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THE EXISTENCE OF HOMOGENEOUS GEODESICS IN SPECIAL HOMOGENEOUS FINSLER SPACES

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Abstract. A well known result by O. Kowalski and J. Szenthe says that any homogeneous Riemannian manifold admits a homogeneous geodesic through any point. This was proved by the algebraic method using the reductive decomposition of the Lie algebra of the isometry group. In previous papers by the author, the existence of a homogeneous geodesic in any homogeneous pseudo-Riemannian manifold and also in any homogeneous affine manifold was proved. In this setting, a new method based on affine Killing vector fields was developed. Using this method, it was further proved that any homogeneous Lorentzian manifold of even dimension admits a light-like homogeneous geodesic and any homogeneous Finsler space of odd dimension admits a homogeneous geodesic. In the present paper, the affine method is further refined for Finsler spaces and it is proved that any homogeneous Berwald space or homogeneous reversible Finsler space admits a homogeneous geodesic through any point.

1. Introduction

Let M be either a pseudo-Riemannian manifold (M, g), or a Finsler space (M, F), or an affine manifold (M, ∇). If there is a connected Lie group G which acts transitively on M as a group of isometries, respectively, of affine diffeomorphisms, then M is called a *homogeneous manifold*. It can be naturally identified with the *homogeneous space* G/H, where H is the isotropy group of the origin $p \in M$.

A geodesic $\gamma(s)$ through the point p is *homogeneous* if it is an orbit of a one-parameter group of isometries, respectively, of affine diffeomorphisms. More explicitly, if s is an affine parameter and $\gamma(s)$ is defined in an open interval J, there exists a diffeomorphism $s = \varphi(t)$ between the real line and the open interval J and a nonzero vector $X \in \mathfrak{g}$ such that $\gamma(\varphi(t)) = \exp(tX)(p)$ for all $t \in \mathbb{R}$. The vector X is called a *geodesic vector*. The diffeomorphism $\varphi(t)$ may be nontrivial only for null geodesics in a properly pseudo-Riemannian manifold or for geodesics in affine manifolds.

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A homogeneous Riemannian manifold (M, g) or a homogeneous Finsler space (M, F) is always a *reductive homogeneous space*: We denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H respectively and consider the adjoint representation $\mathrm{Ad}\colon H \times \mathfrak{g} \to \mathfrak{g}$ of H on $\mathfrak{g}$. There exists a *reductive decomposition* of the form $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ where $\mathfrak{m} \subset \mathfrak{g}$ is a vector subspace such that $\mathrm{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m}$. For a fixed reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ there is the natural identification of $\mathfrak{m} \subset \mathfrak{g} = T_e G$ with the tangent space $T_p M$ via the projection $\pi\colon G \to G/H = M$. Using this natural identification and the scalar product or the Minkowski norm on $T_p M$, we obtain the invariant scalar product $\langle\, , \rangle$ or the invariant Minkowski norm denoted again by F and its fundamental tensor g on $\mathfrak{m}$. It will be clear from the context whether g means the pseudo-Riemannian metric on the manifold or the fundamental tensor on $\mathfrak{m}$ coming from the Finsler metric. In the second case, it is used usually with the subscript in the form g_X. In the pseudo-Riemannian reductive case, geodesic vectors are characterized by the following *geodesic lemma*.

Lemma 1.1 ([9,11,13]). *Let $(G/H, g)$ be a reductive homogeneous pseudo-Riemannian manifold and $X \in \mathfrak{g}$. Then the curve $\gamma(t) = \exp(tX)(p)$ is geodesic with respect to some parameter s if and only if $\langle [X, Z]_{\mathfrak{m}}, X_{\mathfrak{m}} \rangle = k \langle X_{\mathfrak{m}}, Z \rangle$, for all $Z \in \mathfrak{m}$ and for some constant $k \in \mathbb{R}$. If $k = 0$, then t is an affine parameter for this geodesic. If $k \neq 0$, then $s = e^{-kt}$ is an affine parameter for the geodesic. The second case can occur only if the curve $\gamma(t)$ is a null curve in a properly pseudo-Riemannian space.*

In the above formula, the subscript $\mathfrak{m}$ refers to the $\mathfrak{m}$-component of vectors from $\mathfrak{g}$. The Finslerian version of this lemma was proved in [14].

