hf(x_{n_m-1}^m), y) &= d(g_{n_m} g_{n_m-1} f(x_{n_m-1}^m), y) \\ &\leq d(g_{n_m} g_{n_m-1} f(x_{n_m-1}^m), g_{n_m} x_{n_m}^m) + d(g_{n_m} x_{n_m}^m, y) < \frac{\epsilon_1}{3}. \end{aligned}$$

Therefore $y \in \mathcal{CR}_G(f)$. Hence we obtain $f(CR_G(f)) = CR_G(f)$. □

Recall from [20], that a point x in X is said to be a *G-non wandering point* of f if for every neighbourhood U of x there is an integer $n > 0$ and a $g \in G$ such that $gf^n(U) \cap U \neq \emptyset$. If $\Omega_G(f)$ denotes the set of all G-nonwandering points then it is observed in [20] that $\Omega_G(f)$ is a closed (G, f)-invariant subset of X which is non-empty if X is compact. Further, it is easy to observe that every G-nonwandering point is a G-chain recurrent point. However, the converse need not be true, that can be observed from the following example.

EXAMPLE 4.4. Consider $I = [0, 1]$ as a subspace of $\mathbb{R}$ and let $G = \mathbb{Z}_2$ act on I by the usual action. Define a map

$$f(x) = \begin{cases} \sqrt{\frac{x}{3}}, & \text{if } 0 \leq x \leq \frac{1}{3} \\ 2x - \frac{1}{3}, & \text{if } \frac{1}{3} \leq x \leq \frac{2}{3} \\ 3 - 3x, & \text{if } \frac{2}{3} \leq x \leq 1 \end{cases}$$

Then each point of $(0, \frac{1}{3})$ is a $\mathbb{Z}_2$-chain recurrent point but not a $\mathbb{Z}_2$-non wandering point. Here $\Omega_G(f) = \{0, \frac{1}{3}, \frac{3}{4}\}$.

It is known that a chain recurrent point of a map f is a non-wandering point of f if the map has shadowing property [4]. In the following theorem, using the G-shadowing property, we show that every G-chain recurrent point is a G-nonwandering point.

THEOREM 4.5. *Let X be a compact metric G-space with G compact and let $f : X \to X$ be a continuous pseudoequivariant map. If f has the G-shadowing property then $CR_G(f) = \Omega_G(f)$.*

Proof. It is sufficient to show that $CR_G(f) \subset \Omega_G(f)$. For a given $\epsilon > 0$ by Lemma 3.5 there is an $\eta > 0$ such that for all $y \in X$ and $g \in G$, $U_\eta(gy) \subset gU_\epsilon(y)$. The G-shadowing property of f implies that there is a $\delta > 0$ such that every (δ, G)-pseudo orbit for f is η-shadowed by a point of X. Let $x \in CR_G(f)$ and let U be an open set containing x. Then there is a finite (δ, G)-pseudo orbit $\{x = x_0, x_1, \ldots, x_k = x\}$ for f. Therefore there is a point y in X η-tracing $\{x{=}x_0,\ x_1, \ldots\ldots, x_k{=}x\}$. This implies that there exists g_0, g_k in G satisfying $d(y, g_n x) < \eta$ and $d(f^k(y), g_k x) < \eta$, which further implies that there is an $l \in G$ such that $lf^k(U) \cap U \neq \emptyset$. □

It is known that if a continuous map $f : X \to X$ has the G-shadowing property then so does $f_{|\,\Omega_G(f)}$, [20, Theorem 3.8]. Therefore, in view of the above theorem, it follows that if f has G-shadowing property then so does $f_{|CR_G(f)}$. Note that this is one of the key observation used in the proof of Decomposition Theorem proved in [9]. In the following results we relate G-chain recurrent points of f and $f^n, n \geq 1$.

