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ISOMETRY GROUP OF GROMOV–HAUSDORFF SPACE

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Abstract. The present paper is devoted to investigation of the isometry group of the Gromov–Hausdorff space, i.e., the metric space of compact metric spaces considered up to isometry and endowed with the Gromov–Hausdorff metric. The main goal is to present a complete proof of the following result by G. Lowther (2015): the isometry group of the Gromov–Hausdorff space is trivial.

1. Introduction

In the present paper the isometry group of the Gromov–Hausdorff space, i.e., the metric space consisting of isometry classes of compact metric spaces and endowed with the Gromov–Hausdorff metric is investigated. The Main Theorem belonging to G. Lowther [7] states that this group is trivial, in spite of the fact that the Gromov–Hausdorff space is rather symmetric in a local sense.

The interest to "spaces of subsets" is inspired not only by evident mathematical importance, but also by applications such as pattern recognition and comparison, constructing of continuous deformations of one geometrical object into another, etc., see for example [16, 17]. A metric on such spaces can be considered as a "measure of similarity" between two objects. In 1914 F. Hausdorff [6] defined a symmetric, non-negative function on pairs of non-empty subsets of a metric space that gives a metric on the set of all its closed bounded subsets. The Gromov–Hausdorff distance between any metric spaces generalises the Hausdorff metric to the case of arbitrary metric spaces using isometric embeddings in arbitrary metric spaces (see below an exact definition). Let us mention, see a historical review in [18], that this function was defined by D. Edwards [3] in 1975 and then rediscovered and generalized by M. Gromov [5] in 1981. The Gromov–Hausdorff distance converts the set of isometry classes of compact metric spaces into a metric space that is usually referred as the *Gromov–Hausdorff space* or as the *superspace* and below is denoted by $\mathcal{M}$. The geometry of this space is

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rather complicated that attracts attention of many specialists. It is well-known that the space $\mathcal{M}$ is path-connected, complete, separable, see for example [1]. Recently it has been proved that it is also geodesic, i.e., any two points of $\mathcal{M}$ can be connected by a shortest curve, whose length is equal to the distance between these points, see [10]. An introduction to geometry of Gromov–Hausdorff space can be found, for example in [1].

Isometry groups of metric spaces are very important as for understanding of their geometry, so as for such practical problems as distance calculation, geodesic construction, etc. It turns out that the Gromov–Hausdorff space is rather symmetric in the local sense, in particular, sufficiently small balls of the same radii centered at n-point generic spaces (the ones, where all non-zero distances are pairwise distinct and all triangle inequalities are strict) are isometric to each other, see [8]. The isometry group of a sufficiently small ball centered at such space contains a subgroup isomorphic to the permutation group of an n-element set [8]. But it turns out that there are no any global isometries, namely, the following result holds.

Main Theorem. *The isometry group of the Gromov–Hausdorff space is trivial.*

Let us say a few words about the background of this statement and of the proof presented below. In 2015 S. Iliadis (Lomonosov Moscow State University) told us about his conjecture concerning the triviality of the isometry group of the Gromov–Hausdorff space. We have googled this subject and found a blog `mathoverflow.net`, see [7], where N. Schweber (UW-Madison) formulated the same conjecture. Among comments we have found a positive "solution" given by G. Lowther[1]. The text presented by G. Lowther turns out to be a draft, where many proofs are omitted (that is natural for a draft), but the text also contains some wrong statements[2].

The present paper is a result of our critical retreat of the ideas of G. Lowther from [7]. Unfortunately G. Lowther has not published any other text on this subject during more than 2 years passed. We add some new constructions, reformulate some statements from [7] in a correct way and, besides, give geometrical interpretations, making some statements from [7] more clear to our opinion.

The proof turns out to be rather long, so for convenience we give here a short scheme of the paper. In Preliminaries we introduce basic concepts, notations, and list well-known results concerning Hausdorff and Gromov-Hausdorff distances. In particular, we describe technique of irreducible correspondences that turns out to be useful in Gromov–Hausdorff distance studying. As an application we include some formulas for the Gromov–Hausdorff distance between a compact metric space and so-called simplex (a finite metric space such that all non-zero distances in it are the same). In Section 3 we consider pointed metric spaces and their isometries. These objects are applied in Section 4 to prove invariance of the family of all finite metric spaces

[1]According to the same blog: "Apparently, this user prefers to keep an air of mystery about them".

[2]The most striking example: it is stated that the Gromov–Hausdorff distance between finite metric spaces of the same cardinality is attained on a bijective correspondence (but simple computer simulation shows that it is not true in general).

under arbitrary isometries of the Gromov–Hausdorff space $\mathcal{M}$. Section 5 contains necessary information concerning groups actions on metric spaces. In Section 6 we consider a subspace $\mathcal{M}_{[n]} \subset \mathcal{M}$ of n-point metric spaces, and describe small neighbourhoods of each point X from $\mathcal{M}_{[n]}$ in this space. We prove that each sufficiently small neighbourhood $U \subset \mathcal{M}_{[n]}$ is isometric either to the intersection of a ball in the normed space $\mathbb{R}^N_\infty$ with so-called metric cone C, or to a quotient of such intersection over the action of the stabilizer of the permutation subgroup G action. Here $N = n(n-1)/2$, the cone C consists of the vectors corresponding to distance $n \times n$ matrices, and G acts on $\mathcal{M}_{[n]}$ by changing of the numeration of the points from X. If the space X is generic, then the neighbourhood is isometric to the whole ball. Moreover, the subset of generic metric spaces is path connected in $\mathcal{M}_{[n]}$. In Section 7 we prove that the subspaces $\mathcal{M}_{[n]}$ are invariant under an arbitrary isometry of the space $\mathcal{M}$. In Section 8 we apply John's generalisation of Mazur-Uhlam Theorem saying that any local isometry of a real normed space is generated by an affine mapping to prove that the local isometries of the balls from $\mathcal{M}_{[n]}$, generated by a global isometry of the space $\mathcal{M}$ are generated by the same linear mapping H. Such linear mapping generates a permutation of basic vectors from $\mathbb{R}^N_\infty$, and in Section 9 we accumulate necessary information concerning the permutation groups. In Section 10 we demonstrate that the permutation generated by the linear mapping H is generated by changing a numeration of the points from X, and hence, any isometry of $\mathcal{M}$ takes each generic finite metric space to itself. Since generic finite metric spaces are everywhere dense in $\mathcal{M}$, the latter completes the proof of the Main Theorem.

2. Preliminaries

Let X be a set. By $\#X$ we denote the *cardinality* of X.

Now, let X be an arbitrary metric space. The distance between its points x and y is denoted by $|xy|$. If $A, B \subset X$ are non-empty subsets, then put $|AB| = \inf\big\{|ab| : a \in A,\, b \in B\big\}$. If $A = \{a\}$, then we write $|aB| = |Ba|$ instead of $|\{a\}B| = |B\{a\}|$.

Let us fix the notations for the following standard objects related to a metric space X:

- for $x \in X$ and $r > 0$ by $U_r(x) = \{y \in X : |xy| < r\}$ we denote the *open ball centered at x of radius r*;

- for $x \in X$ and $r \ge 0$ by $B_r(x) = \{y \in X : |xy| \le r\}$ and $S_r(x) = \{y \in X : |xy| = r\}$ we denote the *closed ball* and the *sphere centered at x of radius r*, respectively;

- for a non-empty subset $A \subset X$ and $r > 0$ by $U_r(A) = \{x \in X : |xA| < r\}$ we denote the *open neighbourhood of A of radius r*;

- for a non-empty subset $A \subset X$ and $r \ge 0$ by $B_r(A) = \{x \in X : |xA| \le r\}$ and $S_r(A) = \{x \in X : |xA| = r\}$ we denote the *closed neighbourhood* and the *equidistant set of A of radius r*.

2.1 Hausdorff distance and Gromov–Hausdorff distance

For non-empty A, $B \subset X$ put

$$d_H(A,B) = \inf\bigl\{r > 0 : A \subset U_r(B) \text{ and } B \subset U_r(A)\bigr\} = \max\bigl\{\sup_{a\in A}|aB|,\ \sup_{b\in B}|Ab|\bigr\}.$$

This value is called the *Hausdorff distance between A and B*. It is well-known, see [1], that the Hausdorff distance forms a metric on the set of all non-empty closed bounded subsets of X.

Let X and Y be metric spaces. A triple (X', Y', Z) consisting of a metric space Z and two its subsets X' and Y' that are isometric to X and Y, respectively, is called a *realization of the pair (X, Y) in Z*. The *Gromov–Hausdorff distance $d_{GH}(X, Y)$ between X and Y* is the infimum of r such that there exists a realization (X', Y', Z) of the pair (X, Y) with $d_H(X', Y') \le r$.

By $\mathcal{M}$ we denote the set of all compact metric spaces considered up to an isometry.

THEOREM 2.1 ([1, 10]). *Being restricted on $\mathcal{M}$, the distance d_{GH} is a metric. The metric space $\mathcal{M}$ is complete, separable, and geodesic (given any two points, there exists a curve connecting them, whose length equals the distance between the points).*

The next result is an immediate consequence of the definitions.

PROPOSITION 2.2 ([1]). *For an arbitrary non-empty subset Y of a metric space X the inequality $d_{GH}(X, Y) \le d_H(X, Y)$ holds. In particular, if Y is an ε-net in X, then $d_{GH}(X, Y) \le \varepsilon$.*

To calculate the Gromov–Hausdorff distance it is convenient to use the technique of correspondences.

Let X and Y be any non-empty sets. Recall that a *relation* between X and Y is a subset of the Cartesian product $X \times Y$. By $\mathcal{P}(X, Y)$ we denote the set of all **non-empty** relations between X and Y. Let us consider each relation $\sigma \in \mathcal{P}(X, Y)$ as a multivalued mapping, whose domain could be less than the whole set X. Then, similarly to the case of mappings, for each $x \in X$ and any $A \subset X$ their images $\sigma(x)$ and $\sigma(A)$ are defined, and for each $y \in Y$ and any $B \subset Y$ their pre-images $\sigma^{-1}(y)$ and $\sigma^{-1}(B)$ are defined as well.

A relation $R \in \mathcal{P}(X, Y)$ is called a *correspondence* if, being restricted onto R, the canonical projections $\pi_X\colon (x, y) \mapsto x$ and $\pi_Y\colon (x, y) \mapsto y$ are surjective, or, that is equivalent, if $R(X) = Y$ and $R^{-1}(Y) = X$. By $\mathcal{R}(X, Y)$ we denote the set of all correspondences between X and Y.

Let X and Y be arbitrary metric spaces, and σ be a relation between them. The value $\operatorname{dis}\sigma = \sup\Bigl\{\bigl||xx'| - |yy'|\bigr| : (x, y), (x', y') \in \sigma\Bigr\}$ is called the *distortion* $\operatorname{dis}\sigma$ *of the relation σ*.

PROPOSITION 2.3 ([1]). *For any metric spaces X and Y it holds*

$$d_{GH}(X, Y) = \frac{1}{2}\inf\bigl\{\operatorname{dis} R : R \in \mathcal{R}(X, Y)\bigr\}.$$

For any metric space X and a real number $\lambda > 0$ by λX we denote the metric space obtained from X by multiplication of all the distances by λ.

PROPOSITION 2.4 ([1]). *Let X and Y be metric spaces. Then*
(i) if X is a single-point metric space, then $d_{GH}(X,Y) = \frac{1}{2}\operatorname{diam} Y$;

(ii) if $\operatorname{diam} X < \infty$ or $\operatorname{diam} Y < \infty$, then $d_{GH}(X,Y) \ge \frac{1}{2}|\operatorname{diam} X - \operatorname{diam} Y|$;

(iii) $d_{GH}(X,Y) \le \frac{1}{2}\max\{\operatorname{diam} X, \operatorname{diam} Y\}$, in particular, $d_{GH}(X,Y) < \infty$ for bounded X and Y;

(iv) for any $X \in \mathcal{M}$ and any $\lambda \ge 0$, $\mu \ge 0$ we have $d_{GH}(\lambda X, \mu X) = \frac{1}{2}|\lambda - \mu| \operatorname{diam} X$; this immediately implies that the curve $\gamma(t) := t\,X$ is a shortest one for any pair of its points;

(v) for any $X, Y \in \mathcal{M}$ and any $\lambda > 0$ we have $d_{GH}(\lambda X, \lambda Y) = \lambda d_{GH}(X,Y)$. Moreover, for $\lambda \ne 1$ the unique space that remains the same under this operation is the single-point space. In other words, the multiplication of a metric by a number $\lambda > 0$ is a homothety of the space $\mathcal{M}$ with the center at the single-point metric space.

Thus, the Gromov–Hausdorff space can be imagined as a cone with the vertex at the single-point space, and with generators that are geodesics emanate from the vertex, see Figure 1.

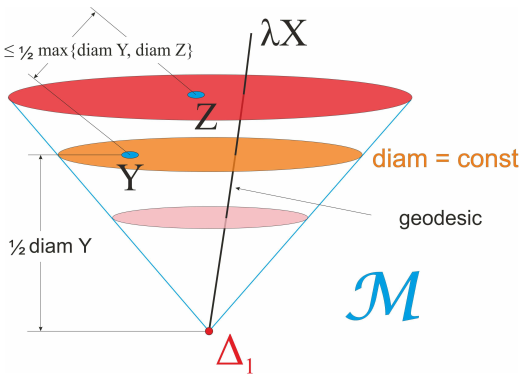

Figure 1: The Gromov–Hausdorff space: some general properties. Here Δ_1 stands for the single-point space.

2.2 Irreducible correspondences

If X and Y are finite metric spaces, then the set $\mathcal{R}(X,Y)$ is finite, thus there exists an $R \in \mathcal{R}(X,Y)$ such that $d_{GH}(X,Y) = \frac{1}{2}\operatorname{dis} R$. Every such correspondence R is called *optimal*. It turns out that optimal correspondences always exist for any compact metric spaces X and Y, see [2,9]. By $\mathcal{R}_{\rm opt}(X,Y)$ we denote the set of all optimal correspondences between X and Y.

PROPOSITION 2.5 ([2,9]). *Let X and Y be compact metric spaces. Then $\mathcal{R}_{\rm opt}(X,Y) \ne \emptyset$.*

The inclusion relation generates the standard partial order on $\mathcal{R}(X,Y)$. The correspondences that are minimal with respect to this order are called *irreducible*. By $\mathcal{R}^0(X,Y)$ we denote the set of all irreducible correspondences between X and Y. It is shown in [11] that each $R \in \mathcal{R}(X,Y)$ contains an irreducible correspondence and, thus, the following result holds.

