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# ON SOME MULTIVARIATE SUMMATORY FUNCTIONS OF THE EULER PHI-FUNCTION

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**Abstract**. In this note we obtain an asymptotic formula with a power saving error term for the summation function of Euler phi-function evaluated at iterated and generalized least common multiples of four integer variables.

## 1. Introduction

In this paper we denote by $[n_1, \dots, n_k]$ the least common multiple and by $(n_1, \dots, n_k)$ the greatest common divisor of positive integers $n_1, \dots, n_k$. In [2], Diaconis and Erdős obtained asymptotic formulas for summatory functions

$$\sum_{m,n \le x} (m,n) \qquad \text{and} \qquad \sum_{m,n \le x} [m,n]$$

of the greatest common divisor and the least common multiple. More recently, Hilberdink in [6] investigated in more details the arithmetic function $\circ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, defined by $m \circ n := \frac{[m,n]}{(m,n)}$, which has several very interesting properties. For example, the set of squarefree positive integers is an abelian group with respect to the operation $\circ$. Moreover, for any squarefree integer $k \in \mathbb{N}$, the set $D(k)$ of all divisors of $k$ is a finite abelian group under the restriction of $\circ$ on $D(k)$. Hilberdink investigated in depth discrete Fourier analysis and multiplicative functions on these finite groups $D(k)$. One particularly interesting feature is that the restriction of Möbius function $\mu$ on $D(k)$ is one of the characters of this group.

Quotients $\frac{[m,n]}{(m,n)}$ of the least common multiple and the greatest common divisor of integers $m$ and $n$ appear in many papers in linear algebra (dealing with "arithmetical matrices") and in number theory, see for example [3–5, 7]. Recently, T. Hilberdink

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and L. Tóth in [8] considered the problem of establishing an asymptotic formula for the summation function of $\frac{[m,n]}{(m,n)}$ and obtained the formula

$$\sum_{m,n\le x}\frac{[m,n]}{(m,n)}=\frac{\pi^2}{60}\,x^4+O(x^3\log x).$$

Moreover, the authors in [8] derived more general asymptotic formulas, where the analogous summation is taken over $k\ge 3$ arguments. For an arithmetic function $f$ from a suitable class of multiplicative functions, the authors of [8] obtained the asymptotic formulas for

$$\sum_{n_1,\dots,n_k\le x} f\left([n_1,\dots,n_k]\right)\qquad\text{and}\qquad \sum_{n_1,\dots,n_k\le x} f\left(\frac{[n_1,\dots,n_k]}{(n_1,\dots,n_k)}\right),$$

with the power saving of $O(x^{1/2-\epsilon})$ in the error terms in both cases.

The author of the present note in [1] considered further summatory function for the following "generalized" least common multiple $\left[\frac{[n_1,\dots,n_k]^a}{(n_1,\dots,n_k)^c},\frac{[n_{k+1},\dots,n_{k+\ell}]^b}{(n_{k+1},\dots,n_{k+\ell})^d}\right]$, for integers $a\ge c\ge 1$ and $b\ge d\ge 0$, which is a multiplicative function of $k+\ell$ variables. Our goal in this note is to give similar generalization for the summation of Euler phi-function $\varphi$, where for simplicity of notation, we restrict ourselves to the case $k=\ell=2$.

Theorem 1.1. *For integers $a,b,c,d\ge 0$, $a,b\ge 1$, $a\ge c$, $b\ge d$ and for any $0<\epsilon<\frac{1}{2}$ we have*

$$\sum_{n_1,n_2,n_3,n_4\le x}\varphi\left(\left[\frac{[n_1,n_2]^a}{(n_1,n_2)^c},\frac{[n_3,n_4]^b}{(n_3,n_4)^d}\right]\right)=\frac{C_{a,c;b,d}}{(a+1)^2(b+1)^2}\,x^{2a+2b+4}+O_\epsilon\left(x^{2a+2b+\frac{7}{2}+\epsilon}\right),$$

*where the implied constant depends only on $\epsilon$ and the constant $C_{a,c;b,d}$ is given by the Euler product*

$$\prod_p\left(1-\frac{1}{p}\right)^4\sum_{\nu_1,\nu_2,\nu_3,\nu_4=0}^{\infty}\frac{\varphi\left(p^{\max\{(a\max-c\min)\{\nu_1,\nu_2\},(b\max-d\min)\{\nu_3,\nu_4\}\}}\right)}{p^{(a+1)(\nu_1+\nu_2)+(b+1)(\nu_3+\nu_4)}}.$$

