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ON GRADED 2-ABSORBING SUBMODULES OVER Gr-MULTIPLICATION MODULES

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Abstract. Let G be a multiplicative group with identity e, R be a G-graded commutative ring and M be a graded R-module. The aim of this article is some investigations of graded 2-absorbing submodules over Gr-multiplication modules. A graded submodule N of R-module M is called graded 2-absorbing if whenever $a, b \in h(R)$ and $m \in h(M)$ with $abm \in N$, then either $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$. We also introduce the concept of graded classical 2-absorbing submodule as a generalization of graded classical prime submodules and show a number of results in this class.

1. Introduction

Throughout this paper all rings are G-graded commutative rings and M is a graded R-module with non-zero identity. Let G be a multiplicative group with identity e. Then R is a *G-graded ring*, if there exist additive subgroups R_g of R such that $R = \bigoplus_{g\in G} R_g$ and $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$. It is denoted by $G(R)$. The elements of R_g are called *homogeneous* of degree g, where R_g are additive subgroups of R indexed by elements $g \in G$. If $a \in R$, then a can be written uniquely as $\sum_{g\in G} a_g$, where a_g is the component of a in R_g. Also, we write $h(R) = \bigcup_{g\in G} R_g$. Let I be an ideal of $G(R)$. Then I is a *graded ideal of* $G(R)$ if $I = \bigoplus_{g\in G}(I \cap R_g)$. Moreover, R_e is a subring of $G(R)$ and $1 \in R_e$.

Let R be a G-graded ring and M be an R-module. Then M is called a *graded R-module* (*G-graded R-module*) if there exists a family of subgroups $\{M_g\}_{g\in G}$ of M such that $M = \bigoplus_{g\in G} M_g$ (as abelian groups) and $R_gM_h \subseteq M_{gh}$ for all $g, h \in G$. Here, R_gM_h denotes the additive subgroup of M consisting of all finite sums of elements r_gm_h with $r_g \in R_g$ and $m_h \in M_h$. We write $h(M) = \bigcup_{g\in G} M_g$ and the elements of $h(M)$ are called *homogeneous*. If $M = \bigoplus_{g\in G} M_g$ is a graded R-module, then for

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all $g \in G$ the subgroup M_g of M is an R_e-module. Let $M = \bigoplus_{g\in G} M_g$ be a graded R-module and N be a submodule of M. Then N is called a *graded submodule of* M if $N = \bigoplus_{g\in G} N_g$ where $N_g = N \cap M_g$ for $g \in G$. In this case, N_g is called the *g-component* of N, (cf. [10]). Moreover, M/N becomes a graded R-module with g-component $(M/N)_g = (M_g + N)/N$ for $g \in G$. Let R be a G-graded ring and M, N be graded R-modules. Then $(N :_R M) = \{r \in h(R) \mid rM \subseteq N\}$ is a graded ideal of $G(R)$ and for every element $a \in h(R)$ and $m \in h(M)$, IN, aN and Rm are graded submodules of M and Ra is a graded ideal, (cf. [3,4]). *The annihilator* of an element m of M which is $ann(m) = (0 :_R m) = \{r \in h(R) \mid rm = 0\}$. A graded R-module M is said to be a *cancellation graded module* of $G(R)$ if for all graded ideals I and J of $G(R)$ such that $IM = JM$, it follows that $I = J$. A graded R-module M is *faithful*, if for every element $r \in h(R)$ such that $rM = 0$, it follows that $r = 0$. A graded R-module M is called *graded multiplication* (*Gr-multiplication*) *R-module*, if for every graded submodule N of M, there exists a graded ideal I of $G(R)$ such that $N = IM$. In this case, we can easily show that if M is a Gr-multiplication R-module, then $N = (N :_R M)M$ for every graded submodule N of M. Let M be a Gr-multiplication R-module, $N = IM$ and $K = JM$ are graded submodules of M where I and J are graded ideals of $G(R)$. The product of N and K is denoted by NK and is defined by $(IJ)M$. Then the product of N and K is independent of presentations of N and K, by [8, Theorem 4]. Moreover, we can define the product of two elements $m, m' \in h(M)$ — if $Rm = IM$ and $Rm' = JM$, then $mm' = (IM)(JM) = (IJ)M$, (cf. [8,11]).

A graded ideal I of $G(R)$ is said to be a *graded prime* (*G-prime*) (resp. *graded weakly prime*) *ideal*, if $I \neq R$ and whenever $a, b \in h(R)$ with $ab \in I$ (resp. $0 \neq ab \in I$), then either $a \in I$ or $b \in I$. The graded radical of I, which is denoted by $Gr(I)$, is the set of all $x \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x_g^{n_g} \in I$. Note that, if r is a homogeneous element of $G(R)$, then $r \in Gr(I)$ if and only if $r^n \in I$ for some $n \in \mathbb{N}$. It is easy to see that if I is a graded ideal of $G(R)$, then $Gr(I)$ is a graded ideal of $G(R)$, [12, Proposition 2.3]. If I is a graded ideal of $G(R)$, then $Gr(I)$ is the intersection of all G-prime ideals of $G(R)$ containing I, [12, Proposition 2.5]. Furthermore, a graded ideal I of $G(R)$ is said to be a *graded primary* (*G-primary*) (resp. *graded weakly primary*) *ideal*, if $I \neq R$ and whenever $a, b \in h(R)$ such that $ab \in I$ $(0 \neq ab \in I)$, then either $a \in I$ or $b \in Gr(I)$. If $Gr(I)$ is a G-prime ideal, then I is called a *graded P-primary* (*G-P-primary*) *ideal*. M is a graded maximal (G-maximal) ideal of G-graded ring R, if $M \neq R$ and there is no graded ideal I of $G(R)$ such that $M \subset I \subset R$.

Graded prime submodules on G-graded commutative rings have been introduced and studied in [3, 4, 8]. Let R be a G-graded ring and M be a graded R-module. A proper graded submodule N of M is called *graded prime submodule* of M, if whenever $a \in h(R)$ and $m \in h(M)$ with $am \in N$, then either $a \in (N :_R M)$ or $m \in N$. In this paper, we generalize the concept of 2-*absorbing ideals* in [5] to the concept of *graded* 2-*absorbing submodules*. Refer that the concept of graded 2-absorbing submodule is a characterization of 2-absorbing submodule which has been explained by A. Yousefian Darani and F. Soheilnia in [15, 16]. Later, K. Al-Zoubi and R. Abu-Dawwas in [1], extended the concept of *graded* 2-*absorbing submodules*. They defined that a graded proper submodule N of M is said to be a *graded* 2-*absorbing submodule*, if whenever $a, b \in h(R)$ and $m \in h(M)$ with $abm \in N$, then either $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$. The concept of graded 2-absorbing ($G(2)$-absorbing) ideal is defined in [13]. Let R be a G-graded ring and I be a graded ideal of $G(R)$. Then I is said to be a *graded* 2-*absorbing* ($G(2)$-*absorbing*) *ideal* if $I \neq G(R)$ and whenever $a, b, c \in h(R)$ with $abc \in I$, then either $ab \in I$ or $bc \in I$ or $ac \in I$.