PROPOSITION 2.6. *For any metric spaces X and Y we have $\mathcal{R}^0(X,Y) \neq \emptyset$.*

The next result describes irreducible correspondences.

PROPOSITION 2.7. *For each $R \in \mathcal{R}^0(X,Y)$ there exist partitions $R_X = \{X_i\}_{i\in I}$ and $R_Y = \{Y_i\}_{i\in I}$ of the spaces X and Y, respectively, such that $R = \bigcup_{i\in I} X_i \times Y_i$.*

Proof. Put $R_X = \bigcup_{y\in Y}\big\{R^{-1}(y)\big\}$, $R_Y = \bigcup_{x\in X}\big\{R(x)\big\}$, and let us show that R_X and R_Y are partitions. Suppose the contrary, and let, say, R_Y be not a partition. Since R is a correspondence, then R_Y is a covering of Y such that some of its elements $R(x)$ and $R(x')$ for $x \neq x'$ intersect each other, but, due to definition of R_Y, do not coincide. Let $y \in R(x)\cap R(x')$, then $(x,y),(x',y) \in R$. Since $R(x) \neq R(x')$, one of these sets contains an element which does not lie in the other one. To be definite, let $y' \in R(x')\setminus R(x)$. Then $(x',y') \in R$, therefore, if we remove (x',y) from R, then we obtain a relation $\sigma \subset R$ such that $y \in \sigma(x)$ and $x' \in \sigma^{-1}(y')$, so σ is a correspondence. The latter contradicts to irreducibility of R. The case of R_X is similar.

Thus, let us re-denote the partition R_X as $R_X = \{X_i\}_{i\in I}$. Notice that for any $x,x' \in X_i$ we have $R(x) = R(x')$. Indeed, if $X_i = R^{-1}(y)$, then $R(x)$ and $R(x')$ contain y and, therefore, they intersect each other. However, R_Y is a partition, so we get $R(x) = R(x')$.

Choose an arbitrary $i \in I$ and an arbitrary $x \in X_i$, and put $Y_i = R(x)$ (this notation is correct, because according to the above reasoning it does not depend on the choice of $x \in X_i$). Now we show that the correspondence $\varphi\colon X_i \mapsto Y_i$ is a bijection between R_X and R_Y.

If φ is not injective, then there exist $x,x' \in X$ belonging to different elements of the partition R_X and such that $R(x) = R(x')$. However, in this case for $y \in R(x)$ it holds $x,x' \in R^{-1}(y) \in R_X$, a contradiction. Further, φ is surjective because for any Y_i, $y \in Y_i$, the set $R^{-1}(y)$ is an element of the partition R_X. Choose an arbitrary $x \in R^{-1}(y)$. Then $R(x) \in R_Y$ contains y, thus $\varphi\big(R^{-1}(y)\big) = Y_i$.

Since for any $x,x' \in X_i$ we have $R(x) = R(x') = Y_i$, then $X_i \times Y_i \subset R$. On the other hand, since R_X is a partition of X, then for any $x \in X$ there exists $X_i \in R_X$ such that $x \in X_i$, therefore, each $(x,y) \in R$ is contained in some $X_i \times Y_i$. □

2.3 Partitions

For any non-empty subsets A and B of a metric space X we put

$$|AB|' = \sup\big\{|ab| : a \in A,\, b \in B\big\}.$$

If $D = \{X_i\}_{i\in I}$ is a partition of a metric space X, then we define the *diameter of this partition* as follows: $\operatorname{diam} D = \sup_{i\in I} \operatorname{diam} X_i$. We also put

$$\alpha(D) = \inf\big\{|X_iX_j| : i,j \in I,\, i \neq j\big\}, \qquad \beta(D) = \sup\big\{|X_iX_j|' : i,j \in I,\, i \neq j\big\}.$$

The next result can be easily obtained from the definition of distortion and from Proposition 2.7.

Proposition 2.8. *Let X and Y be arbitrary metric spaces, $D_X=\{X_i\}_{i\in I}$, $D_Y=\{Y_i\}_{i\in I}$ be some partitions of the spaces X and Y, respectively, and $R=\bigcup_{i\in I}X_i\times Y_i\in\mathcal{R}(X,Y)$. Then*

$$\begin{aligned}\operatorname{dis}R&=\sup\bigl\{|X_iX_j|'-|Y_iY_j|,\,|Y_iY_j|'-|X_iX_j|:i,j\in I\bigr\}\\&=\sup\bigl\{\operatorname{diam}D_X,\,\operatorname{diam}D_Y,\,|X_iX_j|'-|Y_iY_j|,\,|Y_iY_j|'-|X_iX_j|:i,j\in I,\,i\ne j\bigr\}\\&\le\max\bigl\{\operatorname{diam}D_X,\operatorname{diam}D_Y,\beta(D_X)-\alpha(D_Y),\beta(D_Y)-\alpha(D_X)\bigr\}.\end{aligned}$$

In particular, if $R\in\mathcal{R}^0(X,Y)$, then in the previous formula one can take R_X and R_Y from Proposition 2.7 instead of D_X and D_Y.

For a set X and any $n\in\mathbb{N}$ by $\mathcal{D}_n(X)$ we denote the family of all partitions of the set X into n non-empty subsets. Notice that for $n>\#X$ we have $\mathcal{D}_n(X)=\emptyset$, and for $n=\#X$ the family $\mathcal{D}_n(X)$ consists of the unique partition of X into its one-element subsets.

Let X be an arbitrary metric space. The next characteristic of X will be used below:

$$d_n(X)=\begin{cases}\inf\bigl\{\operatorname{diam}D:D\in\mathcal{D}_n(X)\bigr\}, & \text{if } \mathcal{D}_n(X)\ne\emptyset,\\ \infty, & \text{if } \mathcal{D}_n(X)=\emptyset.\end{cases}$$

Remark 2.9. If X is a finite metric space and $n=\#X$, then $d_n(X)=0$.

Remark 2.10. The function $g(n)=d_n(X)$ decreases monotonically on the set of n such that $\mathcal{D}_n(X)\ne\emptyset$.

2.4 Optimal irreducible correspondences

It was proved in [11] that for compact metric spaces X and Y there always exists an optimal irreducible correspondence R. By $\mathcal{R}^0_{\mathrm{opt}}(X,Y)$ we denote the set of all irreducible optimal correspondences between X and Y. Thus, the following result holds.

Proposition 2.11 ([11]). *Let X and Y be arbitrary compact metric spaces, then $\mathcal{R}^0_{\mathrm{opt}}(X,Y)\ne\emptyset$.*

Corollary 2.12. *Let X and Y be arbitrary compact metric spaces, $R\in\mathcal{R}^0_{\mathrm{opt}}(X,Y)$, $R_X=\{X_i\}_{i\in I}$, $R_Y=\{Y_i\}_{i\in I}$, $R=\bigcup_{i\in I}X_i\times Y_i$. Then*

$$2d_{GH}(X,Y)=\sup\bigl\{\operatorname{diam}R_X,\operatorname{diam}R_Y,|X_iX_j|'-|Y_iY_j|,|Y_iY_j|'-|X_iX_j|:i,j\in I,\,i\ne j\bigr\}.$$

2.5 Distances to simplexes

A metric space X is called a *simplex*, if all its non-zero distances are equal to each other. Notice that a simplex is compact, iff it consists of a finite number of points. By Δ_n we denote the simplex consisting of n points with non-zero distances 1. Then for $t>0$ the metric space $t\,\Delta_n$ is a simplex, whose non-zero distances are equal to t. Notice that Δ_1 is the single-point metric space, and that $t\,\Delta_1=\Delta_1$ for all $t>0$. In what follows, we put $\Delta_n=\{1,\ldots,n\}$ for convenience.

For any metric space X, $n\le\#X$, and $D=\{X_1,\ldots,X_n\}\in\mathcal{D}_n(X)$ we put $R_D=\sqcup\bigl(\{i\}\times X_i\bigr)\in\mathcal{R}(t\,\Delta_n,X)$. Let us notice that if $D'\in\mathcal{D}_n(X)$ differs from D by a renumbering of its elements, then $\operatorname{dis}R_D=\operatorname{dis}R_{D'}$.

Proposition 2.13 ([12,13]). *Let X be an arbitrary metric space and $n\in\mathbb{N}$, $n\le\#X$. Then for any $t>0$ and $D\in\mathcal{D}_n(X)$ we have* $\operatorname{dis}R_D=\max\{\operatorname{diam}D,t-\alpha(D),\beta(D)-t\}$.

Proposition 2.14 ([12,13]). *Let X be a compact metric space. Then for each $n\in\mathbb{N}$, $n\le\#X$, and $t>0$ there exists some $R\in\mathcal{R}_{\mathrm{opt}}(t\,\Delta_n,X)$ such that the family $\bigl\{R(i)\bigr\}$ is a partition of the space X. In particular, if $n=\#X$, then this R can be chosen among bijections.*

The next result follows from Propositions 2.13 and 2.14.

Corollary 2.15. *Let X be a compact metric space and $n\in\mathbb{N}$, $n\le\#X$. Then for any $t>0$ we have*

$$2d_{GH}(t\,\Delta_n,X)=\inf\Bigl\{\max\bigl(\operatorname{diam}D,\,t-\alpha(D),\,\beta(D)-t\bigr):D\in\mathcal{D}_n(X)\Bigr\}.$$

Proposition 2.16 ([12,13]). *Let X be a finite metric space, $m=\#X$, $n\in\mathbb{N}$, $t>0$. Denote by $a\le b$ the first and the second smallest distances between different points of the space X (if they are defined). Then*

$$2d_{GH}(t\,\Delta_n,X)=\begin{cases}\max\{t,\,\operatorname{diam}X-t\} & \text{for } m<n,\\ \max\{t-a,\,\operatorname{diam}X-t\} & \text{for } m=n\ge 2,\\ \max\{a,\,t-b,\,\operatorname{diam}X-t\} & \text{for } m=n+1\ge 3,\\ \max\{d_n(X),\,\operatorname{diam}X-t\} & \text{for } m\ge n \text{ and } \operatorname{diam}X\ge 2t.\end{cases}$$

Moreover, for $m=n+1$ there exists an optimal correspondence sending some point of the simplex to a pair of the closest points of X, and forming a bijection between the remaining points.

Proposition 2.16 implies an explicit formula for the Gromov–Hausdorff distance between simplexes.

Corollary 2.17. *For integer $p,q\ge 2$ and real $t,s>0$ we have*

$$2d_{GH}(t\,\Delta_p,s\,\Delta_q)=\begin{cases}|t-s| & \text{for } p=q,\\ \max\{t,s-t\} & \text{for } p>q,\\ \max\{s,t-s\} & \text{for } p<q.\end{cases}$$

In particular, if $p\ne q$, then $2d_{GH}(t\,\Delta_p,s\,\Delta_q)\ge\min\{t,s\}$.

Proposition 2.18. *Let X be a metric space containing a subspace isometric to $t\,\Delta_n$, $n\ge 2$, and suppose that M is a finite metric space, $\#M\le n-1$. Then $2d_{GH}(X,M)\ge t$. If $\operatorname{diam}X=t$ and $\operatorname{diam}M\le t$, then $2d_{GH}(X,M)=t$.*

Proof. Indeed, denote by $C = \{c_1, \dots, c_n\}$ a subspace of X isometric to $t\,\Delta_n$, then for any $R \in \mathcal{R}(X, M)$ there exist $p \in M$ and distinct c_i, c_j such that (c_i, p), $(c_j, p) \in R$, so $\operatorname{dis} R \ge t$. Since R is an arbitrary correspondence, then $2d_{GH}(X, M) \ge t$. If $\operatorname{diam} X = t$ and $\operatorname{diam} M \le t$, then item (iii) of Proposition 2.4 implies that $2d_{GH}(X, M) \le t$. □

3. Isometries of metric spaces

In this section we work out some technique for description of isometries of metric spaces. The main attention is paid to self-isometries.

3.1 Operations with invariant subsets

Let X be a metric space and $f\colon X \to X$ be an isometry. By $\mathcal{P}^f(X)$ we denote the set of all subsets of X invariant with respect to f, namely, $\mathcal{P}^f(X) = \big\{A \subset X : f(A) = A\big\}$. The next statement is evident.

Proposition 3.1. *The family $\mathcal{P}^f(X)$ contains X, $\emptyset$, and it is invariant under the operations of union, intersection, and taking complement. Besides, if $A \in \mathcal{P}^f(X)$, then $U_r(A) \in \mathcal{P}^f(X)$ for any $r > 0$, and $B_r(A), S_r(A) \in \mathcal{P}^f(X)$ for any $r \ge 0$.*

3.2 Isometries of finite pointed spaces

A set X is called *pointed*, if one of its elements is marked. More formally, a *pointed set* is a pair (X, x), where $x \in X$. For a pointed set (X, x) by $p(X)$ we denote its marked point x. Two pointed metric spaces X and Y are called *p-isometric*, if there exists an isometry $f\colon X \to Y$ such that $p(Y) = f\big(p(X)\big)$. For a pointed metric space X by $\operatorname{Gr}(X)$ we denote the class of all pointed metric spaces that are p-isometric to X. By $\mathcal{M}_*$ we denote the set of the classes of p-isometric pointed compact metric spaces. Thus, if X is a pointed compact metric space, then $\operatorname{Gr}(X) \in \mathcal{M}_*$.

Let X be an arbitrary metric space and $x \in X$. For any $n \in \mathbb{N}$ by $\mathcal{P}_n(x)$ we denote the set of all pointed n-point subspaces $Z \subset X$ containing x as a marked point, i.e., such that $p(Z) = x$. Also, we define $\mathcal{M}_*(X, x, n) \subset \mathcal{M}_*$ to be $\operatorname{Gr}\big(\mathcal{P}_n(x)\big) = \big\{\operatorname{Gr}(Z) : Z \in \mathcal{P}_n(x)\big\}$. The following statement is evident.

Proposition 3.2. *Let $f\colon X \to Y$ be an isometry of metric spaces, then for any $n \in \mathbb{N}$ and any point $x \in X$ we have $\mathcal{M}_*(X, x, n) = \mathcal{M}_*\big(Y, f(x), n\big)$. In particular, each isometry $f\colon X \to X$ is invariant on the level sets of the mapping $x \mapsto \mathcal{M}_*(X, x, n)$.*

A triple $\{A, B, C\}$ of different points of a metric space X is called a *triangle* and is denoted by ABC. We write $ABC \subset X$. For such triangles we use school geometry terminology.