Lemma 1.2 ([14]). *Let $(G/H, F)$ be a homogeneous Finsler space. The vector $X \in \mathfrak{g}$ is a geodesic vector if and only if it holds $g_{X_{\mathfrak{m}}}([X, Z]_{\mathfrak{m}}, X_{\mathfrak{m}}) = 0$, for all $Z \in \mathfrak{m}$.*

Another possible approach is to study the manifold M using a more fundamental affine method, which was proposed by the present author, O. Kowalski and Z. Vlášek in [7, 10]. It is based on the well known fact that a homogeneous manifold M with the origin p admits $n = \dim M$ fundamental vector fields (Killing vector fields) which are linearly independent at each point of some neighbourhood of p. Also, in a homogeneous space $M = G/H$ with an invariant affine connection ∇, each regular orbit of a 1-parameter subgroup $g_t \subset G$ on M is an integral curve of an affine Killing vector field on M.

Lemma 1.3 ([10]). *The integral curve γ of a nonvanishing Killing vector field Z on $M = (G/H, \nabla)$ is geodesic if and only if $\nabla_{Z_{\gamma(t)}} Z = k_\gamma \cdot Z_{\gamma(t)}$ holds along γ, where $k_\gamma \in \mathbb{R}$ is a constant. If $k_\gamma = 0$, then t is the affine parameter of geodesic γ. If $k_\gamma \neq 0$, then the affine parameter of this geodesic is $s = e^{k_\gamma t}$.*

In the paper [12], O. Kowalski and J. Szenthe proved that any homogeneous Riemannian manifold admits a homogeneous geodesic through the origin. The generalization to the pseudo-Riemannian (reductive and nonreductive) case was obtained by the present author in [5] in the framework of a more general result, which says that any homogeneous affine manifold (M, ∇) admits a homogeneous geodesic through the origin. Here the affine method from [7, 10], based on the study of integral curves of Killing vector fields, was used. The proof is also using differential topology, namely smooth mappings $\mathbb{S}^n \to \mathbb{S}^n$.

In the paper [6] by the present author, the affine method used in [5, 7, 10] for the study of homogeneous affine manifolds was adapted to the pseudo-Riemannian case and it was shown that any Lorentzian homogeneous manifold of even dimension admits a light-like homogeneous geodesic through the origin.

Recently, in a paper by Z. Yan, the existence of a homogeneous geodesic in homogeneous Finsler space of odd dimension was claimed. Unfortuantely, the proof in this paper, using the algebraic method by O. Kowalski and J. Szenthe, is wrong. The correct proof was given in [8] by adapting the affine method to the Finslerian situation.

In the present paper, the affine method for the Finslerian situation is further refined and used to prove that in a homogeneous Berwald space and in a homogeneous reversible Finsler space, a homogeneous geodesic always exists.

2. Basic settings

Recall that a *Minkowski norm* on the vector space $\mathbb{V}$ is a nonnegative function $F : \mathbb{V} \to \mathbb{R}$ which is smooth on $\mathbb{V} \setminus \{0\}$, positively homogeneous ($F(\lambda y) = \lambda F(y)$ for any $\lambda > 0$) and whose Hessian $g_{ij} = (\frac{1}{2}F^2)_{y^i y^j}$ is positively definite on $\mathbb{V} \setminus \{0\}$. Here (y^i) are the components of a vector $y \in \mathbb{V}$ with respect to a fixed basis B of $\mathbb{V}$ and putting y^i to a subscript means the patrial derivative. Then the pair $(\mathbb{V}, F)$ is called the *Minkowski space*. The tensor g_y with components $g_{ij}(y)$ is the *fundamental tensor*. The *Cartan tensor* C_y has components $C_{ijk}(y) = (\frac{1}{4}F^2)_{y^i y^j y^k}$. A Finsler metric on the smooth manifold M is a function F on TM which is smooth on $TM \setminus \{0\}$ and whose restriction to any tangent space T_xM is a Minkowski norm. Then the pair (M, F) is called the *Finsler space*. On a Finsler space, functions g_{ij} and C_{ijk} depend smoothly on $x \in M$ and on $o \neq y \in T_xM$.