Here and through the paper, $(a\max-c\min)\{\nu_1,\nu_2\}$ denotes $a\cdot\max\{\nu_1,\nu_2\}-c\cdot\min\{\nu_1,\nu_2\}$. We recall that $\varphi$ is a multiplicative function which is on prime powers given by $\varphi(p^a)=p^a-p^{a-1}$. Because of multiplicativity of $\varphi$, the function $(n_1,n_2,n_3,n_4)\mapsto\varphi\left(\left[\frac{[n_1,n_2]^a}{(n_1,n_2)^c},\frac{[n_3,n_4]^b}{(n_3,n_4)^d}\right]\right)$ will be a multiplicative function of 4 variables, enabling us to adapt the method from [8]. We recall that a function $f:\mathbb{N}^4\to\mathbb{C}$ is multiplicative if it satisfies

$$f(m_1n_1,m_2n_2,m_3n_3,m_4n_4)=f(m_1,m_2,m_3,m_4)f(n_1,n_2,n_3,n_4)$$

whenever $(m_1m_2m_3m_4,n_1n_2n_3n_4)=1$.

## 2. Proof of Theorem 1.1

To prove this theorem we need the following lemma:

Lemma 2.1. *For integers $a, b, c, d \geq 0$, $a, b \geq 1$, $a \geq c$, $b \geq d$ and complex numbers $z_j, 1 \leq j \leq 4$ such that*

$$\Re z_1, \Re z_2 > a + \frac{1}{2} \qquad \text{and} \qquad \Re z_3, \Re z_4 > b + \frac{1}{2} \tag{1}$$

*we have*

$$\begin{aligned} L(z_1, z_2, z_3, z_4) :=& \sum_{n_1,n_2,n_3,n_4=1}^{\infty} \frac{\varphi\left(\left[\frac{[n_1,n_2]^a}{(n_1,n_2)^c}, \frac{[n_3,n_4]^b}{(n_3,n_4)^d}\right]\right)}{n_1^{z_1} n_2^{z_2} n_3^{z_3} n_4^{z_4}} \\ =& \zeta(z_1 - a)\zeta(z_2 - a)\zeta(z_3 - b)\zeta(z_4 - b)H(z_1, z_2, z_3, z_4), \end{aligned} \tag{2}$$

*where $H(z_1, z_2, z_3, z_4)$ is a certain multiple Dirichlet series defined in the proof and absolutely convergent in the region* (1).

*Proof.* Because of the multiplicativity of the function

$$(n_1, n_2, n_3, n_4) \longmapsto \varphi\left(\left[\frac{[n_1, n_2]^a}{(n_1, n_2)^c}, \frac{[n_3, n_4]^b}{(n_3, n_4)^d}\right]\right),$$

by [9, Proposition 11] the multiple Dirichlet series $L(z_1,\ z_2, z_3, z_4)$ has the following Euler product expansion:

$$L(z_1, z_2, z_3, z_4) = \prod_p \sum_{\nu_1,\nu_2,\nu_3,\nu_4=0}^{\infty} \frac{\varphi\left(p^{\max\{(a\max -c\min)\{\nu_1,\nu_2\},(b\max -d\min)\{\nu_3,\nu_4\}\}}\right)}{p^{\nu_1 z_1+\nu_2 z_2+\nu_3 z_3+\nu_4 z_4}}.$$

In each Euler's factor corresponding to a prime $p$, we single out the contribution of the terms for which $\nu_1 + \nu_2 + \nu_3 + \nu_4 \leq 1$:

$$\begin{aligned} L(z_1, z_2, z_3, z_4) = \prod_p \Bigg( & 1 + \frac{p^a - p^{a-1}}{p^{z_1}} + \frac{p^a - p^{a-1}}{p^{z_2}} + \frac{p^b - p^{b-1}}{p^{z_3}} + \frac{p^b - p^{b-1}}{p^{z_4}} \\ & + \sum_{\substack{\nu_1,\nu_2,\nu_3,\nu_4\geq 0 \\ \nu_1+\nu_2+\nu_3+\nu_4\geq 2}} \frac{\varphi\left(p^{\max\{(a\max -c\min)\{\nu_1,\nu_2\},(b\max -d\min)\{\nu_3,\nu_4\}\}}\right)}{p^{\nu_1 z_1+\nu_2 z_2+\nu_3 z_3+\nu_4 z_4}} \Bigg). \end{aligned} \tag{3}$$