Proposition 3.3. *Let P and Q be distinct points of a metric space X. Suppose that for each triangle $PBC \subset X$ its side BC cannot be the longest one, but among the triangles $QBC \subset X$ there exists one, whose longest side is BC. Then $\mathcal{M}_*(X, P, 3) \ne \mathcal{M}_*(X, Q, 3)$.*

3.3 Equidistant points families

For any points P and Q of a metric space X by $\mathrm{Mid}(X,P,Q)$ we denote the set of all points $A\in X$ such that $|AP|=|AQ|$. The next statement is evident.

PROPOSITION 3.4. *Let $f\colon X\to Y$ be an arbitrary isometry of metric spaces, then for any $P,Q\in X$ it holds $f\big(\mathrm{Mid}(X,P,Q)\big)=\mathrm{Mid}\big(Y,f(P),f(Q)\big)$. In particular, each isometry $f\colon X\to X$ preserving the points $P,Q\in X$ takes $\mathrm{Mid}(X,P,Q)$ onto itself.*

4. Invariant subsets in $\mathcal{M}$

Several ideas on invariance of some subsets of $\mathcal{M}$ under a self-isometry of $\mathcal{M}$ are taken from [7].

4.1 Invariance of Δ_1

To start with we prove that the single-point space Δ_1 remains fixed under any self-isometry of $\mathcal{M}$.

THEOREM 4.1. *For any $A\in\mathcal{M}$, $A\ne\Delta_1$, it holds $\mathcal{M}_*(\mathcal{M},A,3)\ne\mathcal{M}_*(\mathcal{M},\,\Delta_1,3)$.*

Proof. By items (i) and (iii) of Proposition 2.4, for any $B,C\in\mathcal{M}$ we have
$$d_{GH}(B,C)\le\max\big\{d_{GH}(B,\Delta_1),\,d_{GH}(\Delta_1,C)\big\},$$
thus, if $X=\Delta_1$, then in each triangle $XBC\subset\mathcal{M}$ the side BC cannot be the longest one.

If $A\in\mathcal{M}$, $A\ne\Delta_1$, then, by item (iv) of Proposition 2.4, the curve $\gamma(t)=t\,A$, $t\in[1/2,2]$, is a shortest geodesic for which A is an interior point. Therefore, by Theorem 2.1, for $B=\gamma(1/2)$ and $C=\gamma(2)$ we have $d_{GH}(B,C)=d_{GH}(B,A)+d_{GH}(A,C)$, thus, in such triangle ABC the side BC is the longest one. It remains to apply Proposition 3.3. □

Thus, it remains to apply Theorem 4.1, Proposition 3.2, and item (i) of Proposition 2.4.

COROLLARY 4.2. *Let $f\colon\mathcal{M}\to\mathcal{M}$ be an arbitrary isometry, then $f(\Delta_1)=\Delta_1$. In particular, for any $X\in\mathcal{M}$ we have $\operatorname{diam} f(X)=\operatorname{diam} X$.*

4.2 Invariance of $t\,\Delta_n$, $n\ge2$

By $\mathcal{M}^t$ we denote the set of all $A\in\mathcal{M}$ such that $\operatorname{diam}A\le t$. In other words, $\mathcal{M}^t$ is a ball in $\mathcal{M}$ of radius $t/2$ centered at Δ_1.

THEOREM 4.3. *For $t>0$ and any $A\in\mathcal{M}^t$, $A\ne t\,\Delta_n$, $n=1,2,\dots$, it holds*
$$\mathcal{M}_*(\mathcal{M}^t,A,3)\ne\mathcal{M}_*(\mathcal{M}^t,t\,\Delta_n,3).$$

Proof. Similarly with the proof of Theorem 4.1 we proceed as follows: (1) for each triangle $XBC \subset \mathcal{M}^t$ with $X = t\Delta_n$ we show that the side BC cannot be longer than its other sides; (2) we prove that for each $A \in \mathcal{M}^t$, $A \ne t\Delta_n$, there exists a triangle $ABC \subset \mathcal{M}^t$ such that the side BC is the longest one; after that we apply Proposition 3.3.

(1) The case $X = t\,\Delta_1 = \Delta_1$ is already considered in Theorem 4.1, so we pass to the case $n > 1$.

Suppose otherwise, i.e., for some n there exists a triangle $XBC \subset \mathcal{M}^t$ such that BC is its longest side. By item (iii) of Proposition 2.4, we have $2d_{GH}(B,C) \le t$, therefore, $2d_{GH}(X,B) < t$ and $2d_{GH}(X,C) < t$.

LEMMA 4.4. *Under the above assumptions we have $\#B \ge n$ and $\#C \ge n$.*

Proof. Suppose the contrary, and let, say, $\#B < n$, then, by Proposition 2.16, we have $2d_{GH}(X,B) = \max\{t, \operatorname{diam} B - t\} = t$, however, $2d_{GH}(X,B) < t$, a contradiction. □

Further, by Proposition 2.14, there exist $R \in \mathcal{R}_{\rm opt}(t\,\Delta_n, B)$ and $S \in \mathcal{R}_{\rm opt}(t\,\Delta_n, C)$ such that $D = \{R(i)\}$ and $E = \{S(i)\}$ are partitions of the spaces B and C, respectively. Put $B_i = R(i)$, $C_i = S(i)$, $T = \bigcup_{i=1}^n B_i \times C_i$, then $T \in \mathcal{R}(B,C)$ and, by Proposition 2.8, we have $\operatorname{dis} T \le \max\bigl\{\operatorname{diam} D, \operatorname{diam} E, \beta(D) - \alpha(E), \beta(E) - \alpha(D)\bigr\}$. By Proposition 2.13, it holds

$$\operatorname{dis} R = \max\{\operatorname{diam} D,\, t-\alpha(D),\, \beta(D)-t\},\ \operatorname{dis} S = \max\{\operatorname{diam} E,\, t-\alpha(E),\, \beta(E)-t\},$$

therefore, $\max\{\operatorname{diam} D,\, t-\alpha(D)\} \le \operatorname{dis} R$ and $\max\{\operatorname{diam} E,\, t-\alpha(E)\} \le \operatorname{dis} S$. Since $\operatorname{diam} B \le t$ and $\operatorname{diam} C \le t$, then $\beta(D) \le t$ and $\beta(E) \le t$, thus

$$\begin{aligned} 2d_{GH}(B,C) \le \operatorname{dis} T &\le \max\bigl\{\operatorname{diam} D, \operatorname{diam} E, t - \alpha(E), t - \alpha(D)\bigr\} \\ &\le \max\{\operatorname{dis} R,\, \operatorname{dis} S\} = \max\bigl\{2d_{GH}(X,B),\, 2d_{GH}(X,C)\bigr\}, \end{aligned}$$

and so BC cannot be the longest side of the triangle XBC, a contradiction.

(2) If $\operatorname{diam} A < t$, then, by item (iv) of Proposition 2.4, the curve $\gamma(s) = s\,A$, $s \in [1/2, t/\operatorname{diam} A]$, is a shortest geodesic belonging to $\mathcal{M}^t$, because $\operatorname{diam}\gamma(s) \le (t/\operatorname{diam} A)\operatorname{diam} A = t$. Besides that, A is an interior point of the curve γ. Thus, by Theorem 2.1, for $B = \gamma(1/2)$ and $C = \gamma(t/\operatorname{diam} A)$ we have $d_{GH}(B,C) = d_{GH}(B,A) + d_{GH}(A,C)$, therefore, in such a triangle ABC the side BC is the longest one.

Now, let $\operatorname{diam} A = t$, then $|xx'| \le t$ for all $x, x' \in A$, and for some pair of points the equality holds, but for some other pair we have inequality, because $A \ne t\Delta_n$. In particular, $\#A \ge 3$.

Suppose at first that A is a finite metric space consisting of $m \ge 3$ elements, and let $a \le b$ be the two smallest distances between different points of the space A. Put $B = t\,\Delta_m$ and $C = t\,\Delta_{m-1}$. Then, by Proposition 2.16, we have $2d_{GH}(B,C) = t$ and $2d_{GH}(A,C) = \max\{a, t-b, \operatorname{diam} A - t\}$. Since $a < t$ is the least non-zero distance in A, $b > 0$, and $\operatorname{diam} A \le t$, then $2d_{GH}(A,C) < t$.

Further, Proposition 2.16 implies that $2d_{GH}(A,B) = \max\{t-a, \operatorname{diam} A - t\}$. Since $a > 0$ and $\operatorname{diam} A = t$, then $2d_{GH}(A,B) < t$. Thus, in the case under consideration we have $2\max\{d_{GH}(A,B),\, d_{GH}(A,C)\} < t = 2d_{GH}(B,C)$, so BC is the longest side of the triangle ABC.

Now suppose that A is infinite. Fix an arbitrary $\varepsilon \in (0, t/4)$, choose a finite ε-net $\{b_1, \dots, b_{m-1}\}$ in A, and take it as the space B. Then, by Proposition 2.2, we have $d_{GH}(A,B) \le \varepsilon < t/4$.

Let $b_m \in A$ be an arbitrary point distinct from the chosen b_i. Put $A_i = B_\varepsilon(b_i)$, $i = 1, \dots, m$. Then $A = \bigcup_{i=1}^m A_i$ and $\operatorname{diam} A_i < t/2$ for all i.

To construct C we take a set $\{c_1, \dots, c_{2m}\}$ and define the distances on it as follows: $|c_ic_j| = t$ for all $1 \le i < j \le m$, and all the remaining distances are equal to $t/2$. Since the subspace $\{c_1, \dots, c_m\} \subset C$ is isometric to $t\,\Delta_m$, and the diameters of B and C are at most t, then, by Proposition 2.18, we have $2d_{GH}(B,C) = t$.

Consider the following correspondence $R \in \mathcal{R}(A,C)$:

$$R = \{(b_i, c_i)\}_{i=1}^m \cup \big(A_1 \times \{c_{m+1}\}\big) \cup \dots \cup \big(A_m \times \{c_{2m}\}\big).$$

It is easy to see that $\operatorname{dis} R < t$, thus $2d_{GH}(A,C) < t$, therefore, BC is the longest side of the triangle ABC. The theorem is proved. □

Corollary 4.5. *If $f\colon \mathcal{M} \to \mathcal{M}$ i an arbitrary isometry, then for any integer $n \ge 2$ and real $t > 0$ we have $f(t\,\Delta_n) = t\,\Delta_n$.*

Proof. By Theorem 4.3 and Proposition 3.2, each isometry of the space $\mathcal{M}$ preserves the family $\{t\,\Delta_n\}_{n=1}^\infty$, i.e., for each integer $n \ge 2$ there exists $m \ge 2$ such that $f(t\,\Delta_n) = t\,\Delta_m$. We have to show that $m = n$. To do that, we prove a number of auxiliary statements.

Lemma 4.6. *Suppose that for some $p, q \ge 2$ and $t > 0$ we have $f(t\,\Delta_p) = t\,\Delta_q$. Then $f(s\,\Delta_p) = s\,\Delta_q$ for all $s > 0$.*

Proof. Suppose otherwise, i.e., that for some $s > 0$ it holds $f(s\,\Delta_p) = s\,\Delta_r$, $r \ne q$. Since f is an isometry, then by Corollary 2.17 we have

$$|t - s| = 2d_{GH}(t\,\Delta_p, s\,\Delta_p) = 2d_{GH}\big(f(t\,\Delta_p), f(s\,\Delta_p)\big) = 2d_{GH}(t\,\Delta_q, s\,\Delta_r) \ge \min\{t, s\},$$

that does not hold for $s \in (t - t/2, t + t/2)$. This implies that the function $t \mapsto q$ is locally constant. Since each ray is connected, we get that this function is constant. □

Lemma 4.7. *Suppose that for some $p, q \ge 2$ it holds $f(\Delta_p) = \Delta_q$. Then for each $i < p$, $f(\Delta_i) = \Delta_j$ implies $j < q$.*

Proof. Indeed, suppose otherwise, i.e., that $j > q$ (the case $j = q$ is impossible, because f is bijective). Then, by Corollary 2.17 and Lemma 4.6, for any $t, s > 0$ we have

$$\begin{aligned}\max\{t, s - t\} &= 2d_{GH}(t\,\Delta_p, s\,\Delta_i) = 2d_{GH}\big(f(t\,\Delta_p), f(s\,\Delta_i)\big)\\ &= 2d_{GH}(t\,\Delta_q, s\,\Delta_j) = \max\{s, t - s\}.\end{aligned}$$

To get a contradiction, put $s = t/3$. □

Let us return to the proof that $m = n$. Suppose the contrary. Without loss of generality, we assume that $m < n$ (otherwise we consider f^{-1}). However, in this case the mapping f takes injectively the set of simplexes $\{\Delta_i, 1 < i < n\}$ to the less set of simplexes $\{\Delta_j, 1 < j < m\}$, a contradiction. $\square$

In fact, we have shown that the set of "corner" points of the ball centered at the single-point metric space Δ_1 consists only of Δ_1 together with the simplexes $t\Delta_k$, $k \geq 2$, belonging to the boundary sphere, see Figure 2.

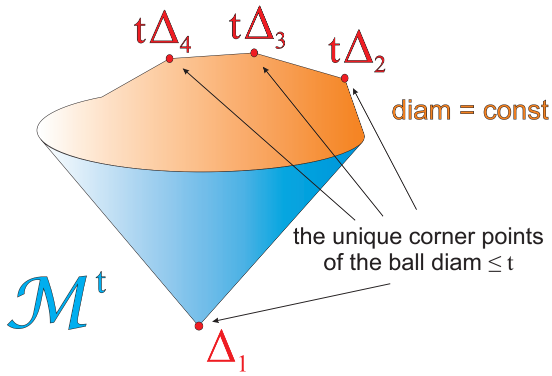

Figure 2: "Corner" points of the ball center at the single-point metric space Δ_1.

4.3 Invariance of the family of finite spaces

For any integer $n \geq 2$ and real $t > 0$ put (see Figure 3)

$$\mathcal{B}_n(t) = \mathrm{Mid}(\mathcal{M}, \Delta_1, t\,\Delta_n) \cap \{B \in \mathcal{M} : \mathrm{diam}\, B \geq 2t\}.$$

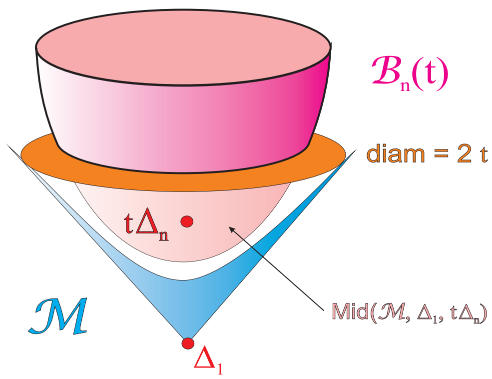

Figure 3: To Definition of $\mathcal{B}_n(t)$.