Further, we recall that the *slit tangent bundle* TM_0 is defined as $TM_0 = TM \setminus \{0\}$. Using the restriction of the natural projection $\pi \colon TM \to M$ to TM_0, we naturally construct the pullback vector bundle π^*TM over TM_0, as indicated in the following diagram:

$$\begin{array}{ccc} \pi^*TM & & TM \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ TM_0 & \xrightarrow{\pi} & M \end{array}$$

For a given local coordinate system $(x^1, \dots, x^n)$ on $U \subset M$, at any $x \in M$, one has a natural basis $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\}$ of T_xM. It is natural to express tangent vectors $y \in T_xM$ with respect to this basis. Then (x^i, y^i) is the *natural coordinate system* on

TU_0. We define further functions on TU_0, namely the *formal Christoffel symbols* γ^i_{jk} and the *nonlinear connection* N^i_j, by the formulas

$$\gamma^i_{jk} = g^{is}\frac{1}{2}\Big(\frac{\partial g_{sj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^s} + \frac{\partial g_{ks}}{\partial x^j}\Big), \quad N^i_j = \gamma^i_{jk}y^k - C^i_{jk}\gamma^k_{rs}y^r y^s. \tag{1}$$

The *Chern connection* is the unique linear connection on the vector bundle π^*TM which is torsion free and almost g-compatible, hence its connection forms satisfy

$$dx^j \wedge \omega^i_j = 0, \quad dg_{ij} - g_{kj}\omega^k_i - g_{ik}\omega^k_j = 2C_{ijs}(dy^s + N^s_k dx^k).$$

It follows that the following holds

$$\omega^i_j = \Gamma^i_{jk}dx^k, \quad \Gamma^i_{jk} = \Gamma^i_{kj}, \quad \Gamma^l_{jk} = \gamma^l_{jk} - g^{li}(C_{ijs}N^s_k - C_{jks}N^s_i + C_{kis}N^s_j), \tag{2}$$

(for details, see for example [1, 3]). If we fix a nowhere vanishing vector field V on M, we obtain an affine connection ∇^V on M. In the local chart, it is expressed with respect to arbitrary vector fields $W_1 = W_1^i\frac{\partial}{\partial x^i}$ and $W_2 = W_2^i\frac{\partial}{\partial x^i}$ by the formula

$$\nabla^V_{W_1}W_2|_x = \big[W_1(W_2^i) + W_2^j {W_1}^k\Gamma^i_{jk}(x,V)\big]\frac{\partial}{\partial x^i}. \tag{3}$$

The affine connection ∇^V on M is torsion free and almost metric compatible, which means

$$\begin{aligned} \nabla^V_{W_1}W_2 - \nabla^V_{W_2}W_1 &= [W_1, W_2], \\ W g_V(W_1, W_2) &= g_V(\nabla^V_W W_1, W_2) + g_V(W_1, \nabla^V_W W_2) + 2C_V(\nabla^V_W V, W_1, W_2), \end{aligned} \tag{4}$$

for arbitrary vector fields W, W_1, W_2. Using the affine connection ∇^V, we define the derivative along a curve $\gamma(t)$ with velocity vector field T. Let W_1, W_2 be vector fields along γ; we define

$$D_{W_1}W_2 = \nabla^{T'}_{W_1'}W_2', \tag{5}$$

where the vector fields T', W_1' and W_2' on the right-hand side are smooth extensions of T, W_1 and W_2 to the neighbourhood of $\gamma(t)$. The definition above does not depend on the particular extension. A regular smooth curve γ with tangent vector field T is a *geodesic* if $D_T(\frac{T}{F(T)}) = 0$. In particular, a geodesic of constant speed satisfies $D_T T = 0$.