The graded classical prime and primary submodules have been introduced and studied in [2] while the concept of classical prime and classical primary submodule were described by Behboodi in [7], M. Baziar and M. Behboodi in [6]. Let R be a G-graded ring and M be a graded R-module. A proper graded submodule N of M is said to be a *graded classical prime* (resp. *graded classical primary*) *submodule*, if whenever $a, b \in h(R)$ and $m \in h(M)$ with $abm \in N$, then either $am \in N$ or $bm \in N$ (resp., $b^n m \in N$ for some positive integer n). Recently, some researchers in [9] have explained and expanded the concept of classical 2-absorbing submodule over commutative rings. Let M be a R-module over commutative ring R and N be a proper submodule. A submodule N of M is called a classical 2-absorbing submodule, if whenever $a, b, c \in R$ and $m \in M$ such that $abcm \in N$, then either $abm \in N$ or $bcm \in N$ or $acm \in N$. Here we introduce the concept of graded classical 2-absorbing submodules of M over G-graded commutative rings. Let R be a G-graded commutative ring, M be a graded R-module and N be a graded proper submodule of M. We say that N is a *classical graded* 2-*absorbing submodule*, if whenever $a, b, c \in h(R)$ and $m \in h(M)$ with $abcm \in N$, then either $abm \in N$ or $bcm \in N$ or $acm \in N$. We show more results about graded 2-absorbing submodules which are generalizations of Gr-multiplication R-modules over G-graded commutative rings in Section 2. In Section 3, we show more results on graded classical 2-absorbing submodules that are generalizations of graded prime submodules and graded 2-absorbing submodules of graded R-modules over G-graded rings.

2. Properties of graded 2-absorbing submodules

Let R be a G-graded ring and M be a graded R-module. A graded proper submodule N of M is called graded 2-absorbing submodule, if whenever $a, b \in h(R)$ and $m \in h(M)$ such that $abm \in N$, then either $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$. We say that $N_r = (N :_M r) = \{m \in h(M) \mid mI \subseteq N\}$ is a graded 2-absorbing submodule of M, (cf. [14]). To start with, we show a result on graded 2-absorbing submodules.

THEOREM 2.1. *Let R be a G-graded commutative ring, M be a graded R-module and N be a graded proper submodule of M. Suppose that N is a graded 2-absorbing submodule. Then the following statements hold:*
(i) For all elements $a, b \in h(R)$ and $m \in h(M)$, $ab \notin (N :_R M)$ implies that $(N :_R abm) = (N :_R am) \cup (N :_R bm)$. Moreover, $(N :_R abm) = (N :_R am)$ or $(N :_R abm) = (N :_R bm)$;

(ii) For all elements $a, b \in h(R)$, $ab \notin (N :_R M)$ implies that $(N :_M ab) = (N :_M a)$ or $(N :_M ab) = (N :_M b)$;
(iii) For every element $a \in h(R)$ and graded submodule K of M, $aK \nsubseteq N$ implies that $(N :_R aK) = (N :_R K)$ or $(N :_R aK) = ((N :_M a) :_R M)$;
(iv) For every element $a \in h(R)$, every element $m \in h(M)$ and every graded ideal I of $G(R)$, $aIm \subseteq N$ implies that $Ia \subseteq (N :_R M)$ or $am \in N$ or $Im \subseteq N$;
(v) For every graded ideal I, J of $G(R)$ and every element $m \in h(M)$, $IJm \subseteq N$ implies that $IJ \subseteq (N :_R M)$ or $Im \subseteq N$ or $Jm \subseteq N$;
(vi) For every element $a, b \in h(R)$ and every graded submodule K of M with $abK \subseteq N$ implies that $ab \in (N :_R M)$ or $aK \subseteq N$ or $bK \subseteq N$;
(vii) For every element $a \in h(R)$, every graded ideal I of $G(R)$ and graded submodule K of M, $aIK \subseteq N$ implies that $aI \subseteq (N :_R M)$ or $aK \subseteq N$ or $IK \subseteq N$;
(viii) For every graded ideal I of $G(R)$ and graded submodule K of M, $IK \nsubseteq N$, implies that $(N :_R IK) = (N :_R K)$ or $(N :_R IK) = ((N :_M I) :_R M)$;
(ix) For every graded ideal I, J of $G(R)$ and graded submodule K of M, $IJK \subseteq N$ implies that $IJ \subseteq (N :_R M)$ or $IK \subseteq N$ or $JK \subseteq N$.