Proposition 4.8. *For each integer $n \geq 2$ and real $t > 0$ the following statements hold*:

(i) let $f\colon \mathcal{M}\to\mathcal{M}$ *be an arbitrary isometry, then* $f\big(\mathcal{B}_n(t)\big)=\mathcal{B}_n(t)$*;*

(ii) for each real $s\ge 2t$ *and integer* $m>n$ *we have* $s\,\Delta_m\in\mathcal{B}_n(t)$*, in particular,* $\mathcal{B}_n(t)\ne\emptyset$*;*

(iii) for any $B\in\mathcal{B}_n(t)$ *we have* $\#B>n$*;*

(iv) for any $B\in\mathcal{B}_n(t)$ *we have* $d_n(B)=\operatorname{diam} B$*.*

Proof. (i) This immediately follows from Proposition 3.4, Corollary 4.2, and Corollary 4.5.

(ii) Indeed, $\operatorname{diam} s\,\Delta_m=s\ge 2t$, and by Corollary 2.17 and item (i) of Proposition 2.4, we get $2d_{GH}(t\,\Delta_n,s\,\Delta_m)=\max\{s,t-s\}=s=\operatorname{diam}(s\,\Delta_m)=2d_{GH}(\Delta_1,s\,\Delta_m)$.

(iii) Suppose otherwise, i.e., that $\#B\le n$. Denote by a the smallest distance between different points of B. Then, by Proposition 2.16 and definition of $\mathcal{B}_n(t)$, we have

$$2d_{GH}(t\,\Delta_n,B)=\max\{t,\operatorname{diam} B-t\}=\operatorname{diam} B-t\qquad\text{for }\#B<n,$$

$$2d_{GH}(t\,\Delta_n,B)=\max\big\{t-a,\ \operatorname{diam} B-t\big\}=\operatorname{diam} B-t\qquad\text{for }\#B=n.$$

However, by definition of $\mathcal{B}_n(t)$ we have $2d_{GH}(t\,\Delta_n,B)=2d_{GH}(\Delta_1,B)=\operatorname{diam} B$, a contradiction.

(iv) Since $\#B>n$ by item (iii), and $\operatorname{diam} B\ge 2t$ by definition, then we can apply Proposition 2.16 which implies that $\operatorname{diam} B=2d_{GH}(t\,\Delta_n,B)=\max\big\{d_n(B),\operatorname{diam} B-t\big\}$, thus, $d_n(B)=\operatorname{diam} B$. □

For an integer $n\ge 2$ and real $t>0$ put

$$\mathcal{F}_n(t)=\big\{A\in\mathcal{M}:\ \operatorname{diam} A=t,\ \text{and } 2d_{GH}(A,B)=2d_{GH}(\Delta_1,B)=\operatorname{diam} B\ \text{for all } B\in\mathcal{B}_n(t)\big\}.$$

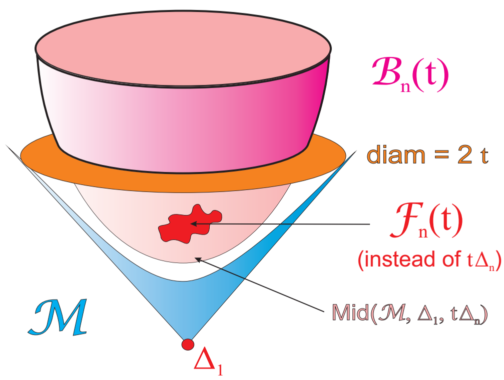

Figure 4: Illustration for $\mathcal{F}_n(t)$.

Remark 4.9. The set $\mathcal{F}_n(t)$ contains the simplex $t\,\Delta_n$ of diameter t. It consists of all A having the same diameter t and such that in definition of $\mathcal{B}_n(t)$ one can change the simplex $t\,\Delta_n$ by any of those A, see Figure 4.

Proposition 4.10. *The set $\mathcal{F}_n(t)$ is invariant under each isometry $f\colon \mathcal{M} \to \mathcal{M}$.*

Proof. For any $r > 0$ put $\mathcal{B}_n(t,r) = \mathcal{B}_n(t) \cap S_r(\Delta_1)$, then $\mathcal{B}_n(t,r)$ is f-invariant by item (i) of Proposition 4.8, Corollary 4.2, and Proposition 3.1. By Proposition 3.1, the equidistant $S_d\big(\mathcal{B}_n(t,r)\big)$ is f-invariant for each $d \ge 0$ as well. It remains to notice that $\mathcal{F}_n(t)$ equals the union over all $r \ge 2t$ of intersections of f-invariant sets $S_{t/2}(\Delta_1)$ and $S_{r/2}\big(\mathcal{B}_n(t,r)\big)$, and apply Proposition 3.1 again. □

Theorem 4.11. *For any integer $n \ge 2$ and real $t > 0$ the set $\mathcal{F}_n(t)$ coincides with the set of all finite metric spaces of diameter t consisting of at most n points.*

Proof. (1) Let us show that each at most n-point metric space A of diameter t belongs to $\mathcal{F}_n(t)$. To do that, take an arbitrary $B \in \mathcal{B}_n(t)$ and verify that $2d_{GH}(A,B) = \operatorname{diam} B$.

By Proposition 2.11, there exists $R \in \mathcal{R}^0_{\text{opt}}(A,B)$. By Proposition 2.7, the family $R_B = \bigcup_{a\in A}\big\{R(a)\big\}$ is a partition of B consisting of at most n elements, i.e., $R_B \in \mathcal{D}_m(B)$ for some $m \le n$. By Remark 2.10, $\operatorname{diam} R_B \ge d_m(B) \ge d_n(B)$. By Corollary 2.12, we have $2d_{GH}(A,B) \ge \operatorname{diam} R_B$, therefore, taking into account item (iv) of Proposition 4.8, we get $2d_{GH}(A,B) \ge d_n(B) = \operatorname{diam} B$. Since $\operatorname{diam} A < \operatorname{diam} B$, then, by item (iii) of Proposition 2.4, we have $\operatorname{diam} B = \max\{\operatorname{diam} A, \operatorname{diam} B\} \ge 2d_{GH}(A,B) \ge \operatorname{diam} B$, thus, $2d_{GH}(A,B) = \operatorname{diam} B$.

(2) Now, let us show that if $A \in \mathcal{F}_n(t)$, then $\#A \le n$. Suppose the contrary, i.e., let $\#A > n$. Put $\varepsilon = t/3$ and choose a finite ε-net $S = \{a_1,\dots,a_m\}$ in A consisting of $m \ge n+1$ points. Let $A_i = B_\varepsilon(a_i)$, then $\operatorname{diam} A_i \le 2\varepsilon < t$ and $A = \bigcup_{i=1}^m A_i$.

Choose an arbitrary $\mu \ge 2t$, put $B = \{b_1,\dots,b_{2m}\}$, and define a metric on B as follows: $|b_ib_j| = \mu$ for $1 \le i < j \le m$, and all the remaining non-zero distances are equal to $\mu/2$. Clearly that $d_n(B) = \mu = \operatorname{diam} B \ge 2t$, therefore, by Proposition 2.16, we have $2d_{GH}(t\,\Delta_n, B) = \max\big\{d_n(B), \operatorname{diam} B - t\big\} = \operatorname{diam} B$, thus, $B \in \mathcal{B}_n(t)$.

Define $R \in \mathcal{R}(A,B)$ as follows:

$$R = \{(a_i,b_i)\}_{i=1}^m \cup \big(A_1 \times \{b_{m+1}\}\big) \cup \dots \cup \big(A_m \times \{b_{2m}\}\big),$$

then $2d_{GH}(A,B) \le \operatorname{dis} R < \mu = \operatorname{diam} B$, a contradiction. Thus, $\#A \le n$. The theorem is proved. □

Corollary 4.12. *Every isometry $f\colon \mathcal{M} \to \mathcal{M}$ takes each n-point metric space to an n-point metric space of the same diameter.*

Proof. By Theorem 4.11, the set $\mathcal{F}_n(t)$ coincides with the family of all metric spaces of diameter $t > 0$, consisting of at most n points. By Proposition 4.10, the set $\mathcal{F}_n(t)$ is invariant under every isometry $f\colon \mathcal{M} \to \mathcal{M}$, thus, each n-point metric space $A \in \mathcal{F}_n(t)$ is mapped to a metric space $B = f(A)$ consisting of at most n point. Suppose that $\#B < n$. Since f^{-1} is an isometry of $\mathcal{M}$ also, then, by the same arguments, we have $\#A = \#f^{-1}(B) < n$, a contradiction. □

5. Groups actions

In what follows, we need some basic facts from the theory of topological group action on topological and metric spaces.

Recall that if a compact group G acts continuously on a topological space X, then its orbits are compact subsets of X. By $G(x)$ we denote the orbit of a point x under such action, and let X/G stand for the set of orbits.

If X is a metric space, and the group G is compact, then the following non-negative symmetric function $(A,B)\mapsto |AB|$, $A,B\in X/G$, does not vanish for any $A\ne B$.

PROPOSITION 5.1. *If a compact group G acts on a metric space X by isometries, then the function $(A,B)\mapsto |AB|$ is a metric on X/G.*

Proof. It remains to verify the triangle inequality. Since the orbits are compact subsets, then for any $A,B,C\in X/G$ there exist $a\in A$, $b_1,b_2\in B$, and $c\in C$ such that $|ab_1|=|AB|$ and $|b_2c|=|BC|$. Since b_1 and b_2 belongs to the same orbit, there exists $g\in G$ such that $g(b_1)=b_2$. However, $g\colon X\to X$ is an isometry, therefore, $\bigl|g(a)g(b_1)\bigr|=|AB|$ and, thus, $|AC|\le|g(a)c|\le\bigl|g(a)g(b_1)\bigr|+|b_2c|=|AB|+|BC|$. □

The metric on the set X/G defined in Proposition 5.1 is called a *quotient-metric.* In what follows, speaking about the metric space X/G, we always mean just this quotient-metric.

PROPOSITION 5.2. *Suppose that a finite group G acts on a metric space X by isometries. Then for every point $x\in X$ the following statements hold.*
(i) For any $\varepsilon>0$ and any g from the stabilizer G_x of the point x we have $g\bigl(B_\varepsilon(x)\bigr)=B_\varepsilon(x)$. Thus, for each $\varepsilon>0$ an action of the stabilizer G_x of the point $x\in X$ on the neighbourhood $B_\varepsilon(x)$ is defined.

(ii) If $G\setminus G_x\ne\emptyset$, then there exists $\varepsilon>0$ such that for all $g\in G\setminus G_x$ it holds $B_\varepsilon(x)\cap g\bigl(B_\varepsilon(x)\bigr)=\emptyset$, in particular, for every point $y\in B_\varepsilon(x)$ its stabilizer G_y is a subgroup of G_x, and also $G(y)\cap B_\varepsilon(x)=G_x(y)$.

(iii) There exists $\varepsilon>0$ such that for any $y_1,y_2\in B_\varepsilon(x)$ the distance between the orbits $G(y_1)$ and $G(y_2)$ is equal to the distance between the orbits $G_x(y_1)$ and $G_x(y_2)$.

Proof. (i) Since for each $g\in G_x$ we have $g(x)=x$, and g is an isometry, then $g\bigl(B_\varepsilon(x)\bigr)=B_\varepsilon(x)$ for any ε.

(ii) Put $Z=\bigl\{g(x):g\in G\setminus G_x\bigr\}$, then $x\notin Z$, and Z is a non-empty finite set (because we assume that $G\setminus G_x\ne\emptyset$), thus $r:=|xZ|>0$. Choose an arbitrary $\varepsilon<r/2$, then for all $g\in G\setminus G_x$ we have $B_\varepsilon(x)\cap g\bigl(B_\varepsilon(x)\bigr)=\emptyset$. In particular, this implies that the stabilizer of each point $y\in B_\varepsilon(x)$ does not intersect $G\setminus G_x$. Besides, for any point $y\in B_\varepsilon(x)$ and each $g\in G_x$ we have $\bigl|x\,g(y)\bigr|=\bigl|g(x)g(y)\bigr|=|xy|\le\varepsilon$, therefore, $B_\varepsilon(x)$ contains exactly that part of the orbit $G(y)$, which is generated by the elements of the stabilizer G_x.

(iii) If $G_x=G$, then we can take an arbitrary ε.

Now, let $G\setminus G_x\ne\emptyset$. Take r from item (ii) and choose an arbitrary $\varepsilon<r/4$, then the distance between any points from $B_\varepsilon(x)$ is less than $r/2$, and the distance

between any point from $B_\varepsilon(x)$ and any point from $B_\varepsilon\big(g(x)\big)$ for $g \in G \setminus G_x$ is greater than $r/2$. Thus, the distance between the orbits $G(y_1)$ and $G(y_2)$, $y_1, y_2 \in B_\varepsilon(x)$, is attained at those points of these orbits that belong to a neighbourhood $B_\varepsilon\big(g(x)\big)$, and this distance is the same in each such neighbourhood (because G acts by isometries). By item (ii), all points of the orbits under consideration that belong to the ball $B_\varepsilon(x)$ form the sets $G_x(y_1)$ and $G_x(y_2)$, respectively. $\square$

Definition 5.3. Under the assumptions and notations of Proposition 5.2, we call the closed ball $B_\varepsilon(x)$ by a *canonical neighbourhood of the point $x \in X$*. Here $\varepsilon > 0$ is arbitrary for $G_x = G$, and $\varepsilon < r/4$ for $G_x \neq G$.