Next, for fixed $\delta_1 > a$ and $\delta_2 > b$, in the region $\Re z_1, \Re z_2 \geq \delta_1 > a$ and $\Re z_3, \Re z_4 \geq \delta_2 > b$, we have that

$$\begin{aligned} & \left| \frac{\varphi\left(p^{\max\{(a\max -c\min)\{\nu_1,\nu_2\},(b\max -d\min)\{\nu_3,\nu_4\}\}}\right)}{p^{\nu_1 z_1+\nu_2 z_2+\nu_3 z_3+\nu_4 z_4}} \right| \\ & \leq \frac{p^{a(\nu_1+\nu_2)+b(\nu_3+\nu_4)}}{p^{\delta_1(\nu_1+\nu_2)+\delta_2(\nu_3+\nu_4)}} = \frac{1}{p^{(\delta_1-a)(\nu_1+\nu_2)+(\delta_2-b)(\nu_3+\nu_4)}}. \end{aligned}$$

Since the number of solutions of $\nu_1 + \nu_2 = m$ in nonnegative integers $\nu_1, \nu_2$ is $m + 1$, the sum over $\nu_1 + \nu_2 + \nu_3 + \nu_4 \geq 2$ in equation (3) is bounded by

$$\sum_{m+n\geq 2} \frac{(m + 1)(n + 1)}{p^{(\delta_1-a)m+(\delta_2-b)n}} = O\left(\frac{1}{p^{2(\delta_1-a)}} + \frac{1}{p^{2(\delta_2-b)}}\right).$$

Now, in the region $\Re z_1, \Re z_2 > \max\{\delta_1, a + 1\}$ and $\Re z_3, \Re z_4 > \max\{\delta_2, b + 1\}$ we can define the function

$$H(z_1,z_2,z_3,z_4):=\frac{L(z_1,z_2,z_3,z_4)}{\zeta(z_1-a)\zeta(z_2-a)\zeta(z_3-b)\zeta(z_4-b)},$$

which in this region has the following Euler product decomposition:

$$\begin{aligned}H(z_1,z_2,z_3,z_4)&=\prod_p\left(1-\frac{1}{p^{z_1-a}}\right)\left(1-\frac{1}{p^{z_2-a}}\right)\left(1-\frac{1}{p^{z_3-b}}\right)\left(1-\frac{1}{p^{z_4-b}}\right)\\&\times\left(1+\frac{1}{p^{z_1-a}}-\frac{1}{p^{z_1-a+1}}+\frac{1}{p^{z_2-a}}-\frac{1}{p^{z_2-a+1}}+\frac{1}{p^{z_3-b}}-\frac{1}{p^{z_3-b+1}}\right.\\&\left.+\frac{1}{p^{z_4-b}}-\frac{1}{p^{z_4-b+1}}+O\left(\frac{1}{p^{2(\delta_1-a)}}+\frac{1}{p^{2(\delta_2-b)}}\right)\right)\\&=\prod_p\left(1+O\left(\frac{1}{p^{\delta_1-a+1}}+\frac{1}{p^{2(\delta_1-a)}}+\frac{1}{p^{\delta_2-b+1}}+\frac{1}{p^{2(\delta_2-b)}}\right)\right),\end{aligned}\tag{4}$$

since the terms $\pm\frac{1}{p^{z_j-a}}$ and $\pm\frac{1}{p^{z_j-b}}$ cancel out. On the other hand, the Euler's product in (4) converges absolutely for any $\delta_1>a+\frac{1}{2}$ and $\delta_2>b+\frac{1}{2}$. Therefore, the identity (2) holds in the wider region (1). □

Now we write the multiple Dirichlet series expansion of the function $H(z_1,z_2,z_3,z_4)$ from Lemma 2.1:

$$H(z_1,z_2,z_3,z_4)=\sum_{n_1,n_2,n_3,n_4=1}^{\infty}\frac{h(n_1,n_2,n_3,n_4)}{n_1^{z_1}n_2^{z_2}n_3^{z_3}n_4^{z_4}}.$$

The function $h(n_1,n_2,n_3,n_4)$ defined in this way is also a multiplicative function of 4 variables. From the identity (2) we infer the following convolution identity between the corresponding multivariate arithmetic functions:

$$\varphi\left(\left[\frac{[n_1,n_2]^a}{(n_1,n_2)^c},\frac{[n_3,n_4]^b}{(n_3,n_4)^d}\right]\right)=\sum_{j_1d_1=n_1,\ldots,j_4d_4=n_4}j_1^aj_2^aj_3^bj_4^b\quad h(d_1,d_2,d_3,d_4),\tag{5}$$

where the sum runs over all 4-tuples $(j_1,j_2,j_3,j_4)$ in which $j_i$ is a positive divisor of $n_i$, for all $1\le i\le 4$.