Proof. (i) Assume that $a, b \in h(R)$ and $m \in h(M)$ with $ab \notin (N :_R M)$. Let $x \in (N :_R abm)$ and so $xabm \in N$. Since N is a graded 2-absorbing submodule and $ab \notin (N :_R M)$, we conclude that $xam \in N$ $\big($so $x \in (N :_R am)\big)$ or $xbm \in N$ $\big($so $x \in (N :_R bm)\big)$. Then $(N :_R abm) = (N :_R am) \cup (N :_R bm)$. Clearly, $(N :_R abm) = (N :_R am)$ or $(N :_R abm) = (N :_R bm)$.
(ii) Assume that $a, b \in h(R)$ with $ab \notin (N :_R M)$. Let $m \in (N :_M ab)$ be such that $abm \in N$. Since N is a graded 2-absorbing submodule and $ab \notin (N :_R M)$, we conclude that $am \in N$ $\big($so $m \in (N :_M a)\big)$ or $bm \in N$ $\big($so $m \in (N :_M b)\big)$. Then $(N :_M ab) = (N :_M a)$ or $(N :_M ab) = (N :_M b)$.
(iii) Assume that $a \in h(R)$ and K is a graded submodule of M such that $aK \nsubseteq N$. Suppose that $m \in (N :_R aK)$. Then $amK \subseteq N$ and so $K \subseteq (N :_R am)$. If $am \in (N :_R M)$, then $m \in ((N :_R M) :_M a)$. We can suppose that $am \notin (N :_R M)$. Then it follows from (ii) that $(N :_R am) \subseteq (N :_R m)$ or $(N :_R am) \subseteq (N :_M a)$. If $(N :_R am) \subseteq (N :_R m)$, then $K \subseteq (N :_M a)$ and so $aK \subseteq N$, which is a contradiction. Thus $(N :_R am) \subseteq (N :_M a)$ and hence $K \subseteq (N :_R m)$. Then $Km \subseteq N$ and so $m \in (N :_R K)$. Hence $(N :_R aK) \subseteq (N :_R K) \cup ((N :_M a) :_R M)$. The converse inclusion is obvious. Therefore, $(N :_R aK) = (N :_R K)$ or $(N :_R aK) = ((N :_M a) :_R M)$.
(iv) Assume that for $a \in h(R)$, $m \in h(M)$ and a graded ideal I of $G(R)$, $aIm \subseteq N$, $Ia \nsubseteq (N :_R M)$ and $am \notin N$ hold. There exists $b \in I \cap h(R)$ such that $ab \notin (N :_R M)$. Since $bam \in N$, N is a graded 2-absorbing submodule of M, $ab \notin (N :_R M)$ and $am \notin N$, we get that $bm \in N$. Now suppose that there exists $i \in I$ such that $(b + i)am \in N$. Then $(b + i)a \in (N :_R M)$ or $(b + i)m \in N$. If $(b + i)a \in N$, then $im \in N$. If $(b + i)m \in (N :_R M)$, then $ia \notin (N :_R M)$. Since $iam \in N$ and N is a graded 2-absorbing submodule of M, we obtain that $im \in N$. Therefore $Im \subseteq N$, as required.
(v) Assume that $IJm \subseteq N$ for some graded ideals I, J of $G(R)$ and $m \in h(M)$ with $Im \nsubseteq N$ and $Jm \nsubseteq N$. Then there exist $a \in I \setminus h(R)$ and $b \in I \setminus h(R)$ such that $am \notin N$ and $bm \notin N$. Since $aJm \subseteq N$, $am \notin N$ and $Jm \nsubseteq N$, (iv) implies

$aJ \subseteq (N :_R M)$. Then $\big(I \setminus (N :_R m)\big)J \subseteq (N :_R M)$. Now, since $Ibm \subseteq N$, $bm \notin N$ and $Im \nsubseteq N$, from (iv) we conclude that $Ib \subseteq (N :_R M)$, thus $I\big(J \setminus (N :_R m)\big) \subseteq (N :_R M)$. Hence there exist $i \in I \cap h(R)$ and $j \in J \cap h(R)$ such that $aj \in (N :_R M)$ or $ib \in (N :_R M)$. Since $(a+i) \in I$ and $(b+j) \in J$, we have $(a+i)(b+j)m \in N$. Then $(a+i)(b+j) \in (N :_R M)$ or $(a+i)m \in N$ or $(b+j)m \in N$. Since N is a graded 2-absorbing submodule of M, if $(a+i)(b+j) = ab+aj+ib+ij \in (N :_R M)$, then $ij \in (N :_R M)$. If $(a+i)m \in N$, then $im \notin N$ and thus $i \in I \setminus (N :_R m)$ and so $ij \in (N :_R M)$. Similarly, if $(b+j)m \in N$, then $ij \in (N :_R M)$. Therefore $IJ \subseteq (N :_R M)$.

(vi) Assume that $abK \in N$ for some graded submodule K of M and $a, b \in h(R)$ with $ab \notin (N :_R M)$. Then $K \subseteq (N :_M ab)$. Since $ab \notin (N :_R M)$, it follows from (ii) that $K \subseteq (N :_M ab) = (N :_M a)$ or $K \subseteq (N :_M ab) = (N :_M b)$. Hence either $aK \subseteq N$ or $bK \subseteq N$.

(vii) Assume that $aIK \subseteq N$ for some graded ideal I of $G(R)$, graded submodule K of M and $a \in h(R)$ such that $aI \nsubseteq (N :_R M)$. Then there exists $i \in I$ such that $ai \notin (N :_R M)$. Since $aiK \subseteq N$ and $ai \notin (N :_R M)$, then either $aK \subseteq N$ or $iK \subseteq N$ $\big($so $IK \subseteq N\big)$.

(viii) Assume that I is a graded ideal and K is a graded submodule of M such that $IK \nsubseteq N$. Suppose that $a \in (N :_R IK)$. Then $aIK \subseteq N$ and so (vii) implies $aI \subseteq (N :_R M)$ or $aK \subseteq N$. Thus $aIM \subseteq N$ $\big($so $a \in ((N :_M I) :_R M)\big)$ or $a \in (N :_R K)$. Hence $(N :_R IK) \subseteq (N :_R K) \cup ((N :_M I) :_R M)$. The converse inclusion is obvious, hence $(N :_R IK) = (N :_R K)$ or $(N :_R IK) = ((N :_M I) :_R M)$.

(ix) Assume that $IJK \subseteq N$ for some graded ideals I, J of $G(R)$ and graded submodule K of M with $IJ \nsubseteq (N :_R M)$. Then there exist $a \in I \cap h(R)$ and $b \in J \cap h(R)$ such that neither $aK \in N$ nor $bK \in N$. Since $abK \in N$ and neither $aK \in N$ nor $bK \in N$, from (vi) we get that $ab \in (N :_R M)$. Since $IJ \nsubseteq (N :_R M)$, we obtain that $rs \notin (N :_R M)$ for some $r \in I \cap h(R)$ and $s \in J \cap h(R)$. Since $rsK \subseteq N$ and $rs \notin (N :_R M)$, it follows from (vi) that $rK \subseteq N$ or $sK \subseteq N$. Let us consider three cases:

Case 1. Let $rK \subseteq N$ but $sK \nsubseteq N$. Since $asK \subseteq N$ but neither $aK \subseteq N$ nor $sK \subseteq N$, we conclude from (vi) that $as \in (N :_R M)$. Since $aK \nsubseteq N$ but $rK \subseteq N$, we conclude that $(a+r)K \nsubseteq N$. Now since $(a+r)sK \subseteq N$ but neither $(a+r)K \subseteq N$ nor $sK \subseteq N$, (vi) implies $(a+r)s = as + rs \in (N :_R M)$. Then $rs \in (N :_R M)$, which is a contradiction.