Corollary 5.4. *Let G be a finite group acting by isometries on a metric space X. Choose an arbitrary point $x \in X$. Then the stabilizer G_x acts on each canonical neighbourhood $B_\varepsilon(x)$, and $\pi_{\varepsilon,x}\colon B_\varepsilon(x)/G_x \to B_\varepsilon\big(G(x)\big) \subset X/G$, where $\pi_{\varepsilon,x}\colon G_x(y) \mapsto G(y)$, is an isometry. Further, for each $g \in G$ the neighbourhood $B_\varepsilon\big(g(x)\big) = g\big(B_\varepsilon(x)\big)$ is also a canonical one, and the mapping g generates an isometry $g_{\varepsilon,x}\colon B_\varepsilon(x)/G_x \to B_\varepsilon\big(g(x)\big)/G_{g(x)}$, where $g_{\varepsilon,x}\colon G_x(y) \mapsto G_{g(x)}\big(g(y)\big)$. Besides, the mappings $g_{\varepsilon,x}$, $\pi_{\varepsilon,x}$, and $\pi_{\varepsilon,g(x)}$ are agreed with each other in the following sense: $\pi_{\varepsilon,x} = \pi_{\varepsilon,g(x)} \circ g_{\varepsilon,x}$. Thus, each mapping $\pi^{-1}_{\varepsilon,g(x)} \circ \pi_{\varepsilon,x}$ is generated by the mapping g.*

In what follows we especially need a version of Corollary 5.4 in the situation, when the stabilizer G_x is trivial (i.e., it consists of the unit element only). Since in this case $G_x(y) = \{y\}$ for any $x, y \in X$, then the mapping $\pi_{\varepsilon,x}\colon G_x(y) \mapsto G(y)$ coincides with the restriction of the canonical projection $\pi\colon y \mapsto G(y)$ onto the canonical neighbourhood $B_\varepsilon(x)$. Similarly, in this case, $g_{\varepsilon,x}\colon B_\varepsilon(x)/G_x \to B_\varepsilon\big(g(x)\big)/G_{g(x)}$ is a mapping between the canonical neighbourhoods $B_\varepsilon(x)$ and $B_\varepsilon\big(g(x)\big)$, and it coincides with the restriction of the mapping g onto the canonical neighbourhood $B_\varepsilon(x)$, thus, in this case Corollary 5.4 can be reformulated as follows.

Corollary 5.5. *Let G be an arbitrary finite group acting on a metric space X by isometries, and let $\pi\colon X \to X/G$ be the canonical projection onto the orbit space, $\pi\colon x \mapsto G(x)$. Suppose that the stabilizers of all points from X are trivial. Then the restriction $\pi_{\varepsilon,x}$ of the projection π onto each canonical neighbourhood $B_\varepsilon(x) \subset X$ of the point x maps isometrically the $B_\varepsilon(x)$ onto $B_\varepsilon\big(G(x)\big) \subset X/G$. Further, for each $g \in G$ the neighbourhood $B_\varepsilon\big(g(x)\big) = g\big(B_\varepsilon(x)\big)$ is also a canonical one. Besides, the restriction $g_{\varepsilon,x}\colon B_\varepsilon(x) \to B_\varepsilon\big(g(x)\big)$ of the mapping g, being isometry, is agreed with the mappings $\pi_{\varepsilon,x}$ and $\pi_{\varepsilon,g(x)}$ in the following sense: $\pi_{\varepsilon,x} = \pi_{\varepsilon,g(x)} \circ g_{\varepsilon,x}$. Thus, each mapping $\pi^{-1}_{\varepsilon,g(x)} \circ \pi_{\varepsilon,x}$ coincides with the restriction of the mapping g onto the canonical neighbourhood $B_\varepsilon(x)$.*

6. The canonical local isometry

For $n \in \mathbb{N}$ put $\circ M_n = \{X \in \mathcal{M} : \#X \le n\}$ and $\mathcal{M}_{[n]} = \{X \in \mathcal{M} : \#X = n\}$, then $\mathcal{M}_{[1]} = \{\Delta_1\}$, $\mathcal{M}_{[2]}$ is isometric to the non-negative ray on the real line, and $\mathcal{M}_{[3]}$

is isometric to the set $\big\{(a,b,c) : 0 < a \le b \le c \le a+b\big\}$ endowed with the metric generated by the norm $\big\|(x,y,z)\big\|_\infty = \frac{1}{2}\max\big\{|x|,|y|,|z|\big\}$ (the latter fact can be found in [11]). In this Section we describe the local geometry of the space $\mathcal{M}_{[}n]$.

6.1 General construction

For $N = n(n-1)/2$ by $\mathbb{R}^N_\infty$ we denote the arithmetic vector space $\mathbb{R}^N$ endowed with the norm $\big\|(x^1,\dots,x^N)\big\|_\infty = \frac{1}{2}\max_{i=1}^N\big\{|x^i|\big\}$. The corresponding distance between points $x, y \in \mathbb{R}^N_\infty$ is denoted by $|xy|_\infty$.

Let $X \in \mathcal{M}_{[n]}$. Enumerate the points of X in an arbitrary way, then $X = \{x_i\}_{i=1}^n$, and let $\rho_{ij} = \rho_{ji} = |x_i x_j|$ be the components of the corresponding distance matrix M_X of the space X. The matrix M_X is uniquely determined by the vector

$$\rho_X = (\rho_{12},\dots,\rho_{1n},\rho_{23},\dots,\rho_{2n},\dots,\rho_{(n-1)n}) \in \mathbb{R}^N.$$

Notice that the set of all possible $\rho_X \in \mathbb{R}^N$, $X \in \mathcal{M}_{[n]}$, consists of all vectors with positive coordinates, which satisfy the following "triangle inequalities": $\rho_{ik} \le \rho_{ij} + \rho_{jk}$ for any pairwise distinct $1 \le i,j,k \le n$ (for convenience we put $\rho_{ij} = \rho_{ji}$ for all i and j). The set of all such vectors is denoted by $\mathcal{C}_n$.

If one changes the numeration of points of the space X, i.e., if one acts by a permutation $\sigma \in S_n$ on X by the rule $\sigma(x_i) = x_{\sigma(i)}$, then the components of the matrix M_X are permuted as follows: $\rho_{ij} \mapsto \sigma(\rho_{ij}) := \rho_{\sigma(i)\sigma(j)}$. By $M_{\sigma(X)}$ and $\rho_{\sigma(X)}$ we denote the resulting matrix and the corresponding vector, respectively.

Notice that this action of the group S_n, in fact, permutes the basis vectors of $\mathbb{R}^N$ in a special way, i.e., $\sigma \in S_n$ sends the basis vector corresponding to the component ρ_{ij} to the basis vector corresponding to the component $\rho_{\sigma(i)\sigma(j)}$ of the vector $\rho \in \mathbb{R}^N$. The group S_N also acts on $\mathbb{R}^N$ by (arbitrary) permutations of the basis vectors, so S_n generates a subgroup G of S_N, which is isomorphic to S_n as $n \ge 3$. *Below by G we always denote this subgroup of S_N.* Its elements are considered either as permutations, or as the corresponding linear transformations of the space $\mathbb{R}^N$.

Since the unit ball in $\mathbb{R}^N_\infty$ is a Euclidean cube centered at the origin, and since each permutation of the basis vectors takes this cube into itself, then the group S_N, together with its subgroup G, acts on $\mathbb{R}^N_\infty$ by isometries. Notice also that, generally speaking, the group S_N does not preserve the cone $\mathcal{C}_n$, because permutations of general type acting on the set of distances of a metric space X can violate a triangle inequality, but G does preserve $\mathcal{C}_n$. In particular, an action of the group G on $\mathcal{C}_n$ is defined.

Further, each orbit of the action of the group G on $\mathbb{R}^N$ contains at most $n!$ points, and each *regular orbit*, i.e., the orbit of an element having trivial stabilizer, consists of $n!$ points exactly. A space X such that the orbit of the corresponding ρ_X is regular, together with all the vectors $g(\rho_X)$, $g \in G$, is called *regular*.

Notice that the cone $\mathcal{C}_n$ is not open in $\mathbb{R}^N$: it contains boundary points, namely, those ρ_X which some triangle inequalities hold as equalities at. Such X and the corresponding ρ_X are referred as *degenerate*, and all the remaining X and ρ_X are called *non-degenerate*.

We say that a space $X \in \mathcal{M}_{[n]}$ and each corresponding $\rho_X \in \mathbb{R}^N$ are *generic* or are *in general position*, if X is regular and non-degenerate. Thus, $X \in \mathcal{M}_{[n]}$ is generic, iff its isometry group is trivial and all triangle inequalities hold strictly. Notice that in [8] we meant by generic spaces a narrow class, demanding in addition that all non-zero distances are pairwise different.

By $\mathcal{C}_n^g$ we denote the subset of $\mathcal{C}_n$ consisting of all vectors in general position, and by $\mathcal{M}_{[n]}^g$ we denote the corresponding subset of $\mathcal{M}_{[n]}$ consisting of all spaces in general position. It is easy to see that both $\mathcal{C}_n^g$ and $\mathcal{M}_{[n]}^g$ are open in $\mathbb{R}^N$ and in $\mathcal{M}_{[n]}$, respectively; besides, these subsets are everywhere dense in $\mathcal{C}_n$ and $\mathcal{M}_{[n]}$, respectively.

Define a mapping $\Pi\colon \mathcal{C}_n \to \mathcal{M}_{[n]} \subset \mathcal{M}$ as $\Pi\big(\rho_X\big) = X$. As it is shown in [8], for any $X \in \mathcal{M}_{[n]}$ there exists a sufficiently small $\varepsilon > 0$, such that for any $Y, Z \in B_\varepsilon(X) \subset \mathcal{M}_{[n]}$ each optimal correspondence $R \in \mathcal{R}(Y, Z)$ is a bijection. Therefore for such Y and Z it holds $d_{GH}(Y,Z) = \min_{\rho_Y,\rho_Z}\big\{|\rho_Y\rho_Z|_\infty\big\} = \big|G(\rho_Y)G(\rho_Z)\big|_\infty$, where in the right hand side of the equality the standard distance between subsets of $\mathbb{R}^N_\infty$ stands, i.e., the infimum (here it is the minimum) of $\mathbb{R}^N_\infty$-distances between their elements.

Thus, we get the following result.

Proposition 6.1. *For any $X \in \mathcal{M}_{[n]}$ there exists $\varepsilon > 0$ such that*

$$d_{GH}(Y,Z) = \big|\Pi^{-1}(Y)\Pi^{-1}(Z)\big|_\infty$$

for every $Y, Z \in B_\varepsilon(X) \subset \mathcal{M}_{[n]}$.

By Proposition 5.1, the action of the group G on $\mathcal{C}_n$ generates a metric space $\mathcal{C}_n/G$. item (iii) of Proposition 5.2 implies the following statement.

Corollary 6.2. *For sufficiently small $\varepsilon > 0$ the ball $B_\varepsilon\big(G(\rho)\big)$ in the space $\mathcal{C}_n/G$ is isometric to the quotient space $\big(B_\varepsilon(\rho)\cap\mathcal{C}_n\big)/G_\rho$, where $B_\varepsilon(\rho)$ is a ball in $\mathbb{R}^N_\infty$, and G_ρ is the stabilizer of the point $\rho \in \mathcal{C}_n$ under the action of the group G.*

Combining Proposition 6.1 and Corollary 6.2, we get the following result.

Corollary 6.3. *The mapping $G(\rho_X) \mapsto X$ is a locally isometric homeomorphism between $\mathcal{C}_n/G$ and $\mathcal{M}_{[n]}$, therefore, for any $X \in \mathcal{M}_{[n]}$ and any $\rho \in \Pi^{-1}(X)$ there exists $\varepsilon > 0$ such that the closed ball $B_\varepsilon(X) \subset \mathcal{M}_{[n]}$ is isometric to $\big(B_\varepsilon(\rho)\cap\mathcal{C}_n\big)/G_\rho$, where $B_\varepsilon(\rho)$ is a ball in $\mathbb{R}^N_\infty$, and G_ρ is the stabilizer of the point $\rho \in \mathcal{C}_n$ under the action of the group G.*

Now we consider all possible types of the spaces $X \in \mathcal{M}_{[n]}$: a generic space, a regular degenerate space, a non-regular non-degenerate space, and, at last, a non-regular degenerate space. All the corresponding results listed below follow from Corollary 6.3.

Generic spaces. Recall that by generic spaces we mean regular non-degenerate spaces $X \in \mathcal{M}_{[n]}$ and the corresponding elements from $\mathcal{C}_n$.

Corollary 6.4. *For each generic space $X \in \mathcal{M}_{[n]}^g$, for all sufficiently small $\varepsilon > 0$ the closed ball $B_\varepsilon(X) \subset \mathcal{M}_{[n]}$ lies in $\mathcal{M}_{[n]}^g$ and is isometric to the ball $B_\varepsilon(\rho_X)$ in $\mathbb{R}^N_\infty$.*

Regular degenerate spaces. For any $n \geq 3$ and $\rho \in \mathcal{C}_n$ by $D(\rho)$ we denote the set of all ordered triples of different indices (i,j,k), $1 \leq i,j,k \leq n$, such that $\rho_{ij}+\rho_{jk} = \rho_{ik}$. Notice that ρ is degenerate, iff $D(\rho) \neq \emptyset$. Further, for non-empty $D(\rho)$ by $T(\rho)$ we denote the polyhedral cone with the vertex at the origin, which is obtained as the intersection of all half-spaces in $\mathbb{R}^N$ defined by the inequalities $\rho_{ij}+\rho_{jk}-\rho_{ik} \geq 0$ over all $(i,j,k) \in D(\rho)$. If $D(\rho) = \emptyset$, then put $T(\rho) = \mathbb{R}^N$. Notice that for a degenerate $\rho \in \mathbb{R}^N_\infty$ and any sufficiently small $\varepsilon > 0$ we have $B_\varepsilon(\rho)\cap\mathcal{C}_n = B_\varepsilon(\rho)\cap T(\rho)$.

Corollary 6.5. *For each regular degenerate space $X \in \mathcal{M}_{[n]}$ and for all sufficiently small $\varepsilon > 0$ the closed ball $B_\varepsilon(X) \subset \mathcal{M}_{[n]}$ is isometric to the intersection $B_\varepsilon(\rho_X) \cap T(\rho_X)$ of the ball $B_\varepsilon(\rho_X)$ in $\mathbb{R}^N_\infty$ and the cone $T(\rho_X)$ defined above.*

We also need the following property of the cone $T(\rho)$.

Proposition 6.6. *If $\rho_X \in \mathcal{C}_n$ is a vector corresponding to a degenerate space $X \in \mathcal{M}_{[n]}$, then for any $\varepsilon > 0$ the set $B_\varepsilon(\rho_X)\setminus\big(B_\varepsilon(\rho_X)\cap T(\rho_X)\big)$ has a non-empty interior.*

Proof. Indeed, since X is a degenerate space, then $T(\rho_X)$ is contained in a half-space Θ bounded by a hyperplane θ of the form $\rho_{ij}+\rho_{jk}-\rho_{ik} = 0$ passing through X. Since both the cube $B_\varepsilon(\rho_X)$ and the hyperplane θ are centrally symmetric with respect to ρ_X, then $B_\varepsilon(\rho_X) \setminus (B_\varepsilon(\rho_X) \cap \Theta)$ contains interior points. It remains to notice that $T(\rho_X) \subset \Theta$. □

Non-regular non-degenerate spaces.

