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GENERALIZED HÖLDER'S INEQUALITY IN MORREY SPACES

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Abstract. The aim of this paper is to present sufficient and necessary conditions for generalized Hölder's inequality in Morrey spaces and generalized Morrey spaces. We also obtain similar results in weak Morrey spaces and generalized weak Morrey spaces. The sufficient and necessary conditions for the generalized Hölder's inequality in these spaces are obtained through estimates for characteristic functions of balls in $\mathbb{R}^d$.

1. Introduction and preliminaries

Several authors have made important observations about Hölder's inequality in the last three decades (see [1, 2, 7, 12]). Recently, Masta *et al.* [6] obtained sufficient and necessary conditions for the generalized Hölder's inequality in Lebesgue spaces. In this paper, we are interested in studying the generalized Hölder's inequality in Morrey spaces and in generalized Morrey spaces. In particular, we shall prove sufficient and necessary conditions for generalized Hölder's inequality in those spaces. In addition, we also prove similar result in weak Morrey spaces and in generalized weak Morrey spaces.

Let us first recall the definition of Morrey spaces. For $1 \le p \le q < \infty$, the *Morrey space* $\mathcal{M}_q^p(\mathbb{R}^d)$ is the set of all p-locally integrable functions f on $\mathbb{R}^d$ such that

$$\|f\|_{\mathcal{M}_q^p} := \sup_{a\in\mathbb{R}^d, r>0} |B(a,r)|^{\frac{1}{q}-\frac{1}{p}} \left(\int_{B(a,r)} |f(y)|^p \, dy \right)^{\frac{1}{p}} < \infty.$$

Here, $B(a,r)$ denotes the open ball centered at $a \in \mathbb{R}^d$ with radius $r > 0$, and $|B(a,r)|$ denotes its Lebesgue measure. One might observe that $\|\cdot\|_{\mathcal{M}_q^p}$ defines a norm on $\mathcal{M}_q^p(\mathbb{R}^d)$, and makes the space complete [9]. Also note that $\mathcal{M}_q^p(\mathbb{R}^d) = L^p(\mathbb{R}^d)$ if $q = p$. Thus, $\mathcal{M}_q^p(\mathbb{R}^d)$ can be viewed as a generalization of the Lebesgue space $L^p(\mathbb{R}^d)$.

The following theorem presents sufficient and necessary conditions for Hölder's inequality in Morrey spaces.

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THEOREM 1.1. *Let $1 \le p \le q < \infty$, $1 \le p_1 \le q_1 < \infty$, and $1 \le p_2 \le q_2 < \infty$. Then the following statements are equivalent:*

(1) $\frac{1}{p_1} + \frac{1}{p_2} \le \frac{1}{p}$ *and* $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$.

(2) $\|fg\|_{\mathcal{M}^p_q} \le \|f\|_{\mathcal{M}^{p_1}_{q_1}} \|g\|_{\mathcal{M}^{p_2}_{q_2}}$ *for every* $f \in \mathcal{M}^{p_1}_{q_1}(\mathbb{R}^d)$ *and* $g \in \mathcal{M}^{p_2}_{q_2}(\mathbb{R}^d)$.

Let us now move to the weak Morrey spaces. For $1 \le p \le q < \infty$, the *weak Morrey space* $w\mathcal{M}^p_q(\mathbb{R}^d)$ is the set of all measurable functions f on $\mathbb{R}^d$ for which $\|f\|_{w\mathcal{M}^p_q} < \infty$, where $\|f\|_{w\mathcal{M}^p_q} := \sup_{a\in\mathbb{R}^d,\, r,\gamma>0} |B(a,r)|^{\frac{1}{q}-\frac{1}{p}}\, \gamma\, |\{x \in B(a,r) \,:\, |f(x)| > \gamma\}|^{\frac{1}{p}}$.