LEMMA 4.6. *Let X be a compact metric G-space and suppose the action of G on X is equicontinuous. For $\epsilon > 0$ and $M \in \mathbb{N}$, there is $\delta > 0$, such that for every infinite (δ, G)-pseudo orbit $\{x_n : n \geq 0\}$ there is $g_n \in G$, $n \in \mathbb{N}$, satisfying $d(g_n f^M(x_n), x_{n+M}) < \epsilon$*

Proof. Let $\epsilon > 0$ and $M \in \mathbb{N}$ be given. Since the action of G on X is an equicontinuous action, it follows that there exists $\eta > 0$ such that for each $g \in G$: $d(a, b) < \eta \Longrightarrow d(ga, gb) < \frac{\epsilon}{M}$. Also, for each $0 \leq i \leq M - 1$, f^i is uniform continuous. Therefore for $\eta > 0$, there is $\delta > 0$ such that for each $0 \leq i \leq M-1$, $d(a, b) < \delta \Longrightarrow d(f^i(a), f^i(b)) < \eta$. Consider a (δ, G)-psuedo orbit $\{x_n : n \geq 0\}$. Then for each $n \geq 0$, there is $h_n \in G$ satisfying $d(g_n f(x_n), x_{n+1}) < \delta$. It is now easy to verify that for each $n \in \mathbb{N}$ and $g_n = h_{n+M} h_{n+M-1} \ldots h_{n+1} h_n$ in G $d(g_n f^M(x_n), x_{n+M}) < \epsilon$. □

Let $k \in \mathbb{N}$. Then it is obvious that every (ϵ, G)-chain for f^k is also an (ϵ, G)-chain for f. By using Lemma 4.6, it follows that if there is an (δ, G)-chain from x to y from length multiple of k for f, then there is an (ϵ, G)-chain from x to y for f^k. Hence we have the following result.

Proposition 4.7. *Let X be a compact metric space and suppose the action of G on X is an equicontinuous action. If $f : X \to X$ is a pseudoequivariant map, then for every $n \in \mathbb{N}$, $\mathcal{CR}_G(f^n) = \mathcal{CR}_G(f)$.*

5. G-chain transitive maps and G-chain recurrent points

Let X be a metric G-space and $f : X \to X$ be a G-chain transitive map. Then for every $x, y \in X$ and every $\epsilon > 0$, there is an (ϵ, G)-chain, say $\{x = x_0, x_1, \ldots, x_n = y\}$, from x to y and an (ϵ, G)-chain, say $\{y = y_0, y_1, \ldots, y_m = x\}$, from y to x. It now follows that $\{x = x_0, x_1, \ldots, x_n = y = y_0, y_1, \ldots, y_m = x\}$ is an (ϵ, G)-chain from x to itself. Hence in this case $CR_G(f) = X$. In the following proposition we show that the converse is true if the space is connected.

Proposition 5.1. *Let X be a compact metric G-space and $f : X \to X$ be a continuous pseudoequivariant surjective map. Suppose $\mathcal{CR}_G(f)$ is a connected subset of X. Then f is G-chain transitive.*

Proof. Let $x, y \in X$ and $\epsilon > 0$ be given. Then we show that there is an (ϵ, G)-chain from x to y. By the compactness of space X it follows that the sequence $\{f^n(x)\}_{n=0}^{\infty}$ has a convergent subsequence, say $\{f^{n_k}(x)\}$, with the limit point p. Therefore $p \in \Omega_G(f)$ and hence $p \in \mathcal{CR}_G(f)$. Now, we show that there is an $\left(\frac{\epsilon}{3}, G\right)$-chain from x to p. Further, for $\frac{\epsilon}{3}$, there is $m_k > 0$ such that $d(f^{m_k}(x), p) < \frac{\epsilon}{3}$. It is now easy to verify that $\{x, f(x), f^2(x), \ldots, f^{m_k-1}(x), p\}$ is an $\left(\frac{\epsilon}{3}, G\right)$-chain from x to p.

Next, using surjectivity of f, there is a sequence $\{y_n\}_{n=0}^{\infty}$ such that $f(y_{n+1}) = y_n$ and $y_0 = y$. Let q be the limit point of a subsequence, say $\{y_{r_k}\}$, of $\{y_n\}$. Then $q \in \mathcal{CR}_G(f)$. For $\frac{\epsilon}{3}$, there is $\delta > 0$ such that $d(t, w) < \delta \Longrightarrow d(f(t), f(w)) < \frac{\epsilon}{3}$. Further, there is $j_k > 0$ such that $d(y_{j_k}, q) < \delta$. It now follows that $\{q, y_{j_k}, y_{j_k-1}, \ldots, y_1, y_0 = y\}$ is an $\left(\frac{\epsilon}{3}, G\right)$-chain from q to y. Since $\mathcal{CR}_G(f)$ is connected, there are finitely many points $\{z_i : 0 \leq i \leq m\}$ in $\mathcal{CR}_G(f)$ such that $z_0 = p$, $z_m = q$ and for all $i, 0 \leq i < m$,