Case 2. Let $rK \nsubseteq N$ but $sK \subseteq N$. Hence the proof is the same as in **Case 1.**

Case 3. Let $rK \subseteq N$ and $sK \subseteq N$. Firstly, we consider that $rK \subseteq N$. Since $rK \subseteq N$ and $aK \nsubseteq N$, we have $(a+r)K \nsubseteq N$. Now since $(a+r)bK \subseteq N$ but neither $(a+r)K \subseteq N$ nor $bK \subseteq N$, it follows from (vi) that $(a+r)b = ab + rb \in (N :_R M)$. Then $rb \in (N :_R M)$. Now we consider that $sK \subseteq N$. Since $sK \subseteq N$ and $bK \nsubseteq N$, we have $(b+s)K \nsubseteq N$. Now since $a(b+s)K \subseteq N$ but neither $aK \subseteq N$ nor $(b+s)K \subseteq N$, from (vi) we have $a(b+s) = ab + as \in (N :_R M)$. Then $as \in (N :_R M)$. Now since $(a+r)(b+s)K \subseteq N$ but neither $(a+r)K \subseteq N$ nor $(b+s)K \subseteq N$, we can conclude that $(a+r)(b+s) = ab + as + rb + rs \in (N :_R M)$ and then $rs \in (N :_R M)$, which is a contradiction. Hence $IK \subseteq N$ or $JK \subseteq N$, as needed. □

Theorem 2.2. *Let R be a G-graded commutative ring, M be a graded R-module* *and N be a graded submodule of M. If N is a graded 2-absorbing submodule of M, then $(N :_R M)$ is a graded 2-absorbing ideal. The converse is true if M is a Gr-multiplication R-module.*

Proof. Assume that N is a graded 2-absorbing submodule. Let $abc \in (N :_R M)$ for some $a, b, c \in h(R)$. Then $abcM \subseteq N$; and put $cM = K$ such that K is a graded submodule of M. Hence $abK \subseteq N$ and so, by Theorem 2.1 (vi), either $ab \in (N :_R M)$ or $aK \subseteq N$ $\big($so $acM \subseteq N\big)$ or $bK \subseteq N$ $\big($so $bcM \subseteq N\big)$. Then $ab \in (N :_R M)$ or $bc \in (N :_R M)$ or $ac \in (N :_R M)$, as needed.

Conversely, suppose that $abm \in N$ for some $a, b \in h(R)$ and $m \in h(M)$ with $ab \notin (N :_R M)$. Since M is a Gr-multiplication R-module, there exists a graded ideal I of $G(R)$ such that $m = IM$. Then $abIM \subseteq N$ and so $abI \subseteq (N :_R M)$. Since $(N :_R M)$ is a graded 2-absorbing ideal and $ab \notin (N :_R M)$, we claim that $aI \subseteq (N :_R M)$ or $bI \subseteq (N :_R M)$. Otherwise, neither $aI \subseteq (N :_R M)$ nor $bI \subseteq (N :_R M)$. Then there exists $i_1, i_2 \in I \cap h(R)$ such that $ai_1 \notin (N :_R M)$ and $bi_2 \notin (N :_R M)$. Since $abi_1 \in (N :_R M)$ but $ab \notin (N :_R M)$, $ai_1 \notin (N :_R M)$ and $(N :_R M)$ is a graded 2-absorbing ideal, we have $bi_1 \in (N :_R M)$. Similarly for the next term, since $abi_2 \in (N :_R M)$ but $ab \notin (N :_R M)$, $bi_2 \notin (N :_R M)$ and $(N :_R M)$ is a graded 2-absorbing ideal, we have $ai_2 \in (N :_R M)$. Now since $ab(i_1 + i_2) \in (N :_R M)$, $ab \notin (N :_R M)$ and $(N :_R M)$ is a graded 2-absorbing ideal, we conclude that $a(i_1 + i_2) \in (N :_R M)$ or $b(i_1 + i_2) \in (N :_R M)$. If $a(i_1 + i_2) = ai_1 + ai_2 \in (N :_R M)$, then $ai_1 \in (N :_R M)$, which is a contradiction. If $b(i_1 + i_2) = bi_1 + bi_2 \in (N :_R M)$, then $bi_2 \in (N :_R M)$, which is a contradiction. Thus $aI \subseteq (N :_R M)$ $\big($so $aIM \subseteq N\big)$ or $bI \subseteq (N :_R M)$ $\big($so $bIM \subseteq N\big)$. Hence $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$. Therefore $(N :_R M)$ is a graded 2-absorbing ideal of R. □

PROPOSITION 2.3. *Let R be a G-graded commutative ring, M be a graded R-module and N be a graded proper submodule of M. If $(N :_R M)$ is a graded 2-absorbing ideal, then $Gr(N :_R M)$ is a graded 2-absorbing ideal.*

Proof. Assume that $abc \in Gr(N :_R M)$ for some $a, b, c \in h(R)$ with $ab \notin Gr(N :_R M)$ and $bc \notin Gr(N :_R M)$. Then there exists a positive integer n such that $(abc)^n = a^n b^n c^n \in Gr(N :_R M)$ with $a^n b^n \notin (N :_R M)$ and $b^n c^n \notin (N :_R M)$. Since $(N :_R M)$ is a graded 2-absorbing ideal, we have $a^n c^n \in (N :_R M)$. Then $ac \in Gr(N :_R M)$. □

The *graded radical* of a graded submodule N of a graded R-module M, which is denoted by $Gr_M(N)$, is defined to be the intersection of all graded prime submodules of M containing N. If N is not contained in any graded prime submodule of M, then $Gr_M(N) = M$.

THEOREM 2.4. *Let R be a G-graded commutative ring, M be a Gr-multiplication R-module and N be a graded submodule of M. If N is a graded 2-absorbing submodule of M, then $Gr_M(N)$ is a graded 2-absorbing submodule of M.*

Proof. Since N is a graded 2-absorbing submodule, by Theorem 2.2, $(N :_R M)$ is a graded 2-absorbing ideal of $G(R)$. Then $Gr(N :_R M)$ is a graded 2-absorbing ideal of $G(R)$, by Proposition 2.3. By [14, Lemma 2], $Gr(N :_R M) = \big(Gr_M(N) :_R M\big)$.

Hence $\big(Gr_M(N) :_R M\big)$ is a graded 2-absorbing ideal of $G(R)$. By Theorem 2.2, since M is a Gr-multiplication R-module, we get that $Gr_M(N)$ is a graded 2-absorbing submodule of M. □

PROPOSITION 2.5. *Let R be a G-graded commutative ring and M be a Gr-multiplication R-module. Suppose that N and K are distinct graded prime submodules of M. Then the following statements hold:*
(i) $(N :_R M) \cap (K :_R M)$ is a graded 2-absorbing ideal of $G(R)$;
(ii) $N \cap K$ is a graded 2-absorbing submodule of M.