*Proof.* (*of Theorem 1.1)* We start by employing the identity (5) in our summation function:

$$\begin{aligned}\sum_{n_1,n_2,n_3,n_4\le x}\varphi\left(\left[\frac{[n_1,n_2]^a}{(n_1,n_2)^c},\frac{[n_3,n_4]^b}{(n_3,n_4)^d}\right]\right)&=\sum_{j_1d_1\le x,\ldots,j_4d_4\le x}j_1^aj_2^aj_3^bj_4^b\quad h(d_1,d_2,d_3,d_4)\\&=\sum_{d_1,d_2,d_3,d_4\le x}h(d_1,d_2,d_3,d_4)\sum_{j_1\le\frac{x}{d_1}}j_1^a\sum_{j_2\le\frac{x}{d_2}}j_2^a\sum_{j_3\le\frac{x}{d_3}}j_3^b\sum_{j_4\le\frac{x}{d_4}}j_4^b\\&=\sum_{d_1,d_2,d_3,d_4\le x}h(d_1,d_2,d_3,d_4)\left(\frac{x^{a+1}}{(a+1)d_1^{a+1}}+O\left(\frac{x^a}{d_1^a}\right)\right)\\&\times\left(\frac{x^{a+1}}{(a+1)d_2^{a+1}}+O\left(\frac{x^a}{d_2^a}\right)\right)\left(\frac{x^{b+1}}{(b+1)d_3^{b+1}}+O\left(\frac{x^b}{d_3^b}\right)\right)\left(\frac{x^{b+1}}{(b+1)d_4^{b+1}}+\left(\frac{x^b}{d_4^b}\right)\right)\end{aligned}$$

$$=\frac{x^{2a+2b+4}}{(a+1)^2(b+1)^2}\sum_{d_1,d_2,d_3,d_4\le x}\frac{h(d_1,d_2,d_3,d_4)}{d_1^{a+1}d_2^{a+1}d_3^{b+1}d_4^{b+1}}+R(x). \tag{6}$$

Here, $R(x)$ is the remainder term, which is bounded by

$$R(x)\ll \sum_{\substack{u_1,u_2\in\{a,a+1\}\\ v_1,v_2\in\{b,b+1\}\\ (u_1,u_2,v_1,v_2)\neq\\ (a+1,a+1,b+1,b+1)}} x^{u_1+u_2+v_1+v_2}\sum_{d_1,d_2,d_3,d_4\le x}\frac{|h(d_1,d_2,d_3,d_4)|}{d_1^{u_1}d_2^{u_2}d_3^{v_1}d_4^{v_2}}, \tag{7}$$

where in the first summation at least one $u_i=a$, $i\in\{1,2\}$, or at least one $v_j=b$, $j\in\{1,2\}$. For one such 4-tuple, for example for $(u_1,u_2,v_1,v_2)=(a,a+1,b+1,b+1)$, the corresponding contribution on the right hand side of (7) is bounded by

$$\ll x^{2a+2b+3}\sum_{d_1,d_2,d_3,d_4\le x}\frac{|h(d_1,d_2,d_3,d_4)|}{d_1^{a}d_2^{a+1}d_3^{b+1}d_4^{b+1}}=x^{2a+2b+3}\sum_{d_1,d_2,d_3,d_4\le x}\frac{|h(d_1,d_2,d_3,d_4)|d_1^{\frac12+\epsilon}}{d_1^{a+\frac12+\epsilon}d_2^{a+1}d_3^{b+1}d_4^{b+1}}$$

$$\le x^{2a+2b+\frac72+\epsilon}\sum_{d_1,d_2,d_3,d_4\le x}\frac{|h(d_1,d_2,d_3,d_4)|}{d_1^{a+\frac12+\epsilon}d_2^{a+1}d_3^{b+1}d_4^{b+1}}, \tag{8}$$

for any $\epsilon>0$. Here the 4-tuple of exponents $(a+\frac12+\epsilon,a+1,b+1,b+1)$ belongs to the region of absolute convergence (1). Therefore, by Lemma 2.1 the multiple Dirichlet series (8) converges to a constant and hence we obtain the bound $O(x^{2a+2b+\frac72+\epsilon})$. We can bound all the other terms in (7) similarly and we get