Note that $\|\cdot\|_{w\mathcal{M}^p_q}$ defines a quasi-norm on $w\mathcal{M}^p_q(\mathbb{R}^d)$. If $q = p$, then $w\mathcal{M}^p_q(\mathbb{R}^d) = wL^p(\mathbb{R}^d)$. Here, $w\mathcal{M}^p_q(\mathbb{R}^d)$ can be viewed as a generalization of the weak Lebesgue space $wL^p(\mathbb{R}^d)$. The relation between $w\mathcal{M}^p_q(\mathbb{R}^d)$ and $\mathcal{M}^p_q(\mathbb{R}^d)$ is shown in the following lemma.

LEMMA 1.2. *([5]) Let $1 \le p \le q < \infty$. Then $\mathcal{M}^p_q(\mathbb{R}^d) \subseteq w\mathcal{M}^p_q(\mathbb{R}^d)$ with $\|f\|_{w\mathcal{M}^p_q} \le \|f\|_{\mathcal{M}^p_q}$ for every $f \in \mathcal{M}^p_q(\mathbb{R}^d)$.*

This lemma will be useful for us to study sufficient and necessary conditions for generalized Hölder's inequality in weak Morrey spaces.

Next we present the definition of generalized Morrey spaces and generalized weak Morrey spaces. For $1 \le p \le q < \infty$, let $\mathcal{G}_p$ be the set of all functions $\phi : (0,\infty) \to (0,\infty)$ such that ϕ is *almost decreasing* (i.e. there exists $C > 0$ such that $\phi(r) \ge C\,\phi(s)$ for every $0 < r < s < \infty$) and $r^{\frac{d}{p}}\phi(r)$ is *almost increasing* (i.e. there exists $C > 0$ such that $r^{\frac{d}{p}}\phi(r) \le C\, s^{\frac{d}{p}}\phi(s)$ for every $0 < r < s < \infty$). Note that if $\phi \in \mathcal{G}_p$, then ϕ satisfies the *doubling condition*, that is, there exists $C > 0$ such that $\frac{1}{C} \le \frac{\phi(r)}{\phi(s)} \le C$ whenever $1 \le \frac{r}{s} \le 2$. For $\phi \in \mathcal{G}_p$, the *generalized Morrey space* $\mathcal{M}^p_\phi(\mathbb{R}^d)$ is defined as the set of measurable functions f on $\mathbb{R}^d$ for which

$$\|f\|_{\mathcal{M}^p_\phi} := \sup_{a\in\mathbb{R}^d, r>0} \frac{1}{\phi(r)} \left(\frac{1}{|B(a,r)|} \int_{B(a,r)} |f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty.$$

Note that $\mathcal{M}^p_\phi(\mathbb{R}^d) = \mathcal{M}^p_q(\mathbb{R}^d)$ for $\phi(r) := r^{-\frac{d}{q}}$, $1 \le p \le q < \infty$. Meanwhile, for $\phi \in \mathcal{G}_p$, the *generalized weak Morrey space* $w\mathcal{M}^p_\phi(\mathbb{R}^d)$ is defined to be the set of all measurable functions f on $\mathbb{R}^d$ such that

$$\|f\|_{w\mathcal{M}^p_\phi} := \sup_{a\in\mathbb{R}^d,\, r,\gamma>0} \frac{\gamma\, |\{x \in B(a,r) \,:\, |f(x)| > \gamma\}|^{\frac{1}{p}}}{\phi(r)|B(a,r)|^{\frac{1}{p}}} < \infty.$$

Here $\|\cdot\|_{w\mathcal{M}^p_\phi}$ is a quasi-norm on $w\mathcal{M}^p_\phi(\mathbb{R}^d)$. Furthermore, $w\mathcal{M}^p_\phi(\mathbb{R}^d) = w\mathcal{M}^p_q(\mathbb{R}^d)$ for $\phi(r) := r^{-\frac{d}{q}}$. The relation between the generalized Morrey spaces and their weak type is given in the following lemma.