Corollary 6.7. *For each non-regular non-degenerate space $X \in \mathcal{M}_{[n]}$, for a sufficiently small $\varepsilon > 0$ the closed ball $B_\varepsilon(X) \subset \mathcal{M}_{[n]}$ is isometric to the space $B_\varepsilon(\rho_X)/G_{\rho_X}$ obtained from the ball $B_\varepsilon(\rho_X)$ in $\mathbb{R}^N_\infty$ by factorisation over the action of the stabilizer G_{ρ_X} of the point ρ_X.*

Non-regular degenerate spaces. Now, let $X \in \mathcal{M}_{[n]}$ be a non-regular degenerate space, then the stabilizer G_{ρ_X} is non-trivial, and the cone $T(\rho_X)$ differs from the entire space. Notice that each motion $g \in G_{\rho_X}$ takes $T(\rho_X)$ into itself. Indeed, since $g(\rho_X) = \rho_X$, then the set of degenerate triangles in X is mapped into itself by any such permutation g of points of the space X. Further, as above $B_\varepsilon(\rho_X) \cap T(\rho_X) = B_\varepsilon(\rho_X) \cap \mathcal{C}_n$ for small $\varepsilon > 0$. Thus, for sufficiently small $\varepsilon > 0$ the stabilizer G_{ρ_X} acts on the set $B_\varepsilon(\rho_X) \cap T(\rho_X) = B_\varepsilon(\rho_X) \cap \mathcal{C}_n$.

Corollary 6.8. *For each non-regular degenerate space $X \in \mathcal{M}_{[n]}$, for all sufficiently small $\varepsilon > 0$ the closed ball $B_\varepsilon(X) \subset \mathcal{M}_{[n]}$ is isometric to the space $\big[B_\varepsilon(\rho_X) \cap T(\rho_X)\big]/G_{\rho_X}$ obtained from the intersection of the ball $B_\varepsilon(\rho_X)$ in $\mathbb{R}^N_\infty$ with the cone $T(\rho_X)$ by factorisation over action of the stabilizer G_{ρ_X} of the point ρ_X.*

6.2 More on generic spaces

Now, let us apply Corollary 5.5.

Corollary 6.9. *For any $X \in \mathcal{M}^g_{[n]}$ there exists $\varepsilon > 0$ such that* *(i) For any $\rho \in \Pi^{-1}(X)$ the ball $B_\varepsilon(\rho)$ in $\mathbb{R}^N_\infty$ lies entirely in $\mathcal{C}^g_n$, and the set $\Pi^{-1}\big(B_\varepsilon(X)\big)$ equals the disjoint union of the balls $\big\{B_\varepsilon(\rho)\big\}_{\rho\in\Pi^{-1}(X)}$.*

(ii) The restriction $\pi_{\varepsilon,\rho}\colon B_\varepsilon(\rho)\to B_\varepsilon(X)$ of the projection Π is an isometry.

(iii) The restriction $g_{\varepsilon,\rho}\colon B_\varepsilon(\rho)\to B_\varepsilon\big(g(\rho)\big)$ of the mapping $g\in G$ is also an isometry.

(iv) The mappings $\pi_{\varepsilon,\rho}$ and $\pi_{\varepsilon,g(\rho)}$ are agreed with each other in the following sense: $\pi_{\varepsilon,\rho}=\pi_{\varepsilon,g(\rho)}\circ g_{\varepsilon,\rho}$, thus each mapping $\pi^{-1}_{\varepsilon,g(\rho)}\circ\pi_{\varepsilon,\rho}$ coincides with the restriction of the mapping $g\in G$ onto the ball $B_\varepsilon(\rho)$.

Definition 6.10. Each neighbourhood $B_\varepsilon(X)$ from Corollary 6.9, together with all neighbourhoods $B_\varepsilon(\rho)$, is called *canonical*.

Proposition 6.11. *The subsets $\mathcal{C}^g_n\subset\mathbb{R}^N$ are path-connected for all $n\ne 3$; moreover, each pair of points in $\mathcal{C}^g_n$ can be connected by a polygonal line lying in $\mathcal{C}^g_n$. For $n=3$ the subset $\mathcal{C}^g_n\subset\mathbb{R}^3$ is not path-connected. The subsets $\mathcal{M}^g_{[n]}\subset\mathcal{M}_{[n]}$ are path-connected for all n.*

Proof. If $n=1$ or $n=2$, then $\mathcal{C}^g_n=\mathcal{C}_n$ and $M^g_{[n]}=M_{[n]}$, thus the path-connectivity follows from the above remarks.

Let $n=3$. Show that $\mathcal{C}^g_n$ is not path-connected. Take, for instance, two points $\rho_0=(3,4,5)$ and $\rho_1=(4,3,5)\in\mathcal{C}^g_3$, and suppose that there exists a continuous curve $\rho_t=\big(\rho_{12}(t),\rho_{13}(t),\rho_{23}(t)\big)$, $t\in[0,1]$, that lies in $\mathcal{C}^g_3$ and connects these points. Then the continuous function $f(t)=\rho_{12}(t)-\rho_{13}(t)$ satisfies $f(0)<0$ and $f(1)>0$, therefore there exists $s\in(0,1)$ such that $\rho_{12}(s)=\rho_{13}(s)$. But then the stabilizer of the point ρ_s is nontrivial, thus $\rho_s\notin C^g_3$.

Now, show that $M^g_{[3]}$ is path-connected. To start with, notice that a triple of real numbers $a\le b\le c$ are the lengths of a triangle $X\in\mathcal{M}^g_{[3]}$, iff $0<a<b<c<a+b$. Choose $X_0,X_1\in\mathcal{M}^g_{[3]}$, and let $0<a_i<b_i<c_i<a_i+b_i$ be non-zero distances in X_i. Then for each $t\in[0,1]$ the triple $\{a_t=(1-t)a_0+t\,a_1,\ b_t=(1-t)b_0+t\,b_1,\ c_t=(1-t)c_0+t\,c_1\}$ also satisfies $0<a_t<b_t<c_t<a_t+b_t$ and, thus, it generates a metric space X_t belonging to $\mathcal{M}^g_{[3]}$. It is easy to see that $t\mapsto X_t$ is a continuous curve in $\mathcal{M}^g_{[3]}$, therefore, $\mathcal{M}^g_{[3]}$ is path-connected.

Consider the case $n\ge 4$. Notice that the cone $\mathcal{C}_n$ is convex, because it is the intersection of half-spaces corresponding to the positivity conditions of metric components, and to triangle inequalities. This implies that for any $\rho_0,\rho_1\in\mathcal{C}_n$ the segment $\rho_t=(1-t)\rho_0+t\,\rho_1$, $t\in[0,1]$, belongs to $\mathcal{C}_n$. Further, if $\rho_0,\rho_1\in\mathcal{C}_n$ are non-degenerate, then all ρ_t are non-degenerate as well. Thus, the set of all non-degenerate vectors $\rho\in\mathcal{C}_n$ is convex. Moreover, the set of all non-degenerate vectors $\rho\in\mathcal{C}_n$ is open and everywhere dense in $\mathcal{C}_n$.

Now, let us investigate the structure of the set of all non-regular $\rho\in\mathcal{C}_n$. The condition of non-regularity of $\rho\in\mathcal{C}_n$ means that there exists a non-identical transformation $\sigma\in G$, such that $\sigma(\rho)=\rho$. Put $X=\Pi(\rho)$ and let $\rho=\rho_X$ for some numeration $X=\{x_1,\dots,x_n\}$ of points of X, i.e., $\rho_{ij}=|x_ix_j|$. Since the permutation σ is not identical, then there exists $i\in\{1,\dots,n\}$ such that $j=\sigma(i)\ne i$. Since

$n \ge 4$, then there exist at least two distinct $p, q \in \{1, \dots, n\}$ different from i such that $r = \sigma(p) \ne i$, $s = \sigma(q) \ne i$. This implies that $\{i, p\} \ne \{j, r\}$ and $\{i, q\} \ne \{j, s\}$, therefore, since $\sigma\bigl(\rho_{ip}\bigr) = \rho_{jr}$ and $\sigma\bigl(\rho_{iq}\bigr) = \rho_{js}$, the condition $\sigma(\rho) = \rho$ implies at least two non-identical conditions, namely, $\rho_{ip} = \rho_{jr}$ and $\rho_{iq} = \rho_{js}$. Moreover, by assumption i differs from j, r, and s, therefore all the four pairs $\{i, p\}$, $\{i, q\}$, $\{j, r\}$, and $\{j, s\}$ are pairwise distinct, and hence these two conditions are independent. Therefore, the set of the vectors $\rho \in \mathcal{C}_n$ such that $\sigma(\rho) = \rho$ is contained in a finite number of linear subspaces of $\mathbb{R}^N$, whose codimension is at least 2. Those linear subspaces are referred as *irregularity subspaces.*

Take two arbitrary $\rho_1, \rho_2 \in \mathcal{C}_n^g$, and for ρ_1 and each irregularity subspace consider their linear hull. We get a collection of subspaces of non-zero codimensions. This implies that the union W of those subspaces does not cover any open set in $\mathbb{R}^N$. Thus, since $\mathcal{C}_n^g$ is open, there exists $\rho_2' \in U_\varepsilon(\rho_2) \subset \mathcal{C}_n^g$, which does not belong to W. Therefore, the segment $[\rho_1, \rho_2']$ does not intersect W, and hence, the polygonal line $\rho_1\rho_2'\rho_2$ does not intersect W as well. So, $\mathcal{C}_n^g$ is path-connected and each two its points can be connected by a polygonal line lying in $\mathcal{C}_n^g$. Since the path-connectivity is preserved under continuous mappings, the set $\mathcal{M}_{[n]}^g$ is path-connected also. □

6.3 Coverings and generic spaces

By $\Pi^g\colon \mathcal{C}_n^g \to \mathcal{M}_{[n]}^g$ we denote the restriction of the mapping $\Pi\colon \mathcal{C}_n \to \mathcal{M}_{[n]}$ onto $\mathcal{C}_n^g$. Recall a definition of a covering, see [4] for details.

Let T and B be path-connected topological spaces, F be a discrete topological space, $n = \#F$. Then each continuous surjective mapping $\pi\colon T \to B$ is called an *n-sheeted covering with the total space T, the base B, and the fiber F*, if each point $b \in B$ has a neighbourhood U such that $\pi^{-1}(U)$ is homeomorphic to $U \times F$, and, if $\varphi\colon \pi^{-1}(U) \to U \times F$ is the corresponding homeomorphism and $\pi_1\colon U \times F \to U$ is the projection, $\pi_1\colon (u, f) \mapsto u$, then $\pi = \pi_1 \circ \varphi$ (i.e., the corresponding diagram is commutative). If we omit the path-connectivity condition, then the mapping π is called a *covering in the wide sense.*

Corollary 6.12. *The mapping $\Pi^g\colon \mathcal{C}_n^g \to \mathcal{M}_{[n]}^g$ is an $n!$-sheeted locally isometric covering (in a wide sense for $n = 3$, because $\mathcal{C}_3$ is not path-connected).*

We use Corollary 6.12 for constructing the lift of paths.

Proposition 6.13 (Lifting of paths [4]). *Let $\pi\colon T \to B$ be an arbitrary covering in a wide sense, $\gamma\colon [a, b] \to B$ be a continuous mapping (a* path *in B), and $t \in T$ be an arbitrary point in $\pi^{-1}\bigl(\gamma(a)\bigr)$. Then there exists unique continuous mapping $\Gamma\colon [a, b] \to T$, such that $\Gamma(a) = t$ and $\gamma = \pi \circ \Gamma$.*

Definition 6.14. The mapping Γ from Proposition 6.13 is called the *lift of γ*.

7. Invariancy of $\mathcal{M}^g_{[n]}$

In this Section we prove that the sets $\mathcal{M}^g_{[n]}$ are invariant under any isometry of the space $\mathcal{M}$. To do that, we use the technique elaborated above together with the invariance of the Hausdorff measure on a metric space under its isometries. Recall the corresponding concepts and facts.

Let X be an arbitrary set, and 2^X be the set of all subsets of X.

Definition 7.1. An *outer measure* on a set X is a mapping $\mu\colon 2^X \to [0,+\infty]$ such that
(i) $\mu(\emptyset) = 0$;

(ii) for any at most countable family $\mathcal{C}$ of subsets of X and any $A \subset X$ such that $A \subset \bigcup_{B\in\mathcal{C}} B$, it holds $\mu(A) \le \sum_{B\in\mathcal{C}} \mu(B)$ (subadditivity).

Definition 7.2. A subset $A \subset X$ is called *measurable with respect to* μ, or simply μ*-measurable* (*in the sense of Carathéodory*), if for any $Y \subset X$ it holds $\mu(Y) = \mu(Y \cap A) + \mu(Y \setminus A)$.

Definition 7.3. A family $\mathcal{S}$ of subsets of X is called a σ*-algebra on* X, if it contains $\emptyset$, X, and it is closed under taking the complement and countable union operations.

It is well-known that for any outer measure μ on a set X the set of all μ-measurable subsets of X is a σ-algebra. Also it is well-known that the intersection of any σ-algebras is a σ-algebra. The latter allows one to define the smallest σ-algebra containing a given family of subsets of X.

If X is a topological space, then the smallest σ-algebra containing the topology is called a *Borel* σ*-algebra*, and its elements are called *Borel sets*. An outer measure μ on a topological space is said to be *Borel*, if all Borel sets are μ-measurable. An outer measure μ on a topological space X is said to be *Borel regular*, if it is Borel and for any set $A \subset X$ the value $\mu(A)$ is equal to the infimum of the values $\mu(B)$ over all Borel sets B^A.

Let X be an arbitrary metric space. For our purposes it suffices to define the Hausdorff measure up to a multiplicative constant. For the standard definition of this measure see, for instance [1].

Definition 7.4. For $\delta > 0$ and $A \subset X$, a family $\{A_i\}_{i\in I}$ of subsets of X is called a δ*-covering of the set* A, if $A \subset \bigcup_{i\in I} A_i$ and $\operatorname{diam} A_i < \delta$ for all $i \in I$ (if $A_i = \emptyset$, then put $\operatorname{diam} A_i = 0$).

Definition 7.5. For any $\delta > 0$, $k > 0$, and $A \subset X$ put

$$H^k_\delta(A) = \inf\Big\{\sum_{i=1}^{\infty}(\operatorname{diam} A_i)^k \;:\; \{A_i\}_{i=1}^{\infty} \text{ is a } \delta\text{-covering of } A\Big\},$$

$$H^k(A) = \sup_{\delta>0} H^k_\delta(A).$$

The next results are well-known, see, for example [1].

Proposition 7.6. *For any $k > 0$ and any positive integer N the following statements hold.*
(i) For any metric space X the functions H^k are Borel regular outer measures on X.