Proof. (i) Since N and K are graded prime submodules of M, by [4, Proposition 2.5], $(N :_R M)$ and $(K :_R M)$ are graded prime ideals of $G(R)$. Then $(N :_R M) \cap (K :_R M)$ is a graded 2-absorbing ideal of $G(R)$, by [13, Theorem 2.5].
(ii) Since N and K are graded prime submodules of M, by [4, Proposition 2.5], $(N :_R M)$ and $(K :_R M)$ are graded prime ideals of $G(R)$. Then (i) implies that $(N :_R M) \cap (K :_R M) = (N \cap K :_R M)$ is a graded 2-absorbing ideal, and hence $N \cap K$ is a graded 2-absorbing submodule by Theorem 2.2. □

THEOREM 2.6. *Let R be a G-graded commutative ring and M be a graded R-module. Suppose that $N_1, \ldots, N_k$ are graded submodules of M. If $N_1, \ldots, N_k$ are graded 2-absorbing submodules of M, then $\bigcap_{i=1}^k N_i$ is a graded 2-absorbing submodule of M.*

Proof. Let $a, b \in h(R)$ and $m \in h(M)$ such that $abm \in \bigcap_{i=1}^k N_i$ with $ab \notin (\bigcap_{i=1}^k N_i :_R M) = \bigcap_{i=1}^k (N_i :_R M)$. Then there exists i such that $ab \notin (N_i :_R M)$. Since $abm \in N_i$, $ab \notin (N_i :_R M)$ and N_i is a graded 2-absorbing submodule of M, we conclude that $am \in N_i$ or $bm \in N_i$. Then $am \in \bigcap_{i=1}^k N_i$ or $bm \in \bigcap_{i=1}^k N_i$. Therefore $\bigcap_{i=1}^k N_i$ is graded 2-absorbing. □

THEOREM 2.7. *Let R be a G-graded commutative ring, M be a graded R-module and N be a graded submodule of M. Suppose that N is a graded 2-absorbing submodule and $Gr(N :_R M) = P$ such that P is a graded prime ideal of $G(R)$. Then the following statements hold:*
(i) If $m \in h(M) \setminus N$, then $Gr(N :_R m)$ is a graded prime ideal of $G(R)$ containing P;
(ii) If $m, m' \in h(M) \setminus N$, then either $Gr(N :_R m) \subseteq Gr(N :_R m')$ or $Gr(N :_R m') \subseteq Gr(N :_R m)$.

Proof. (i) Assume that $a, b \in h(R)$ such that $ab \in Gr(N :_R m)$. Then there exists some positive integer n such that $a^n b^n m \in N$. Since N is a graded 2-absorbing submodule of M, we conclude that $a^n b^n \in (N :_R M)$ or $a^n m \in N$ or $b^n m \in N$. If both these conditions hold, then we are done. Suppose that $a^n b^n \in (N :_R M)$. Since $a = \sum_{g \in G} a_g$ and $b = \sum_{g \in G} b_g$, and it is $a^n b^n = \sum_{g \in G} a_g^n b_g^n \in (N :_R M)$ for some smallest positive integer $n > 0$. Thus $ab \in Gr(N :_R M) = P$. Since P is a graded prime ideal, we have $a \in P$ or $b \in P$. Since $P = Gr(N :_R M) \subseteq Gr(N :_R m)$, we conclude that $a \in Gr(N :_R m)$ or $b \in Gr(N :_R m)$, as needed.
(ii) Assume that $Gr(N :_R m) \nsubseteq Gr(N :_R m')$. Let $a \in Gr(N :_R m)$ and $b \in Gr(N :_R m') \setminus Gr(N :_R m)$. Then there exists a smallest positive integer $n > 0$ such that $a_g^n m \in N$, $b_g^n m' \in N$ and $b_g^n m \notin N$ for $g \in G$. If $a_g^n (m + m') \in N$, then $a_g^n m' \in N$

for $g \in G$ and so $a^n m' = \sum_{g\in G} a_g^n m' \in N$. Thus $a \in Gr(N :_R m')$. Suppose that $a_g^n(m+m') \notin N$. Since N is a graded 2-absorbing submodule, $a_g^n b_g^n(m+m') \in N$ but $a_g^n(m+m') \notin N$ and $b_g^n(m+m') \notin N$, then $a^n b^n = \sum_{g\in G} a_g^n b_g^n \in (N :_R M)$ and so $ab \in Gr(N :_R M)$. Then $ab \in P$ where P is a graded prime ideal. Thus $a \in P$ or $b \in P$. If $b \in P$, then $b^n m \in N$, which is a contradiction. If $a \in P$, then $a \in Gr(N :_R m')$, as needed. □

Theorem 2.8. *Let R be a G-graded commutative ring, M be a graded R-module and N be a graded submodule of M. Suppose that N is a graded 2-absorbing submodule and $Gr(N :_R M) = P \cap Q$ such that P and Q are graded prime ideals of $G(R)$. Then the following statements hold:*
(i) If $m \in h(M) \setminus N$ and $P \subseteq Gr(N :_R m)$, then $Gr(N :_R m)$ is a graded prime ideal of $G(R)$;
(ii) If $m, m' \in h(M) \setminus N$ and $P \subseteq Gr(N :_R m) \cap Gr(N :_R m')$, then either $Gr(N :_R m) \subseteq Gr(N :_R m')$ or $Gr(N :_R m') \subseteq Gr(N :_R m)$.

Proof. The proof is similar to the proof of Theorem 2.7. □

Let R be a G-graded ring, M be a graded R-module and N be a graded submodule of M. Recall that $(N :_M I) = \{m \in h(M) \mid mI \subseteq N\}$ and $(N :_M I^\infty)$ are graded submodules of graded R-module M.

Lemma 2.9. *Let R be a G-graded commutative ring, M be a graded R-module and N be a graded submodule of M. Then $((N :_M I) :_R M) = ((N :_R M) :_M I)$.*

Theorem 2.10. *Let R be a G-graded commutative ring, M be a graded R-module and N be a graded submodule of M. If N is a graded 2-absorbing submodule of M, then $(N :_M I)$ is a graded 2-absorbing submodule of M.*

Proof. Assume that $abm \in (N :_M I)$ for some $a, b \in h(R)$ and $m \in h(M)$. Then $Iabm \in N$ and hence, by Theorem 2.1 (iv), $Im \subseteq N$ or $abm \in N$ or $Iab \subseteq (N :_R N)$. If $Im \subseteq N$, then we are done by definition. If $Iab \in (N :_R M)$, then by Lemma 2.9, $ab \in ((N :_R M) :_M I) = ((N :_M I) :_R M)$. If $abm \in N$, then $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$. Since N is a graded 2-absorbing submodule, hence $Iab \subseteq (N :_R M)$ $\big($so $ab \in ((N :_M I) :_R M)\big)$ or $Iam \in N$ $\big($so $am \in (N :_M I)\big)$ or $Ibm \in N$ $\big($so $bm \in (N :_M I)\big)$. Therefore $(N :_M I)$ is a graded 2-absorbing submodule. □