$$d(z_i, z_{i+1}) < \frac{\epsilon}{6}. \tag{5}$$

Now, $z_0, z_1 \in CR_G(f)$. Therefore there are $\left(\frac{\epsilon}{6}, G\right)$-chains $\{z_0 = z_0^0, z_0^1, \ldots, z_0^{k_0-1}, z_0^{k_0} = z_0\}$ and $\{z_1 = z_1^0, z_1^1, \ldots, z_1^{k_1} = z_1\}$. Using the equation (5), this now implies that $\{z_0 = z_0^0, z_0^1, \ldots, z_0^{k_0-1}, z_1 = z_1^0, z_1^1, \ldots, z_1^{k_1} = z_1\}$ is an $\left(\frac{\epsilon}{3}, G\right)$-chain from z_0 to z_1. By a similar argument we obtain an $\left(\frac{\epsilon}{3}, G\right)$-chain from z_1 to z_2. Further, combining these two chains one gets an $\left(\frac{\epsilon}{3}, G\right)$-chain from z_0 to z_2. Continuing this process we obtain an $\left(\frac{\epsilon}{3}, G\right)$-chain from $z_0 = p$ and $z_m = q$. Thus, we have $\left(\frac{\epsilon}{3}, G\right)$-chains from x to p, p to q and q to y. Therefore there is an (ϵ, G)-chain from x to y. □

COROLLARY 5.2. *Let X be a compact connected metric G-space and let $f : X \to X$ be a continuous pseudoequivariant surjective map. Then f is G-chain transitive if and only if $\mathcal{CR}_G(f) = X$.*

In the following theorem we obtain some more conditions under which f is G-chain transitive.

PROPOSITION 5.3. *Let X be a compact metric G-space with G compact, $f : X \to X$ be a continuous pseudoequivariant map and let Y be a subset of X. Suppose $\Omega_G(f) \subseteq Y$ and f is G-chain transitive on Y. Then f is G-chain transitive on X.*

Proof. Let $x, y \in X$ and $\epsilon > 0$ be given. If $x, y \in \Omega_g(f)$, then there is an (ϵ, G)-chain from x to y as every G-nonwandering point is a G'-chain recurrent point. Assume $x, y \in X \setminus \Omega_G(f)$. We show that there is an (ϵ, G)-chain from y to x. Set

$$K = \left\{ w \;\middle|\; d\left(w, \Omega_G(f)\right) \geq \frac{\epsilon}{4} \right\}.$$

Then K is a non-empty closed subset of X as both x, y are in K. Also, if $w \in K$, then w is not a G-nonwandering point. Therefore for every $w \in K$ there is an open neighborhood $U(w)$ of w such that $gf^m(U(w)) \cap U(w) = \emptyset$, for all $m \geq 1$ and all $g \in G$. Since K is a compact set, there is $\{w_1, w_2, \ldots, w_k\} \subseteq K$, such that $K \subseteq \bigcup_{i=1}^k U(w_i)$. Take $p \in f^{-k}(x)$. Since for all $n \geq 1$, $f^n(U(w_j)) \cap U(w_j) = \emptyset$, there is an n, $0 \leq n \leq k$, such that $f^n(p) \notin K$. But this implies that $d(f^n(p), \Omega_G(f)) < \frac{\epsilon}{4}$. Choose $a \in \Omega_G(f)$ with $d(f^n(p), a) < \frac{\epsilon}{4}$. Consider the set $F = \{gf^n(y) : g \in G, n \in \mathbb{N}\}$ in X and let b be an accumulation point of F. Then there exist $m \in \mathbb{N}$ and $g_m \in G$ such that $d(g_m f^m(y), b) < \frac{\epsilon}{4}$. Since b is a limit point it follows that $b \in \Omega_G(f)$. Therefore there is an $\left(\frac{\epsilon}{4}, G\right)$-chain from a to b. Using this fact, inequality $d(f^n(p), a) < \frac{\epsilon}{4}$, $0 \leq n \leq k$, and $p \in f^{-k}(x)$ we can obtain an (ϵ, G)-chain from y to x. □

Since $\Omega_G(f) \subset CR_G(f)$, it follows from the above theorem that if $f_{|CR_G(f)}$ is G-chain transitive then f is G-chain transitive.

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