$$R(x)\ll x^{2a+2b+\frac72+\epsilon}. \tag{9}$$

Finally, we return to the main term in (6). We have:

$$\sum_{d_1,d_2,d_3,d_4\le x}\frac{h(d_1,d_2,d_3,d_4)}{d_1^{a+1}d_2^{a+1}d_3^{b+1}d_4^{b+1}}=$$

$$\sum_{d_1,d_2,d_3,d_4=1}^{\infty}\frac{h(d_1,d_2,d_3,d_4)}{d_1^{a+1}d_2^{a+1}d_3^{b+1}d_4^{b+1}}-\sum_{\substack{I\subseteq\{1,2,3,4\}\\ I\neq\emptyset}}\sum_{\substack{d_i>x,i\in I\\ d_j\le x,j\notin I}}\frac{h(d_1,d_2,d_3,d_4)}{d_1^{a+1}d_2^{a+1}d_3^{b+1}d_4^{b+1}}. \tag{10}$$

The complete multiple Dirichlet series in (10) converges by Lemma 2.1 and its sum is equal $H(a{+}1,a{+}1,b{+}1,b{+}1)$. All 15 terms for subsets $I\neq\emptyset$ can be bounded similarly. For illustration, we bound the contribution in (10) corresponding to $I=\{1,3\}$:

$$\sum_{\substack{d_1,d_3>x\\ d_2,d_4\le x}}\frac{|h(d_1,d_2,d_3,d_4)|}{d_1^{a+1}d_2^{a+1}d_3^{b+1}d_4^{b+1}}=\sum_{\substack{d_1,d_3>x\\ d_2,d_4\le x}}\frac{|h(d_1,d_2,d_3,d_4)|d_1^{-\frac12+\epsilon}d_3^{-\frac12+\epsilon}}{d_1^{a+\frac12+\epsilon}d_2^{a+1}d_3^{b+\frac12+\epsilon}d_4^{b+1}}$$

$$\le x^{-1+2\epsilon}\sum_{d_1,d_2,d_3,d_4=1}^{\infty}\frac{|h(d_1,d_2,d_3,d_4)|}{d_1^{a+\frac12+\epsilon}d_2^{a+1}d_3^{b+\frac12+\epsilon}d_4^{b+1}}.$$

Here again the multiple Dirichlet series converges to a constant by Lemma 2.1, and we get the bound $O(x^{-1+2\epsilon})$. In general we get that the contribution of the terms corresponding to a subset $I\subseteq\{1,2,3,4\}$, $I\neq\emptyset$ is bounded by $O\left(x^{(-\frac12+\epsilon)|I|}\right)$, where

$|I|$ denotes the cardinality of the subset $I$. Therefore the total error obtained by completing the main term in (6) is $O(x^{2a+2b+\frac{7}{2}+\epsilon})$, i.e. it is the same as in (9). This finishes the proof of the required asymptotic formula with the constant $C_{a,c;b,d} = H(a+1, a+1, b+1, b+1)$. □

Remark 2.2. Theorem 1.1 can be generalized by similar methods to other situations, for example for summation functions of arithmetic functions of the form

$$(n_1, \ldots, n_{k+\ell+m}) \mapsto f\left(\left[\frac{[n_1, \ldots, n_k]^A}{(n_1, \ldots, n_k)^a}, \frac{[n_{k+1}, \ldots, n_{k+\ell}]^B}{(n_{k+1}, \ldots, n_{k+\ell})^b}, \frac{[n_{k+\ell+1}, \ldots, n_{k+\ell+m}]^C}{(n_{k+\ell+1}, \ldots, n_{k+\ell+m})^c}\right]\right)$$

for non-negative integers $A \geq a, B \geq b, C \geq c$ and for any complex valued multiplicative arithmetic functions $f$ which for some real $r > 0$ satisfy $|f(p) - p^r| = O(p^{r-\frac{1}{2}})$ for all primes $p$ and $|f(p^\nu)| = O(p^{\nu r})$ for all $p$ and all $\nu \geq 2$. Examples of such functions are $n \mapsto n^r$, the sum-of-divisors function $\sigma_r(n) = \sum_{d|n} d^r$ or the generalized Euler function $\varphi_r(n) = \sum_{d|n} \mu(\frac{n}{d}) d^r$.

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