Corollary 2.11. *Let N be a graded submodule of graded R-module M. If N is a graded 2-absorbing submodule, then N_r is a graded 2-absorbing submodule for every $r \in h(R) \setminus (N :_M r)$. Moreover, $(N :_M I^n) = (N :_M I^{n+1})$ for all $n \geq 2$.*

Proof. Assume that $abm \in N_r = (N :_M r)$ for some $a, b \in h(R)$, $m \in h(M)$ and $r \in h(R) \setminus (N :_M r)$. Then $ab(rm) \in N$. Since N is a graded 2-absorbing submodule, we conclude that $ab \in (N :_R M)$ or $a(rm) \in N$ or $b(rm) \in N$. If both these conditions hold, then we are done. If $ab \in (N :_R M) \subseteq (N_r :_R M)$, then $ab \in (N_r :_R M)$. Therefore N_r is a graded 2-absorbing submodule of M for every $r \in h(R) \setminus (N :_M r)$. Now suppose that $m \in (N :_M I^3)$ for $m \in h(M)$. Then $I^3 m \subseteq N$. By Theorem 2.1 (v), $I^2 m \subseteq N$ or $Im \subseteq N$ or $I^3 \subseteq (N :_R M)$. If the first two conditions hold, then we are done. If $I^3 \subseteq (N :_R M)$, then $I^2 \subseteq (N :_R M)$. Since $(N :_R M)$ is a graded 2-absorbing ideal, by Theorem 2.2, then $(N :_M I^3) \subseteq (N :_M I^2)$. The converse inclusion is obvious, so $(N :_M I^n) = (N :_M I^{n+1})$ for all $n \geq 2$. □

THEOREM 2.12. *Let R be a G-graded commutative ring, M be a graded R-module and N be a graded submodule of M. Suppose that $Gr(N :_R M)$ is a graded prime ideal with $Gr(N :_R M) \neq (N :_R M)$. If $(N_r :_R M)$ is a graded prime ideal such that $r \in Gr(N :_R M) \setminus (N :_R M)$, then $(N :_R M)$ is a graded 2-absorbing ideal of $G(R)$.*

Proof. Assume that $rst \in (N :_R M)$ for some $r, s, t \in h(R)$ with $st \notin (N :_R M)$. Since $Gr(N :_R M)$ is a graded prime ideal, we may assume that $r \in Gr(N :_R M)$. If $r \in (N :_R M)$, then we are done. So we can suppose that $r \notin (N :_R M)$, and then $r \in Gr(N :_R M) \setminus (N :_R M)$. Since $(N_r :_R M)$ is a graded prime ideal and $rst \in (N_r :_R M)$, we conclude that $s \in (N_r :_R M)$ or $t \in (N_r :_R M)$. Hence $rs \in (N :_R M)$ or $rt \in (N :_R M)$. □

THEOREM 2.13. *Let R be a G-graded commutative ring, M be a graded R-module and N be a graded submodule of M. Suppose that $Gr(N :_R m)$ is a graded prime ideal with $Gr(N :_R m) \neq (N :_R m)$ for all $m \in h(M) \setminus N$. If N_r is a graded prime submodule for $r \in Gr(N :_R m) \setminus (N :_R m)$, then N is a graded 2-absorbing submodule of M.*

Proof. Assume that $r, s \in h(R)$ and $m \in h(M)$ such that $rsm \in N$. Then $rs \in (N :_R m)$. Since $Gr(N :_R m)$ is a graded prime ideal, we may suppose that $r \in Gr(N :_R m)$. If $r \in (N :_R m)$, then we are done. Let $r \notin (N :_R m)$. Then $r \in Gr(N :_R m) \setminus (N :_R m)$. Since N_r is a graded prime submodule and $sm \in (N :_M r) = N_r$, we have $s \in (N_r :_R M)$ or $m \in N_r$ and so $rs \in (N :_R M)$ or $rm \in N$. Hence N is a graded 2-absorbing submodule of M. □

THEOREM 2.14. *Let R be a G-graded commutative ring, M be a graded R-module and N be a submodule of M. Then the following statements are equivalent:*
(i) N is a graded 2-absorbing submodule;
(ii) For every graded ideal I, J of $G(R)$ and every graded submodule K of M with $(K + IL) \cap S \neq \emptyset$, $(K + JL) \cap S \neq \emptyset$ and $(K + IJM) \cap S \neq \emptyset$ such that $S = M \setminus N$, implies that $(K + IJL) \cap S \neq \emptyset$.
Proof. (i) ⇒ (ii) Let N be a graded 2-absorbing submodule. Suppose that I, J are graded ideals of $G(R)$ and K, L are graded submodules of M such that $(K+IL)\cap S \neq \emptyset$, $(K+JL)\cap S \neq \emptyset$ and $(K+IJM)\cap S \neq \emptyset$. If $(K+IJL)\cap S = \emptyset$, then $(K+IJL) \subseteq N$ and so $IJL \subseteq N$. Since N is a graded 2-absorbing submodule, we conclude that $IL \subseteq N$ or $JL \subseteq N$ or $IJM \subseteq N$, by Theorem 2.1 (ix). Then $(K + IL) \cap S = \emptyset$ or $(K + JL) \cap S = \emptyset$ or $(K + IJM) \cap S = \emptyset$, which are contradictions. Hence $(K + IJL) \cap S \neq \emptyset$, as needed.
(ii) ⇒ (i) Suppose that $IJL \subseteq N$ for some graded ideals I, J of $G(R)$ and some graded submodule L of M. We may assume that neither $IL \subseteq N$ nor $JL \subseteq N$ nor $IJM \subseteq N$. Then $(K + IL) \cap S \neq \emptyset$, $(K + JL) \cap S \neq \emptyset$ and $(K + IJM) \cap S \neq \emptyset$ and hence $(K + IJL) \cap S \neq \emptyset$, which is a contradiction. Therefore N is a graded 2-absorbing submodule of M. □

3. Graded classical 2-absorbing submodules

In this section we will define the concept of graded classical 2-absorbing submodules as a generalization of graded classical prime submodules. Also, we show a number of results of graded classical 2-absorbing submodules.

DEFINITION 3.1. Let R be a G-graded ring and M be a graded R-module. A proper graded submodule N of M is called a *graded classical* 2*-absorbing submodule*, if whenever $a, b, c \in h(R)$ and $m \in h(M)$ with $abcm \in N$, then either $abm \in N$ or $bcm \in N$ or $acm \in N$. It is denoted as ${}^{gr}\mathcal{C}l(2)$-absorbing submodule.