3. Homogeneous Finsler spaces

First, let us formulate simple observations which follow from homogeneity of the Finsler metric F.

PROPOSITION 3.1 ([8]). *Let (M,F) be a homogeneous Finsler space, G be a group of isometries acting transitively on M, X^* be a Killing vector field generated by the vector $X \in \mathfrak{g}$, $\phi(t) = \exp(tX)$ and $\gamma(t)$ be the integral curve of X^* through $p \in M$. Along the curve $\gamma(t)$, it holds*

$$\phi(t)(p) = \gamma(t), \qquad \phi(t)_*(X^*(p)) = X^*(\gamma(t))$$

and

$$F(\phi(t)(p), \phi(t)_* V) = F(p, V),$$
$$g_{(\gamma(t), X^*(\gamma(t)))}(\phi(t)_* U, \phi(t)_* V) = g_{(p, X^*(p))}(U, V),$$

for all $t \in \mathbb{R}$ and for all $U, V \in T_p M$.

Proposition 3.2 ([8]). *With the same assumptions as in Proposition* 3.1*, along the curve $\gamma(t)$, it holds*

$$g_{(\gamma(t), X^*(\gamma(t)))}(D_{X^*} X^*\big|_{\gamma(t)}, \phi(t)_* U) = g_{(p, X^*(p))}(D_{X^*} X^*\big|_p, U),$$

for all $t \in \mathbb{R}$ and for all $U \in T_p M$. Consequently, if

$$D_{X^*} X^*\big|_p = 0, \tag{6}$$

then the curve $\gamma(t)$ is a homogeneous geodesic.

Let us now recall that the Finsler metric F is called a *Berwald metric* if the Christoffel symbols $\Gamma^i_{jk}(x, y)$ of the Chern connection in natural coordinates do not depend on the direction y, hence $\Gamma^i_{jk}(x, y) = \Gamma^i_{jk}(x)$. We further recall that the Finsler metric F is *reversible* if, for any point $x \in M$ and for any vector $y \in T_x M$, it holds $F(x, y) = F(x, -y)$.

Proposition 3.3. *Let (M, F) be a homogeneous Berwald space or a homogeneous reversible Finsler space. With the same assumptions as in Proposition* 3.1*, for any Killing vector field X^*, it holds*

$$D_{X^*} X^* = \nabla^{X^*}_{X^*} X^* = \nabla^{-X^*}_{-X^*} -X^* = D_{-X^*} -X^*. \tag{7}$$

Proof. If the Finsler metric F is Berwald, using formulas (3) and (5) we easily deduce that for any Killing vector field X^*, formula (7) is valid.

For a reversible Finsler metric, one can check by the straightforward calculations that it holds $g_{ij}(x, y) = g_{ij}(x, -y)$, $C_{ijk}(x, y) = -C_{ijk}(x, -y)$. Using formula (1), we obtain $\gamma^i_{jk}(x, y) = \gamma^i_{jk}(x, -y)$, $N^i_j(x, y) = -N^i_j(x, -y)$ and further, using formula (2), we obtain $\Gamma^i_{jk}(x, y) = \Gamma^i_{jk}(x, -y)$. Finally, using formula (3), we see again that formula (7) is valid also in this situation. □

4. The main result

Theorem 4.1. *Let (M, F) be a homogeneous Berwald space or a homogeneous reversible Finsler space and let $p \in M$. Then M admits a homogeneous geodesic through p.*