LEMMA 1.3. *Let $1 \le p < \infty$ and $\phi \in \mathcal{G}_p$. Then $\mathcal{M}^p_\phi(\mathbb{R}^d) \subseteq w\mathcal{M}^p_\phi(\mathbb{R}^d)$ with $\|f\|_{w\mathcal{M}^p_\phi} \le \|f\|_{\mathcal{M}^p_\phi}$ for every $f \in \mathcal{M}^p_\phi(\mathbb{R}^d)$.*

In Section 2 we state our main results, and in Section 3 we present the proofs.

2. Statement of the results

Our main results are presented in the following theorems. The first theorem is more general than Theorem 1.1.

THEOREM 2.1. *Let $m \geq 2$. If $1 \leq p \leq q < \infty$ and $1 \leq p_i \leq q_i < \infty$ for $i = 1, \dots, m$, then the following statements are equivalent:*

(1) $\sum_{i=1}^{m} \frac{1}{p_i} \leq \frac{1}{p}$ *and* $\sum_{i=1}^{m} \frac{1}{q_i} = \frac{1}{q}$.

(2) $\left\| \prod_{i=1}^{m} f_i \right\|_{\mathcal{M}_q^p} \leq \prod_{i=1}^{m} \|f_i\|_{\mathcal{M}_{q_i}^{p_i}}$ *for every* $f_i \in \mathcal{M}_{q_i}^{p_i}(\mathbb{R}^d)$, $i = 1, \dots, m$.

THEOREM 2.2. *Let $m \geq 2$. If $1 \leq p \leq q < \infty$ and $1 \leq p_i \leq q_i < \infty$ for $i = 1, \dots, m$, then the following statements are equivalent:*

(1) $\sum_{i=1}^{m} \frac{1}{p_i} \leq \frac{1}{p}$ *and* $\sum_{i=1}^{m} \frac{1}{q_i} = \frac{1}{q}$.

(2) $\left\| \prod_{i=1}^{m} f_i \right\|_{w\mathcal{M}_q^p} \leq m \prod_{i=1}^{m} \|f_i\|_{w\mathcal{M}_{q_i}^{p_i}}$ *for every* $f_i \in w\mathcal{M}_{q_i}^{p_i}(\mathbb{R}^d)$, $i = 1, \dots, m$.

For generalized Morrey spaces, we have the following theorems.

THEOREM 2.3. *Let $m \geq 2$, $1 \leq p, p_i < \infty$ with $\sum_{i=1}^{m} \frac{1}{p_i} \leq \frac{1}{p}$, $\phi \in \mathcal{G}_p$, and $\phi_i \in \mathcal{G}_{p_i}$ for $i = 1, \dots, m$.*

(1) If $\prod_{i=1}^{m} \phi_i(r) \leq \phi(r)$ *for every* $r > 0$, *then* $\left\| \prod_{i=1}^{m} f_i \right\|_{\mathcal{M}_\phi^p} \leq \prod_{i=1}^{m} \|f_i\|_{\mathcal{M}_{\phi_i}^{p_i}}$ *for every* $f_i \in \mathcal{M}_{\phi_i}^{p_i}(\mathbb{R}^d)$, $i = 1, \dots, m$.

(2) If $\left\| \prod_{i=1}^{m} f_i \right\|_{\mathcal{M}_\phi^p} \leq \prod_{i=1}^{m} \|f_i\|_{\mathcal{M}_{\phi_i}^{p_i}}$ *for every* $f_i \in \mathcal{M}_{\phi_i}^{p_i}(\mathbb{R}^d)$, $i = 1, \dots, m$, *then there exists* $C > 0$ *such that* $\prod_{i=1}^{m} \phi_i(r) \leq C\,\phi(r)$ *for every* $r > 0$.