LEMMA 3.2. *Let R be a G-graded ring and M be a graded R-module. Then the following statements hold:*
(i) Every graded classical prime submodule is a graded 2-absorbing submodule and a ${}^{gr}\mathcal{C}l(2)$-absorbing submodule;
(ii) If N is a graded classical prime submodule, then $(N :_R M)$ is a graded prime ideal, [2, Lemma 3.1].

In what follows, we show the basic theorem on ${}^{gr}\mathcal{C}l(2)$-absorbing submodules.

THEOREM 3.3. *Let R be a G-graded ring, M be a graded R-module and N be a proper graded submodule of M. Suppose that N is a ${}^{gr}\mathcal{C}l(2)$-absorbing submodule. Then the following statements hold:*
(i) For all elements $a, b, c \in h(R)$ either $(N :_M abc) = (N :_M ab)$ or $(N :_M abc) = (N :_M bc)$ or $(N :_M abc) = (N :_M ac)$;
(ii) For all elements $a, b, c \in h(R)$ and every graded submodule K of M, $abcK \subseteq N$ implies that $abK \subseteq N$ or $bcK \subseteq N$ or $acK \subseteq N$;
(iii) For all elements $a, b \in h(R)$ and every graded submodule K of M, $abK \nsubseteq N$ implies that $(N :_R abK) = (N :_R aK)$ or $(N :_R abK) = (N :_R bK)$;
(iv) For all elements $a, b \in h(R)$, every graded ideal I of $G(R)$ and every graded submodule K of M, $abIK \subseteq N$ implies that $abK \subseteq N$ or $aIK \subseteq N$ or $bIK \subseteq N$;
(v) For every element $a \in h(R)$, every graded ideal I of $G(R)$ and every graded submodule K of M, $aIK \nsubseteq N$ implies that $(N :_R aIK) = (N :_R aK)$ or $(N :_R aIK) = (N :_R IK)$;
(vi) For every element $a \in h(R)$, every graded ideal I, J of $G(R)$ and every graded submodule K of M, $aIJK \subseteq N$ implies that $aIK \subseteq N$ or $aJK \subseteq N$ or $IJK \subseteq N$;
(vii) For every graded ideal I, J of $G(R)$ and every graded submodule K of M, $IJK \nsubseteq N$ implies that $(N :_R IJK) = (N :_R IK)$ or $(N :_R IJK) = (N :_R JK)$;
(viii) For every graded ideal I, J, P of $G(R)$ and every graded submodule K of M, $IJPK \subseteq N$ implies that $IJK \subseteq N$ or $JPK \subseteq N$ or $IPK \subseteq N$.

Proof. (i) Assume that $abcm \in N$ for some $m \in h(M)$. Since N is a ${}^{gr}\mathcal{C}l(2)$-absorbing submodule, we have $abm \in N$ or $bcm \in N$ or $acm \in N$. Then either $m \in (N :_M ab)$ or $m \in (N :_M bc)$ or $m \in (N :_M ac)$ and hence $(N :_M abc) = (N :_M ab)$ or $(N :_M abc) = (N :_M bc)$ or $(N :_M abc) = (N :_M ac)$.
(ii) Assume that $abcK \subseteq N$ for some $a, b, c \in h(R)$ and some graded submodule K of M. Then $K \subseteq (N :_M abc)$ and so either $K \subseteq (N :_M ab)$ or $K \subseteq (N :_M bc)$ or

$K \subseteq (N :_M ac)$, it follows from (i). Hence either $abK \subseteq N$ or $bcK \subseteq N$ or $acK \subseteq N$, as needed.
(iii) Assume that K is a graded submodule of M and $a, b \in h(R)$ are such that $abK \nsubseteq N$. Let $x \in (N :_R abK)$ for some $x \in h(R)$ and thus $xabK \subseteq N$. Since $abK \nsubseteq N$, we have $xaK \subseteq N$ or $xbK \subseteq N$, what follows from (ii). Then either $x \in (N :_R aK)$ or $x \in (N :_R bK)$ and hence $(N :_R abK) = (N :_R aK)$ or $(N :_R abK) = (N :_R bK)$.
(iv) Assume that $abIK \subseteq N$ for some $a, b \in h(R)$, some graded ideal I and some graded submodule K of M; thus $I \subseteq (N :_R abK)$. If $abK \subseteq N$, then we are done, what follows from (iii). If $abK \nsubseteq N$, then $I \subseteq (N :_R aK)$ or $I \subseteq (N :_R bK)$, what follows from (iii). Hence either $aIK \subseteq N$ or $bIK \subseteq N$, as required.

The proof of the remaining parts is similar to the previous ones, so we omit it. □

THEOREM 3.4. *Let R be a G-graded ring, M be a graded R-module and N be a proper graded submodule of M. Then N is a ${}^{gr}\mathcal{C}l(2)$-absorbing submodule of M if and only if for every graded submodule K of M such that $K \nsubseteq N$, $(N :_R K)$ is a graded 2-absorbing ideal.*

Proof. Assume that N is a ${}^{gr}\mathcal{C}l(2)$-absorbing submodule of M. Let $a, b, c \in h(R)$ be such that $abc \in (N :_R K)$. Then $abcK \subseteq N$ and so $abK \subseteq N$ or $bcK \subseteq N$ or $acK \subseteq N$, by Theorem 3.3 (ii). Hence either $ab \in (N :_R K)$ or $bc \in (N :_R K)$ or $ac \in (N :_R K)$. Therefore $(N :_R K)$ is a graded 2-absorbing ideal of $G(R)$. Conversely, suppose that $abcL \subseteq N$ for some $a, b, c \in h(R)$ and some graded submodule L of M. If $L \subseteq N$, then we are done. If $L \nsubseteq N$, then $abc \in (N :_R L)$. Since $(N :_R L)$ is a graded 2-absorbing ideal, we conclude that $ab \in (N :_R L)$ or $bc \in (N :_R L)$ or $ac \in (N :_R L)$. Then either $abm \in N$ or $bcm \in N$ or $acm \in N$ for some $m \in h(M) \cap L$, as needed. □

COROLLARY 3.5. *Let R be a G-graded ring, M be a graded R-module and N be a proper graded submodule of M. If N is a ${}^{gr}\mathcal{C}l(2)$-absorbing submodule of M, then $(N :_R M)$ is a graded 2-absorbing ideal of $G(R)$. Moreover, for every $m \in h(M) \setminus N$, $(N :_R m)$ is a graded 2-absorbing ideal of $G(R)$.*

THEOREM 3.6. *Let R be a G-graded ring, M be a graded R-module and N be a graded proper submodule of M. If N is a graded 2-absorbing submodule of M, then N is a ${}^{gr}\mathcal{C}l(2)$-absorbing submodule. The converse is true if M is a Gr-multiplication R-module.*