Proof. Let us consider the Killing vector fields $K_1, \dots, K_n$ which are linearly independent at each point of some neighbourhood $\mathcal{U}$ of p and denote by B the basis $\{K_1(p), \dots, K_n(p)\}$ of $T_p M$. Any tangent vector $X \in T_p M$ has coordinates $(x_1, \dots x_n)$ with respect to the basis B. These coordinates determine the Killing vector field $X^* = x_1 K_1 + \dots + x_n K_n$ and an integral curve γ of X^* through p. We are going to show that there exists a vector $\bar{X} \in T_p M$ such that the integral curve γ of $\bar{X}^*$ through p is geodesic.

Let us consider the sphere $\mathbb{S}^{n-1}$ of vectors $X \in T_pM$ whose coordinates $(x_1, \dots, x_n)$ with respect to B have the norm equal to 1 with respect to the standard Euclidean scalar product $\langle , \rangle$ on $\mathbb{R}^n$. In other words, the scalar product $\langle , \rangle$ is chosen in a way that the above basis B is orthonormal. We stress that this scalar product does not come from any Finslerian product g_X.

For each $X \in \mathbb{S}^{n-1}$, we denote by $v(X)$ the derivative $D_{X^*_{\gamma(t)}} X^*|_{t=0}$. Further, we denote by $t(X)$ the vector $v(X) - \langle v(X), X\rangle X$. Then, for each $X \in \mathbb{S}^{n-1}$, it holds $t(X) \perp X$ with respect to the above Euclidean scalar product. We can interpret the map $X \mapsto t(X)$ as a smooth mapping $t\colon \mathbb{S}^{n-1} \to T_pM$, or, as a smooth tangent vector field on the sphere $\mathbb{S}^{n-1}$.

The situation when n is odd was treated in the mentioned paper [8] for general Finsler homogeneous space. Hence our particular focus here is the case when n is even.

Let us assume that $t(X) \neq 0$ everywhere. Putting $f(X) = t(X)/\|t(X)\|$, where the norm comes from the Euclidean scalar product $\langle , \rangle$, we obtain a smooth map $f\colon \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ without fixed points. According to a well known statement from differential topology, the degree of f is $\deg(f) = (-1)^n$, because f is homotopic to the antipodal mapping. On the other hand, according to formula (7), we have $v(X) = v(-X)$ and hence it holds also $f(X) = f(-X)$ for each $X \in \mathbb{S}^{n-1}$. If Y is a regular value of f, then the inverse image $f^{-1}(Y)$ consists of even number of elements. It follows that $\deg(f)$ is an even number, which is a contradiction. Hence, the assumption $t(X) \neq 0$ was wrong and there exists a vector $\bar{X}$ such that $t(\bar{X}) = 0$.

Now we use the standard fact that $C_{X^*}(X^*, X^*, X^*) = 0$ for any Killing vector field X^*. Applying this to formula (4), we observe that it holds

$$g_{(p,X)}(v(X), X) = g_{(p,X)}(D_{X^*_{\gamma(t)}} X^*\big|_{t=0}, X^*_p) = 0,$$

for each $X \in \mathbb{S}^{n-1} \subset T_pM$, and hence $v(X)$ lies in the orthogonal complement of X in T_pM with respect to the scalar product $g_{(p,X)}$.

To finish the proof, we consider the above vector $\bar{X}$, which satisfies $t(\bar{X}) = 0$. Because the vector $t(\bar{X})$ is the projection of the vector $v(\bar{X})$ to another complementary subspace to $\operatorname{span}(\bar{X})$ in T_pM, it follows that $v(\bar{X}) = 0$ if and only if $t(\bar{X}) = 0$. We obtain that $D_{\bar{X}^*} \bar{X}^*\big|_p = v(\bar{X}) = 0$.

We see, using Proposition 3.2 and formula (6), that the integral curve of the vector field $\bar{X}^*$ through p is a homogeneous geodesic. □

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