THEOREM 2.4. *Let $m \geq 2$ and $1 \leq p, p_i < \infty$ for $i = 1, \dots, m$. If $\phi \in \mathcal{G}_p$ and $\phi_i \in \mathcal{G}_{p_i}$ such that $\prod_{i=1}^{m} \phi_i(r) = \phi(r)$ for every $r > 0$ and there exists $\epsilon > 0$ such that $r^{\frac{\epsilon}{p_i}} \phi_i(r)$ are almost decreasing for $i = 1, \dots, m$, then the following statements are equivalent:*

(1) $\sum_{i=1}^{m} \frac{1}{p_i} \leq \frac{1}{p}$.

(2) $\left\| \prod_{i=1}^{m} f_i \right\|_{\mathcal{M}_\phi^p} \leq \prod_{i=1}^{m} \|f_i\|_{\mathcal{M}_{\phi_i}^{p_i}}$ *for every* $f_i \in \mathcal{M}_{\phi_i}^{p_i}(\mathbb{R}^d)$, $i = 1, \dots, m$.

Remark 2.5. In [10,11], Sugano states that $\left\|\prod_{i=1}^{m} f_i\right\|_{\mathcal{M}^p_\phi} \le \prod_{i=1}^{m} \|f_i\|_{\mathcal{M}^{p_i}_{\phi_i}}$ holds for every $f_i \in \mathcal{M}^{p_i}_{\phi_i}(\mathbb{R}^d)$, $i = 1, \ldots, m$, provided that $\sum_{i=1}^{m} \frac{1}{p_i} = \frac{1}{p}$ and $\prod_{i=1}^{m} \phi_i(r) = \phi(r)$. Theorems 2.3 and 2.4 may be viewed as counterparts of Sugano's results.

Finally, for generalized weak Morrey spaces, the following theorems hold.

Theorem 2.6. *Let $m \ge 2$, $1 \le p, p_i < \infty$ with $\sum_{i=1}^{m} \frac{1}{p_i} \le \frac{1}{p}$, $\phi \in \mathcal{G}_p$, and $\phi_i \in \mathcal{G}_{p_i}$ for $i = 1, \ldots, m$.*

(1) If $\prod_{i=1}^{m} \phi_i(r) \le \phi(r)$ for every $r > 0$, then $\left\|\prod_{i=1}^{m} f_i\right\|_{w\mathcal{M}^p_\phi} \le m \prod_{i=1}^{m} \|f_i\|_{w\mathcal{M}^{p_i}_{\phi_i}}$ for every $f_i \in w\mathcal{M}^{p_i}_{\phi_i}(\mathbb{R}^d)$, $i = 1, \ldots, m$.

(2) If $\left\|\prod_{i=1}^{m} f_i\right\|_{w\mathcal{M}^p_\phi} \le m \prod_{i=1}^{m} \|f_i\|_{w\mathcal{M}^{p_i}_{\phi_i}}$ for every $f_i \in w\mathcal{M}^{p_i}_{\phi_i}(\mathbb{R}^d)$, $i = 1, \ldots, m$, then there exists $C > 0$ such that $\prod_{i=1}^{m} \phi_i(r) \le C\,\phi(r)$ for every $r > 0$.

Theorem 2.7. *Let $m \ge 2$ and $1 \le p, p_i < \infty$ for $i = 1, \ldots, m$. If $\phi \in \mathcal{G}_p$ and $\phi_i \in \mathcal{G}_{p_i}$ such that $\prod_{i=1}^{m} \phi_i(r) = \phi(r)$ for every $r > 0$ and there exists $\epsilon > 0$ such that $r^{\frac{\epsilon}{p_i}} \phi_i(r)$ are almost decreasing for $i = 1, \ldots, m$, then the following statements are equivalent:*

(1) $\sum_{i=1}^{m} \frac{1}{p_i} \le \frac{1}{p}$.