Proof. Assume that N is a graded 2-absorbing submodule. Let $a, b, c \in h(R)$ and $m \in h(M)$ be such that $abcm \in N$. Since N is a graded 2-absorbing submodule, we conclude that $ab \in (N :_R M)$ or $bcm \in N$ or $acm \in N$. If both cases are true, then we are done. If the first case holds, then $abm \in N$ and so N is ${}^{gr}\mathcal{C}l(2)$-absorbing. Conversely, suppose that N is a ${}^{gr}\mathcal{C}l(2)$-absorbing submodule of M. Then $(N :_R M)$ is a graded 2-absorbing ideal of $G(R)$, by Corollary 3.5. Since M is a Gr-multiplication R-module, by Theorem 2.2, N is a graded 2-absorbing submodule. □

THEOREM 3.7. *Let R be a G-graded ring, M be a graded R-module and N be a proper graded submodule of M. Suppose that N is a ${}^{gr}\mathcal{C}l(2)$-absorbing submodule. Then the following statements hold:*

(i) For all elements $a, b, c \in h(R)$ either $(N :_M abc) = (N :_M ab)$ or $(N :_M abc) = (N :_M bc)$ or $(N :_M abc) = (N :_M ac)$;
(ii) For all elements $a, b \in h(R)$ and $m \in h(M)$, $abm \nsubseteq N$ implies that $(N :_R abm) = (N :_R am)$ or $(N :_R abm) = (N :_R bm)$;
(iii) For all elements $a, b \in h(R)$, $m \in h(M)$ and every graded ideal I of $G(R)$, $abIm \subseteq N$ implies that $abm \subseteq N$ or $aIm \subseteq N$ or $bIm \subseteq N$;
(iv) For all elements $a \in h(R)$, $m \in h(M)$ and every graded ideal I of $G(R)$, $aIm \nsubseteq N$ implies that $(N :_R aIm) = (N :_R am)$ or $(N :_R aIm) = (N :_R Im)$;
(v) For all elements $a \in h(R)$, $m \in h(M)$ and all graded ideals I, J of $G(R)$, $aIJm \subseteq N$ implies that $aIm \subseteq N$ or $aJm \subseteq N$ or $IJm \subseteq N$;
(vi) For all graded ideals I, J of $G(R)$ and every element of submodule $m \in h(M)$, $IJm \nsubseteq N$ implies that $(N :_R IJm) = (N :_R Im)$ or $(N :_R IJm) = (N :_R Jm)$;
(vii) For all graded ideals I, J, P of $G(R)$ and every element $m \in h(M)$, $IJPm \subseteq N$ implies that $IJm \subseteq N$ or $JPm \subseteq N$ or $IPm \subseteq N$.

Proof. The complete proof is similar to the proof of Theorem 3.3. □

THEOREM 3.8. *Let R be a G-graded ring, M be a Gr-multiplication R-module and N be a graded submodule of M. Then the following statements are equivalent:*
(i) N is a ${}^{gr}\mathcal{C}l(2)$-absorbing submodule;
(ii) For all graded submodules N_1, N_2 and N_3 of M and every element $m \in h(M)$, $N_1N_2N_3m \in N$ implies that $N_1N_2m \subseteq N$ or $N_2N_3m \subseteq N$ or $N_1N_3m \subseteq N$.

Proof. (i) ⇒ (ii) Suppose that $N_1N_2N_3m \in N$ for some graded submodules N_1, N_2 and N_3 of M and some $m \in h(M)$. Since M is a Gr-multiplication R-module, there exist graded ideals I_1, I_2 and I_3 of $G(R)$ such that $N_1 = I_1M$, $N_2 = I_2M$ and $N_3 = I_3M$. Then $I_1I_2I_3m \in N$ and so, by Theorem 3.7 (vii), either $I_1I_2m \subseteq N$ or $I_2I_3m \subseteq N$ or $I_1I_3m \subseteq N$. Hence either $N_1N_2m \subseteq N$ or $N_2N_3m \subseteq N$ or $N_1N_3m \subseteq N$.
(ii) ⇒ (i) Assume that $I_1I_2I_3m \in N$ for some graded ideals I_1, I_2 and I_3 of $G(R)$ and some $m \in h(M)$. Since M is a Gr-multiplication R-module, we obtain that $I_1M = N_1$, $N_2 = I_2M$ and $N_3 = I_3M$. Then the conclusion follows from (ii). □

THEOREM 3.9. *Let R be a G-graded ring, M be a graded R-module and N be a submodule of M. Then the following statements are equivalent:*
(i) N is a ${}^{gr}\mathcal{C}l(2)$-absorbing submodule;
(ii) For all graded ideals I, J, P of $G(R)$ and all graded submodules K, L of M, with $(K + IJL) \cap S \neq \emptyset$, $(K + JPL) \cap S \neq \emptyset$ and $(K + IPL) \cap S \neq \emptyset$, $S = M \setminus N$, implies that $(K + IJPL) \cap S \neq \emptyset$.

Proof. (i) ⇒ (ii) Assume that N is a ${}^{gr}\mathcal{C}l(2)$-absorbing submodule. Suppose that I, J, P are graded ideals and K, L are graded submodules such that $(K + IJL) \cap S \neq \emptyset$, $(K + JPL) \cap S \neq \emptyset$ and $(K + IPL) \cap S \neq \emptyset$. Let $(K + IJPM) \cap S = \emptyset$. Then $K + IJPL \subseteq N$ and so $K \subseteq N$ and $IJPL \subseteq N$. Since N is a ${}^{gr}\mathcal{C}l(2)$-absorbing submodule, Theorem 3.3 (viii) implies that $IJL \subseteq N$ or $JPL \subseteq N$ or $IPL \subseteq N$. Suppose that $IJL \subseteq N$, then $(K + IJL) \cap S = \emptyset$, which is a contradiction. In the next two cases we can obtain a contradiction in a similar way.
(ii) ⇒ (i) Suppose that $IJPL \subseteq N$ for some graded ideals I, J, P of $G(R)$ and some graded submodules K, L of M. If neither $IJL \subseteq N$ nor $JPL \subseteq N$ nor $IPL \subseteq N$,

then $IJL \cap S \neq \emptyset$, $JPL \cap S \neq \emptyset$ and $IPL \cap S \neq \emptyset$ and thus $IJPL \cap S \neq \emptyset$, which is a contradiction. Hence either $IJL \subseteq N$ or $JPL \subseteq N$ or $IPL \subseteq N$. Therefore N is a ${}^{gr}\mathcal{C}l(2)$-absorbing submodule. □

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