(ii) If $f\colon X \to Y$ is an isometry of arbitrary metric spaces, then $H^k\big(f(A)\big) = H^k(A)$ for any subset $A \subset X$.

(iii) In any N-dimensional normed space, the H^N-measure of a unit ball is non-zero and finite, thus, the H^N-measure of any bounded subset with non-empty interior is non-zero and finite.

Proposition 7.7. *Suppose that a compact group F acts continuously by isometries on a metric space X, and let Y be a subset of X that is invariant with respect to the action of the group G. Suppose also that*
(i) For some $k > 0$ we have $H^k(Y) \in (0, \infty)$;

(ii) There exist $f \in F$ and $A \subset Y$ such that $H^k(A) > 0$ and $A \cap f(A) = \emptyset$.
Then $H^k(Y/F) < H^k(Y)$.

Proposition 7.8. *Let $f\colon \mathcal{M} \to \mathcal{M}$ be an arbitrary isometry, then $\mathcal{M}^g_{[n]} = f(\mathcal{M}^g_{[n]})$.*

Proof. Choose an arbitrary $X \in \mathcal{M}^g_{[n]}$ and let $Y = f(X)$. At first suppose that Y is a regular degenerate space, then, by Corollaries 6.4 and 6.5, there exists $\varepsilon > 0$ such that the ball $B_\varepsilon(X) \subset \mathcal{M}_{[n]}$ is isometric to $B_\varepsilon(\rho_X) \subset \mathbb{R}^N_\infty$, and the ball $B_\varepsilon(Y)$ is isometric to the intersection $B_\varepsilon(\rho_Y) \cap T(\rho_Y)$ of the ball $B_\varepsilon(\rho_Y) \subset \mathbb{R}^N_\infty$ and the cone $T(\rho_Y)$. Since the translations in $\mathbb{R}^N_\infty$ are isometries, then, by items (ii) and (iii) of Proposition 7.6, we have

$$H^N\big(B_\varepsilon(\rho_Y)\big)=H^N\big(B_\varepsilon(\rho_X)\big)=H^N\big(B_\varepsilon(X)\big)=H^N\big(B_\varepsilon(Y)\big)=H^N\big(B_\varepsilon(\rho_Y)\cap T_{\rho_Y}\big) > 0.$$

By Proposition 6.6 and item (iii) of Proposition 7.6, is holds $H^N\Big(B_\varepsilon(\rho_Y)\setminus\big(B_\varepsilon(\rho_Y)\cap T(\rho_Y)\big)\Big)>0$, therefore, since the outer measure H^N is a Borel one, we get $H^N\big(B_\varepsilon(\rho_Y)\cap T(\rho_Y)\big)<H^N\big(B_\varepsilon(\rho_Y)\big)$, a contradiction. Thus, the balls $B_\varepsilon(X)$ and $B_\varepsilon(Y)$ are not isometric, so Y cannot be a regular degenerate space.

Next, let Y be a non-regular non-degenerate space. Then, by Corollary 6.7, the ball $B_\varepsilon(Y)$ is isometric to $B_\varepsilon(\rho_Y)/G_{\rho_Y}$, where G_{ρ_Y} is the stabilizer of the point ρ_Y, which is a non-trivial group, because the space Y is non-regular. Let $g \in G_{\rho_Y}$ be an element different from the unity. Since the generic spaces are everywhere dense in $\mathcal{M}_{[n]}$, there exists $\rho_Z \in U_\varepsilon(\rho_Y)$ corresponding to a generic space $Z \in \mathcal{M}_{[n]}$. Since the stabilizer of the point ρ_Z is trivial, item (ii) of Proposition 5.2 implies that there exists $\delta > 0$ such that $U_\delta(\rho_Z) \subset U_\varepsilon(\rho_Y)$ and $g\big(U_\delta(\rho_Z)\big) \cap U_\delta(\rho_Z) = \emptyset$. However, $U_\delta(\rho_Z)$ is an open ball in $\mathbb{R}^N_\infty$, therefore, by item (iii) of Proposition 7.6, we have $0 < H^N\big(U_\delta(\rho_Z)\big) < \infty$. Further, by item (i) of Proposition 5.2, it holds $g\big(U_\delta(\rho_Z)\big) \subset U_\varepsilon(\rho_Y)$, thus, by item (ii) of Proposition 7.7 we conclude that $H^N\big(B_\varepsilon(\rho_Y)/G_{\rho_Y}\big) < H^N\big(B_\varepsilon(\rho_Y)\big)$ and, so,

$$H^N\big(B_\varepsilon(Y)\big) = H^N\big(B_\varepsilon(\rho_Y)/G_{\rho_Y}\big) < H^N\big(B_\varepsilon(\rho_Y)\big) = H^N\big(B_\varepsilon(\rho_X)\big) = H^N\big(B_\varepsilon(X)\big).$$

Thus, Y cannot be a non-regular non-degenerate space.

The case of a non-regular degenerate space Y can be proceeded by a combination of the above arguments. □

8. Local affinity property

In 1968 F. John [14] obtained a generalisation of the Mazur–Ulam Theorem [15] on affinity property of isometries of normed vector spaces.

Proposition 8.1 ([14, Theorem IV, p. 94]). *Let $U \subset X$ be a connected open subset of a real complete normed space X, and $h\colon U \to W$ be an isometry that maps U onto an open subset W of a real complete normed space Y. Then h is the restriction of an affine isometry $H\colon X \to Y$.*

Proposition 8.1 implies that all the isometries of the space $\mathbb{R}^d_\infty$ are affine. Describe these isometries in more details.

Proposition 8.2. *Let $h\colon \mathbb{R}^d_\infty \to \mathbb{R}^d_\infty$ be an isometry. Then $h(x) = (S \cdot P)\,x + b$, where $b \in \mathbb{R}^d$ is a translation vector, P is a permutation matrix of the vectors from the standard basis, and S is a diagonal matrix with ± 1 on its diagonal.*

Proof. Due to Proposition 8.1, h is affine. Any affine mapping is a composition of a linear mapping $x \mapsto Ax$ with a translation. Since the distance in a normed space is invariant under any translation, it suffices to describe all linear isometries $h(x) = Ax$. Every such mapping takes the unit ball centered at the origin onto itself. Notice that the unit ball in $\mathbb{R}^d_\infty$ is the cube, whose 2^d vertices are the points with coordinates ± 1. The hyper-faces (i.e., the facets) of this cube are given by the equations $x_i = \pm 1$, and h maps them into each other. This implies that the faces of the cube (of any dimension) are transferred by h into the faces of the same dimension.

The radius-vector of the center of a hyper-face $x_i = \pm 1$ is the vector $\pm e_i$, where e_i is a vector from the standard basis of the arithmetic space $\mathbb{R}^d$. Notice that this center is equal to the sum of the radius-vectors of the vertices of the corresponding hyper-face, up to the factor 2^{d-1}. Thus the mapping h takes each vector e_i into a vector $\pm e_j$, i.e., h is the composition of a basic vectors permutation with their signs changes. □

Let $f\colon \mathcal{M} \to \mathcal{M}$ be an arbitrary isometry, $X \in \mathcal{M}^g_{[n]}$ and $Y = f(X)$. By Proposition 7.8, we have $Y \in \mathcal{M}^g_{[n]}$. Choose $\varepsilon > 0$ in such a way that the balls $B_\varepsilon(X)$ and $B_\varepsilon(Y)$ in $\mathcal{M}_{[n]}$ are canonical neighbourhoods. Then for any $\rho_X \in \Pi^{-1}(X)$ and $\rho_Y \in \Pi^{-1}(Y)$ we have $B_\varepsilon(\rho_X) \subset \mathcal{C}^g_n$, $B_\varepsilon(\rho_Y) \subset \mathcal{C}^g_n$, and the restrictions π_{ε,ρ_X} and π_{ε,ρ_Y} of the mapping Π^g onto these neighbourhoods are isometries $B_\varepsilon(\rho_X) \to B_\varepsilon(X)$ and $B_\varepsilon(\rho_Y) \to B_\varepsilon(Y)$, respectively. Further, the mapping

$$h_{\varepsilon,\rho_X,\rho_Y} = \pi^{-1}_{\varepsilon,\rho_Y} \circ f \circ \pi_{\varepsilon,\rho_X} : U_\varepsilon(\rho_X) \to U_\varepsilon(\rho_Y)$$

is also an isometry. By Proposition 8.1, the mapping h is affine. So, we get the following result.

Corollary 8.3. *Under the above notations, if $\varepsilon > 0$ is such that $B_\varepsilon(X)$ and $B_\varepsilon(Y)$ are canonical neighbourhoods, then the mapping*

$$h_{\varepsilon,\rho_X,\rho_Y} = \pi^{-1}_{\varepsilon,\rho_Y} \circ f \circ \pi_{\varepsilon,\rho_X} : U_\varepsilon(\rho_X) \to U_\varepsilon(\rho_Y)$$

has the form $h_{\varepsilon,\rho_X,\rho_Y}(\rho) = (S \cdot P)\,\rho + b$, where $b \in \mathbb{R}^N$ is a translation vector, P is a permutation matrix of the standard basic vectors, and S is a diagonal matrix with ± 1 on its diagonal.

The next lemma will be used in what follows.

Lemma 8.4. *If two affine mappings $x \mapsto A_i\,x + b_i$, $i = 1, 2$, defined on intersecting open subsets of the space $\mathbb{R}^d$ coincide with each other in their intersection, then $A_1 = A_2$ and $b_1 = b_2$.*

Construction 8.5. Let $X, X' \in \mathcal{M}^g_{[n]}$ and the corresponding $\rho_X, \rho_{X'} \in \mathcal{C}^g_n$ be such that the straight segment $L = [\rho_X, \rho_{X'}]$ belongs to $\mathcal{C}^g_n$. Let us consider the straight segment L as a continuous curve, and denote by γ the image of the curve L under the mapping Π^g. Then γ is a curve in $\mathcal{M}^g_{[n]}$ connecting X and X'. Let γ' be the image of the curve γ under the isometry f, then γ' connects $Y := f(X)$ and $Y' := f(X')$. Choose an arbitrary $\rho_Y \in \mathcal{C}^g_n$. By Corollary 6.12, the mapping $\Pi^g \colon \mathcal{C}^g_n \to \mathcal{M}^g_{[n]}$ is a covering in a wide sense, therefore, by Proposition 6.13, there exists a unique continuous curve L' in $\mathcal{C}^g_n$ starting at ρ_Y and such that its Π^g-image is the curve γ'. Since the second endpoint of the curve L' is projected to Y', this endpoint coincides with $\rho_{Y'}$ for some numeration of points of the space Y'.

Now, choose $\varepsilon > 0$ such that all the balls $U_\varepsilon(X)$, $U_\varepsilon(X')$, $U_\varepsilon(Y)$, and $U_\varepsilon(Y')$ are canonical neighbourhoods simultaneously. Then, under the notations of Corollary 6.9, the isometries π_{ε,ρ_X}, $\pi_{\varepsilon,\rho_{X'}}$, π_{ε,ρ_Y}, $\pi_{\varepsilon,\rho_{Y'}}$ generate two other isometries $h_{\varepsilon,\rho_X,\rho_Y} = \pi^{-1}_{\varepsilon,\rho_Y} \circ f \circ \pi_{\varepsilon,\rho_X}$ and $h_{\varepsilon,\rho_{X'},\rho_{Y'}} = \pi^{-1}_{\varepsilon,\rho_{Y'}} \circ f \circ \pi_{\varepsilon,\rho_{X'}}$. By Proposition 8.1, the mappings $h_{\varepsilon,\rho_X,\rho_Y}$ and $h_{\varepsilon,\rho_{X'},\rho_{Y'}}$ are the restrictions of some affine isometries $H \colon \mathbb{R}^N_\infty \to \mathbb{R}^N_\infty$ and $H' \colon \mathbb{R}^N_\infty \to \mathbb{R}^N_\infty$, respectively.

Lemma 8.6. *Under the above notations, the affine isometries H and H' coincide.*

Proof. Let the segment L together with the curves γ, γ', and L' be parameterised by a parameter $t \in [a, b]$, $L(a) = \rho_X$ and $L(b) = \rho_{X'}$.

For each $t \in [a, b]$ choose $\varepsilon_t > 0$ such that $B_{\varepsilon_t}\big(L(t)\big)$ and $B_{\varepsilon_t}\big(L'(t)\big)$ are canonical neighbourhoods. The family of balls $\Big\{U_{\varepsilon_t}\big(L(t)\big)\Big\}$ is an open covering of the segment $[\rho_X, \rho_{X'}]$. Let $\{U_i\}_{i=1}^m$ be a finite subcovering that exists because the segment is compact. Without loss of generality, suppose that the family $\{U_i\}_{i=1}^m$ is minimal in the sense that no one U_i is contained in another U_j; besides, assume that the centres ρ_i of the balls U_i are ordered along the segment $[\rho_X, \rho_{X'}]$. These two assumptions imply that the consecutive U_i intersect each other, in particular, the distance between each ρ_i and ρ_{i+1} is less than the sum of radii ε_i and ε_{i+1} of the balls U_i and U_{i+1}, respectively. Since the balls U_i are open, then each intersection $U_i \cap U_{i+1}$ is open as well. Further, since $|\rho_i\rho_{i+1}|_\infty < \varepsilon_i + \varepsilon_{i+1}$, then there exists $\rho'_i \in (\rho_i, \rho_{i+1})$ such that

$\rho_i' \in U_i \cap U_{i+1}$; besides, since the set $U_i \cap U_{i+1}$ is open, one can choose an open ball U_i' with center ρ_i' and radius ε_i' in such a way that $U_i' \subset U_i \cap U_{i+1}$. As a result, we have constructed a new covering $\{U_1, U_1', U_2, U_2', \ldots\}$ of the segment $[\rho_X, \rho_{X'}]$. By $\{V_i\}_{i=1}^{2m-1}$ we denote the consecutive elements of this new covering. Introduce new notations: let $\rho_i = L(t_i)$ be the center of the ball V_i, and ε_i be the radius of this ball. Thus, $V_i = U_{\varepsilon_i}(\rho_i)$.

Further, put $\nu_i = L'(t_i)$ and consider the family of open balls $\big\{W_i := U_{\varepsilon_i}(\nu_i)\big\}_{i=1}^{2m-1}$, then, by definition, each of these balls lies in $\mathcal{C}_n^g$, and the restrictions $\pi_{\varepsilon_i,\nu_i}$ of the mapping Π^g onto these balls are isometries. Besides, the restriction $\pi_{\varepsilon_i,\rho_i}$ of the mapping Π^g onto each V_i is an isometry as well. Put $h_i = h_{\varepsilon_i,\rho_i,\nu_i} = \pi_{\varepsilon_i,\nu_i}^{-1} \circ f \circ \pi_{\varepsilon_i,\rho_i}$, then $h_i \colon V_i \to W_i$ is an isometry for each i.