(2) $\left\|\prod_{i=1}^{m} f_i\right\|_{w\mathcal{M}^p_\phi} \le m \prod_{i=1}^{m} \|f_i\|_{w\mathcal{M}^{p_i}_{\phi_i}}$ for every $f_i \in w\mathcal{M}^{p_i}_{\phi_i}(\mathbb{R}^d)$, $i = 1, \ldots, m$.

3. Proofs of the theorems

Here, the letter C denotes a constant that may change from line to line. To prove our results, we shall use Lemma 1.2, Lemma 1.3, and the following lemma.

Lemma 3.1. *([3–5]) Let $1 \le p < \infty$ and $\phi \in \mathcal{G}_p$. Then there exists $C > 0$ (depending on ϕ) such that*

$$\frac{1}{\phi(R)} \le \|\chi_{B(a_0,R)}\|_{w\mathcal{M}^p_\phi} \le \|\chi_{B(a_0,R)}\|_{\mathcal{M}^p_\phi} \le \frac{C}{\phi(R)}$$

for every $a_0 \in \mathbb{R}^d$ and $R > 0$. In particular, we have

$$R^{\frac{d}{q}} \le \|\chi_{B(a_0,R)}\|_{w\mathcal{M}^p_q} \le \|\chi_{B(a_0,R)}\|_{\mathcal{M}^p_q} \le C\,R^{\frac{d}{q}}$$

for every $a_0 \in \mathbb{R}^d$ and $R > 0$.

Proof. This fact is proved in [3–5]; we rewrite it here for convenience. Let $B_0 := B(a_0, R) \subseteq \mathbb{R}^d$ where $a_0 \in \mathbb{R}^d$ and $R > 0$. If $r \le R$, then $\phi(r) \ge C\,\phi(R)$, so that

$$\frac{1}{\phi(r)}\left(\frac{1}{|B(a,r)|}\int\limits_{B(a,r)} |\chi_{B_0}(x)|^p dx\right)^{\frac{1}{p}} \le \frac{C}{\phi(R)}\left(\frac{|B(a,r)\cap B_0|}{|B(a,r)|}\right)^{\frac{1}{p}} \le \frac{C}{\phi(R)}$$

for every $a \in \mathbb{R}^d$. If $r \ge R$, then $r^{\frac{d}{p}}\phi(r) \ge C\,R^{\frac{d}{p}}\phi(R)$, so that

$$\frac{1}{\phi(r)}\left(\frac{1}{|B(a,r)|}\int\limits_{B(a,r)} |\chi_{B_0}(x)|^p dx\right)^{\frac{1}{p}} \le \frac{C}{\phi(R)R^{\frac{d}{p}}}|B(a,r)\cap B_0|^{\frac{1}{p}} \le \frac{C}{\phi(R)}$$

for every $a \in \mathbb{R}^d$. Hence we conclude that $\|\chi_{B_0}\|_{\mathcal{M}^p_\phi} \le \frac{C}{\phi(R)}$.

Next, by Lemma 1.3, we have $\|\chi_{B_0}\|_{w\mathcal{M}^p_\phi} \le \|\chi_{B_0}\|_{\mathcal{M}^p_\phi}$. Finally, by using the definition of $\|\cdot\|_{w\mathcal{M}^p_\phi}$, we have

$$\|\chi_{B_0}\|_{w\mathcal{M}^p_\phi} \ge \frac{\gamma}{\phi(R)}\left(\frac{|\{x \in B_0 : |\chi_{B_0}(x)| > \gamma\}|}{|B_0|}\right)^{\frac{1}{p}} = \frac{\gamma}{\phi(R)}\left(\frac{|B_0|}{|B_0|}\right)^{\frac{1}{p}} = \frac{\gamma}{\phi(R)},$$

for every $\gamma \in (0,1)$. Therefore $\|\chi_{B_0}\|_{w\mathcal{M}^p_\phi} \ge \frac{1}{\phi(R)}$, and the lemma is proved. □

3.1 The proof of Theorem 2.1

Proof. (1)⇒(2) Let $\sum_{i=1}^m \frac{1}{p_i} \le \frac{1}{p}$ and $\sum_{i=1}^m \frac{1}{q_i} = \frac{1}{q}$ hold. Put $\frac{1}{p^*} := \sum_{i=1}^m \frac{1}{p_i}$. Clearly $p^* \ge p$. Now take an arbitrary $B := B(a,R) \subseteq \mathbb{R}^d$ and $f_i \in \mathcal{M}^{p_i}_{q_i}(\mathbb{R}^d)$, where $i = 1, \dots, m$. By the generalized Hölder's inequality in Lebesgue spaces [2], we have

$$\begin{aligned} |B|^{\frac{1}{q}-\frac{1}{p}}\left(\int_B \prod_{i=1}^m |f_i(x)|^p dx\right)^{\frac{1}{p}} &\le |B|^{\frac{1}{q}-\frac{1}{p^*}}\left(\int_B \prod_{i=1}^m |f_i(x)|^{p^*} dx\right)^{\frac{1}{p^*}} \\ &\le \prod_{i=1}^m |B|^{\frac{1}{q_i}-\frac{1}{p_i}}\left(\int_B |f(x)|^{p_i} dx\right)^{\frac{1}{p_i}}. \end{aligned}$$

Taking the supremum over B, we obtain $\left\|\prod_{i=1}^m f_i\right\|_{\mathcal{M}^p_q} \le \prod_{i=1}^m \|f_i\|_{\mathcal{M}^{p_i}_{q_i}}$.

(2)⇒(1) Suppose that $\left\|\prod_{i=1}^m f_i\right\|_{\mathcal{M}^p_q} \le \prod_{i=1}^m \|f_i\|_{\mathcal{M}^{p_i}_{q_i}}$ for every $f_i \in \mathcal{M}^{p_i}_{q_i}(\mathbb{R}^d)$, $i = 1, \dots, m$. Take an arbitrary $R > 0$ and choose $f_i := \chi_{B(0,R)}$ for $i = 1, \dots, m$. It follows from the hypothesis that $\|\chi_{B(0,R)}\|_{\mathcal{M}^p_q} = \left\|\prod_{i=1}^m f_i\right\|_{\mathcal{M}^p_q} \le \prod_{i=1}^m \|f_i\|_{\mathcal{M}^{p_i}_{q_i}} = \prod_{i=1}^m \|\chi_{B(0,R)}\|_{\mathcal{M}^{p_i}_{q_i}}$. Hence, by Lemma 3.1, we have $R^{\frac{d}{q}-\sum_{i=1}^m \frac{d}{q_i}} \le C$. Since $R > 0$ is arbitrary, we conclude that $\sum_{i=1}^m \frac{1}{q_i} = \frac{1}{q}$.

Next, choose $0 < \epsilon < \min\{\frac{dp_1}{q_1}, \dots, \frac{dp_m}{q_m}\}$. Clearly $\epsilon < d$. For arbitrary $K \in \mathbb{N}$, we define $g_{\epsilon,K}(x) := \chi_{\{0\le|x|<1\}}(x) + \sum_{j=1}^K \chi_{\{j\le|x|\le j+j^{-\epsilon}\}}(x)$. (If one prefers, one may reduce to the case of $d = 1$ and then consider the tensor product $G_{\epsilon,K}(x_1, x_2, \dots, x_n) =$

$g_{\epsilon,K}(x_1)g_{\epsilon,K}(x_2)\cdots g_{\epsilon,K}(x_n)$ for the general case, working with cubes instead of balls.) We define $f_i := g_{\epsilon,K}$, $i = 1, \ldots, m$. Note that $\prod_{i=1}^m f_i = g_{\epsilon,K}$ and so $\left|\prod_{i=1}^m f_i\right|^p = g_{\epsilon,K}$. Hence, we obtain