Since, by the construction $V_{2k} \subset V_{2k-1}$, the fact that h_{2k-1} is an isometry implies that $|\rho_{2k}\rho_{2k-1}|_\infty = |\nu_{2k}\nu_{2k-1}|_\infty$, therefore, $W_{2k} \subset W_{2k-1}$ and $\pi_{\varepsilon_{2k},\nu_{2k}} = \pi_{\varepsilon_{2k-1},\nu_{2k-1}}|_{W_{2k}}$, because the both mappings are the restrictions of Π^g. Thus, $h_{2k} = h_{2k-1}|_{V_{2k}}$. Similarly, one can show that $h_{2k} = h_{2k+1}|_{V_{2k}}$.

By Proposition 8.1, for every i there exists an affine mappings $H_i \colon \mathbb{R}^N_\infty \to \mathbb{R}^N_\infty$ such that $h_i = H_i|_{V_i}$. As we have shown above, the consecutive mappings h_i's coincide on open sets that are the intersections of the domains of the corresponding mappings, thus, by Lemma 8.4, all these H_i coincide, in particular, $H_1 = H_{2m-1}$. By the same Lemma, $H = H_1$ and $H' = H_{2m-1}$. □

Corollary 8.7. *If in Construction* 8.5 *one changes the straight segment L by a finite polygonal line, then Lemma* 8.6 *remains true.*

Proposition 8.8. *Under the notations of Corollary* 8.3*, the matrix S is unit, and $b = 0$.*

Proof. Under the notations of Construction 8.5, let us choose an arbitrary $0 < \delta < 1$ and take $X' = \delta\, X$. By Corollary 8.7, the mappings

$$h_{\varepsilon,\rho_X,\rho_Y} \colon U_\varepsilon(\rho_X) \to U_\varepsilon(\rho_Y) \quad \text{and} \quad h_{\varepsilon,\rho_{X'},\rho_{Y'}} \colon U_\varepsilon(\rho_{X'}) \to U_\varepsilon(\rho_{Y'})$$

are the restrictions of the same affine isometry H which does not depend on the choice of δ. By Proposition 8.2, we have $H(\rho) = (S \cdot P)\,\rho + b$, where $b \in \mathbb{R}^N_\infty$ is a translation vector, P is a permutation matrix, and S is a diagonal matrix with ± 1 on its diagonal. Notice that $\|\delta\,\rho_X\|_\infty \to 0$ as $\delta \to 0$, hence, by Corollary 4.2, we have $\big\|H(\delta\,\rho_X)\big\|_\infty \to 0$ as $\delta \to 0$. However, if $b \neq 0$, then for δ such that $\|\delta\,\rho_X\|_\infty < \frac{1}{2}\|b\|_\infty$ and $\big\|(S \cdot P)(\delta\,\rho_X)\big\|_\infty < \frac{1}{2}\|b\|_\infty$, we get

$$\big\|H(\delta\,\rho_X)\big\|_\infty = \big\|(S \cdot P)(\delta\,\rho_X) + b\big\|_\infty \geq -\big\|(S \cdot P)(\delta\,\rho_X)\big\|_\infty + \|b\|_\infty > \frac{1}{2}\|b\|_\infty,$$

a contradiction.

Thus, we have shown that $b = 0$. It remains to notice that all the components of the vectors ρ_X and $H(\rho_X)$ are positive, hence, S cannot contain negative elements. □

9. Some necessary facts on permutation groups

We have shown above that each mapping $h_{\varepsilon,\rho_X,\rho_Y}$ is the restriction of an affine mapping $x \mapsto Px$ of the space $\mathbb{R}^N$ onto itself, where P is a permutation matrix of the basis vectors, i.e., in fact, an element from the permutation group S_N. To complete the proof, we show that $P \in G$, where G, as above, is the subgroup of S_N, that is isomorphic to S_n and generated by permutation (i.e., renumeration) of points of metric spaces. The latter implies that $\Pi(x) = \Pi(Px)$ and, thus, locally f is an identical mapping, see details in Section 10. To do that, we need some facts on permutation groups.

Put $V = \{1, \dots, n\}$, $n \geq 3$. Let E be the set of the basis vectors $e_{ij} = e_{ji}$ of the space $\mathbb{R}^N$, $i \neq j$. Identify e_{ij} with the corresponding two-element subset $\{i, j\} \subset V$. Then $K_n = (V, E)$ is a complete graph with n vertices and N edges, and hence, the actions of G and S_N can be considered as actions on the set of edges E of the graph K_n; note that the action of the group G is generated by permutations on the vertices set V. Notice that for $n = 3$ we also have $N = 3$, and hence $G = S_N$ in this case, i.e., all six permutations of the edges are generated by the permutations of vertices.

Lemma 9.1. *Let $n \geq 5$. A permutation $\alpha \in S_N$ belongs to the subgroup G, iff α takes adjacent edges of K_n to adjacent ones.*

Proof. It is easy to see that each permutation $\alpha \in G$ takes adjacent edges to adjacent ones. Now, let us prove the converse statement.

Suppose that α takes all pairs of adjacent edges of the graph K_n to adjacent ones. Consider all the edges incident to some fixed vertex $v \in V$ (the number of such edges is $n - 1$, in particular, it is not less than 4). Then, by assumption, their images are pairwise adjacent. Let us show that the edges–images also have a common vertex.

Consider images of any three different edges from the chosen ones. Their images form a connected three-edge subgraph H of K_n. Each such subgraph is either a cycle, or a star, or a simple path. The latter case is impossible, because the first and the last edges of the path are not adjacent.

Consider now the image of a fourth edge. It has to be adjacent with all three edges of the subgraph H. Therefore, H cannot be a three-edge cycle and, thus, H is a star, and the image of the fourth edge has to be incident to the common vertex of the star.

Arguing in a similar way, we come to conclusion that the images of all the edges incident with the vertex v are incident to some common vertex. Thus, it is defined a mapping σ from the set V onto itself taking each vertex $v \in V$ to the unique common vertex of the α-images of all the edges incident to v. This mapping is injective: indeed, if v and w are mapped to the same vertex, then their image is common for $2n - 2$ edges that is impossible. Besides, σ induces a mapping on the edges of the graph K_n which coincides with α, therefore, $\alpha \in G$. □

Remark 9.2. For $n = 4$ the condition of Lemma 9.1 is not sufficient. For instance, the next permutation α takes triples of edges having common vertex to triples of edges, which form cycles:

$$\alpha = \begin{pmatrix} \{1,2\} & \{1,3\} & \{1,4\} & \{2,3\} & \{2,4\} & \{3,4\} \\ \{2,3\} & \{3,4\} & \{2,4\} & \{1,3\} & \{1,2\} & \{1,4\} \end{pmatrix}.$$

Evidently, α takes adjacent edges to adjacent ones, but it is not generated by a permutation of vertices.

ASSERTION 9.3. *For $n \ge 8$ the normalizer of the subgroup G in S_N coincides with the group G.*

Proof. By F we denote the set of all pairs of different edges of the graph K_n. Then $F = F_0 \sqcup F_1$, where F_0 consists of the pairs of non-adjacent edges, and F_1 consists of the pairs of adjacent edges (i.e., of edges having a common vertex).

LEMMA 9.4. *Under the above notations,*

$$\#F_0 = \frac{n(n-1)(n-2)(n-3)}{8}, \qquad \#F_1 = \frac{n(n-1)(n-2)}{2}.$$

In particular, for $n \ge 8$ the number of pairs of non-adjacent edges is greater than the number of pairs of adjacent ones, i.e., $\#F_0 > \#F_1$.

Proof. Indeed, consider the graph $E(K_n)$, whose vertices are the edges of the graph K_n, and two its vertices are adjacent, iff the corresponding edges of K_n are adjacent. Then $E(K_n) = (E, F_1)$. Each edge $\{i,j\}$ of K_n is adjacent in $E(K_n)$ with $n-2$ edges by the vertex i, and with $n-2$ edges by the vertex j, thus the degree of each vertex of the graph $E(K_n)$ equals $2n-4$. Since the number of vertices of the graph $E(K_n)$ equals $N = n(n-1)/2$, then by Handshaking Lemma we get:

$$\#F_1 = \frac{1}{2} \cdot \frac{n(n-1)}{2} \cdot (2n-4) = \frac{n(n-1)(n-2)}{2}.$$

To calculate the number of the pairs of non-adjacent edges, we have to subtract $\#F_1$ from the number of all pairs:

$$\begin{aligned} \#F_0 =& \#F - \#F_1 = \frac{\frac{n(n-1)}{2}\Big(\frac{n(n-1)}{2} - 1\Big)}{2} - \frac{n(n-1)(n-2)}{2} \\ =& \frac{n(n-1)}{2}\Big(\frac{n(n-1)}{4} - \frac{1}{2} - (n-2)\Big) = \frac{n(n-1)}{2} \cdot \frac{n^2-5n+6}{4} \\ =& \frac{n(n-1)(n-2)(n-3)}{8}. \end{aligned}$$

In particular,

$$\#F_0 = \frac{(n-3)}{4} \cdot \#F_1,$$

therefore, since $(n-3)/4 > 1$ for $n > 7$, we have $\#F_0 > \#F_1$ for $n \ge 8$. □

Each permutation $P \in S_N$ acts not only on the edges of the graph K_n, but also on the set F of pairs of edges. Lemma 9.4 implies the following result.

LEMMA 9.5. *For $n \ge 8$ each permutation $P \in S_N$ takes some pair of non-adjacent edges of the graph K_n to a pair of its non-adjacent edges.*

Proof. Indeed, one does not have enough elements in F_1 to map all elements of F_0 onto them. $\square$

Return to the proof of Assertion 9.3. Consider a permutation $P \in S_N$ and assume that $P \notin G$. Then, by Lemma 9.1, the permutation P takes some two adjacent edges $\{i,j\}$ and $\{i,k\}$ of the graph K_n to a pair of non-adjacent edges $\{a,b\}$ and $\{c,d\}$. Besides, by Lemma 9.5, there exist two non-adjacent edges $\{i_1,j_1\}$ and $\{i_2,j_2\}$ which are mapped by P to non-adjacent edges $\{a',b'\}$ and $\{c',d'\}$. Since a, b, c, and d, as well as a', b', c', and d' are pairwise distinct, then there exists a permutation $g \in S_n$ such that $g(a)=a'$, $g(b)=b'$, $g(c)=c'$, $g(d)=d'$. Then $\{i,j\} \xmapsto{P} \{a,b\} \xmapsto{g} \{a',b'\} \xmapsto{P^{-1}} \{i_1,j_1\}$ and $\{i,k\} \xmapsto{P} \{c,d\} \xmapsto{g} \{c',d'\} \xmapsto{P^{-1}} \{i_2,j_2\}$, i.e., the composition $P^{-1}g\,P$ takes some adjacent edges to non-adjacent ones, and, thus, is does not belong to G by Lemma 9.1. Therefore, P does not belong to the normalizer of G, that completes the proof of the assertion. $\square$

10. Completion of the Main Theorem proof

Let $f\colon \mathcal{M} \to \mathcal{M}$ be an isometry, $X \in \mathcal{M}^g_{[n]}$, and $f(X)=Y$. Choose $\varepsilon > 0$ such that $B_\varepsilon(X)$ and $B_\varepsilon(Y)$ are canonical neighbourhoods. Fix some $\rho_X \in \Pi^{-1}(X)$ and $\rho_Y \in \Pi^{-1}(Y)$, then the mapping $h_{\varepsilon,\rho_X,\rho_Y}\colon U_\varepsilon(\rho_X) \to U_\varepsilon(\rho_Y)$ from Corollary 8.3 is the restriction of a linear mapping $\mathbb{R}^N \to \mathbb{R}^N$ with permutation matrix $P \in S_N$ (this linear mapping we denote by the same letter P).

Lemma 10.1. *For $n \ge 4$ the permutation $P \in S_N$ belongs to the normalizer of the subgroup G, i.e., $P^{-1}gP \in G$ for every $g \in G$.*

Proof. It is easy to see that the subset of $\mathcal{M}^g_{[n]}$ consisting of all spaces such that all their non-zero distances are pairwise distinct, is everywhere dense in $\mathcal{M}^g_{[n]}$. Besides, if Z is such a space, then for any numeration of the points from Z, all the components of the vector ρ_Z are pairwise distinct, therefore, each $Q \in S_N$ is uniquely defined by the Q-image of such point ρ_Z.

Chose $X \in \mathcal{M}^g_{[n]}$ in such a way that all non-zero distances in X are pairwise distinct. By Proposition 6.11 and Corollary 8.7, for any $\rho \in \Pi^{-1}(X)$ there exists $\rho' \in \Pi^{-1}(Y)$ such that the mapping $h_{\varepsilon,\rho,\rho'}\colon U_\varepsilon(\rho) \to U_\varepsilon(\rho')$ is the restriction of a linear mapping with the same matrix P. Therefore, $P\big(\Pi^{-1}(X)\big) \subset \Pi^{-1}(Y)$. Since the matrix P is non-degenerate, then for any distinct $\rho_1,\rho_2 \in \Pi^{-1}(X)$ we have $P(\rho_1) \ne P(\rho_2)$. At last, since $\#\Pi^{-1}(X) = \#\Pi^{-1}(Y)$, then P maps $\Pi^{-1}(X)$ bijectively onto $\Pi^{-1}(Y)$.

Take any $\rho_X \in \Pi^{-1}(X)$, any $g \in G$, and put $\rho'_X := P^{-1}gP(\rho_X)$. Then $\rho'_X \in \Pi^{-1}(X)$, and hence there exists $g' \in G$ such that $\rho'_X = g'(\rho_X)$. However, as we mentioned above, the mapping $P^{-1}gP \in S_N$ is uniquely defined by the image of ρ_X. Thus, $P^{-1}gP = g' \in G$. $\square$

Now, Lemma 10.1 and Assertion 9.3 imply that for $n \geq 8$ the permutation P is contained in G, therefore the vectors ρ_X and $\rho_{f(X)}$ differ by a renumeration of vertices, i.e., $X = f(X)$. Thus, we have shown that the isometry f is trivial on an everywhere dense subset of the space $\mathcal{M}^g_{[n]}$ and, thus, on the entire $\mathcal{M}^g_{[n]}$. It remains to notice that the union $\bigcup_{n\geq 8} \mathcal{M}^g_{[n]}$ is everywhere dense in $\mathcal{M}$, and hence f is trivial on the whole $\mathcal{M}$. The Main Theorem is proved.

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