$$\begin{aligned}
\left\|\prod_{i=1}^m f_i\right\|_{\mathcal{M}_q^p} &= \sup_{a\in\mathbb{R}^d, r>0} |B(a,r)|^{\frac{1}{q}-\frac{1}{p}} \left(\int_{B(a,r)} \left|\prod_{i=1}^m f_i(x)\right|^p dx\right)^{\frac{1}{p}} \\
&\geq |B(0, K+K^{-\epsilon})|^{\frac{1}{q}-\frac{1}{p}} \left(\int_{B(0,K+K^{-\epsilon})} g_{\epsilon,K}(x)\,dx\right)^{\frac{1}{p}} \\
&= C(K+K^{-\epsilon})^{\frac{d}{q}-\frac{d}{p}} \left(|B(0,1)| + \sum_{j=1}^K \int_{j\leq|x|\leq j+j^{-\epsilon}} dx\right)^{\frac{1}{p}} \\
&= C(K+K^{-\epsilon})^{\frac{d}{q}-\frac{d}{p}} \left(|B(0,1)| + \sum_{j=1}^K (|B(0,j+j^{-\epsilon})| - |B(0,j)|)\right)^{\frac{1}{p}} \\
&\geq C(K+K^{-\epsilon})^{\frac{d}{q}-\frac{d}{p}} \left(\sum_{j=1}^K \left[(j+j^{-\epsilon})^d - j^d\right]\right)^{\frac{1}{p}} \geq C(K+K^{-\epsilon})^{\frac{d}{q}-\frac{d}{p}} \left(\sum_{j=1}^K j^{d-\epsilon-1}\right)^{\frac{1}{p}}.
\end{aligned}$$

The last inequality follows from the Mean Value Theorem applied to the function $t \mapsto t^d$ on the interval $[j, j+j^{-\epsilon}]$ for each $j = 1, \ldots, K$. Observe now that for $K \geq 2$, we have

$$\begin{aligned}
\sum_{j=1}^K j^{d-\epsilon-1} \sim \int_1^K t^{d-\epsilon-1} dt &= C(K^{d-\epsilon} - 1) = C\Big(1 - \frac{1}{K^{d-\epsilon}}\Big)K^{d-\epsilon} \\
&\geq C\,K^{d-\epsilon} \geq C(K+K^{-\epsilon})^{d-\epsilon}.
\end{aligned}$$

We note that this inequality also holds for $K = 1$ provided that $C \leq 1$. With this observation, we obtain

$$\left\|\prod_{i=1}^m f_i\right\|_{\mathcal{M}_q^p} \geq C(K+K^{-\epsilon})^{\frac{d}{q}-\frac{d}{p}}(K+K^{-\epsilon})^{\frac{d}{p}-\frac{\epsilon}{p}} = C(K+K^{-\epsilon})^{\frac{d}{q}-\frac{\epsilon}{p}}.$$

Meanwhile, for each $i = 1, \ldots, m$, we claim that

$$\sup_{a\in\mathbb{R}^d, r>0} |B(a,r)|^{\frac{1}{q_i}-\frac{1}{p_i}} \Big(\int_{B(a,r)} f_i(x)^{p_i}\,dx\Big)^{\frac{1}{p_i}} \lesssim |B(0,L)|^{\frac{1}{q_i}-\frac{1}{p_i}} \Big(\int_{B(0,L)} f_i(x)^{p_i}\,dx\Big)^{\frac{1}{p_i}}$$

for some integer L with $2 \leq L \leq K+1$. Observe that $f_i = g_{\epsilon,K}$ is symmetrical about 0 and has most mass around 0, and so for each $a \in \mathbb{R}^d$ and $r > 0$, we have

$$|B(a,r)|^{\frac{1}{q_i}-\frac{1}{p_i}} \Big(\int_{B(a,r)} f_i(x)^{p_i}\,dx\Big)^{\frac{1}{p_i}} \leq |B(0,r)|^{\frac{1}{q_i}-\frac{1}{p_i}} \Big(\int_{B(0,r)} f_i(x)^{p_i}\,dx\Big)^{\frac{1}{p_i}}.$$

Now, as a function of r only, the value of the last expression on the right-hand side gets larger and larger as r grows from 0 to 2 but decreases for $r > K + K^{-\epsilon}$. This verifies our claim about the supremum.

Since $\frac{1}{q_i} - \frac{1}{p_i} \le 0$ for $i = 1, \dots, m$ and $j + j^{-\epsilon} \le 2j$ for $j = 1, \dots, K$, we have

$$\|f_i\|_{\mathcal{M}^{p_i}_{q_i}} = \sup_{a\in\mathbb{R}^d, r>0} |B(a,r)|^{\frac{1}{q_i}-\frac{1}{p_i}} \left(\int_{B(a,r)} f_i(x)^{p_i}\,dx\right)^{\frac{1}{p_i}}$$

$$\le C\,|B(0,L)|^{\frac{1}{q_i}-\frac{1}{p_i}} \left(\int_{B(0,L)} f_i(x)^{p_i}\,dx\right)^{\frac{1}{p_i}} \le C\,L^{\frac{d}{q_i}-\frac{d}{p_i}} \left(|B(0,1)| + \sum_{j=1}^{L}\left[(j+j^{-\epsilon})^d - j^d\right]\right)^{\frac{1}{p_i}}$$

$$\le C\,L^{\frac{d}{q_i}-\frac{d}{p_i}} \left(|B(0,1)| + \sum_{j=1}^{L} j^{d-\epsilon-1}\right)^{\frac{1}{p_i}} \le C\,L^{\frac{d}{q_i}-\frac{d}{p_i}} L^{\frac{d}{p_i}-\frac{\epsilon}{p_i}} = C\,L^{\frac{d}{q_i}-\frac{\epsilon}{p_i}}.$$

Moreover, since $L \le K+1 \le 2(K+K^{-\epsilon})$, we obtain $\|f_i\|_{\mathcal{M}^{p_i}_{q_i}} \le C(K+K^{-\epsilon})^{\frac{d}{q_i}-\frac{\epsilon}{p_i}}$ for $i = 1, \dots, m$.

Knowing that $\sum\limits_{i=1}^{m} \frac{d}{q_i} = \frac{d}{q}$ and $\left\|\prod\limits_{i=1}^{m} f_i\right\|_{\mathcal{M}^p_q} \le \prod\limits_{i=1}^{m} \|f_i\|_{\mathcal{M}^{p_i}_{q_i}}$, we conclude from the two inequalities above that $(K+K^{-\epsilon})^{-\frac{\epsilon}{p}+\sum_{i=1}^{m}\frac{\epsilon}{p_i}} \le C$ for every $K \in \mathbb{N}$. Therefore $\sum_{i=1}^{m} \frac{1}{p_i} \le \frac{1}{p}$, as desired. $\square$

Remark 3.2. For $m = 2$, we obtain the proof of Theorem 1.1.

3.2 The proof of Theorem 2.2

Proof. If (1) holds, then by similar arguments as in [8] we can prove that (1) implies (2). It thus remains to prove that (2) implies (1). To do so, take an arbitrary $R > 0$ and let $f_i := \chi_{B(0,R)}$ for $i = 1, \dots, m$. By the hypothesis, we then have

$$\|\chi_{B(0,R)}\|_{w\mathcal{M}^p_q} = \left\|\prod_{i=1}^{m} f_i\right\|_{w\mathcal{M}^p_q} \le m \prod_{i=1}^{m} \|f_i\|_{w\mathcal{M}^{p_i}_{q_i}} = m \prod_{i=1}^{m} \|\chi_{B(0,R)}\|_{w\mathcal{M}^{p_i}_{q_i}}.$$

Hence $R^{\frac{d}{q}-\sum_{i=1}^{m}\frac{d}{q_i}} \le C$. Since this holds for every $R > 0$, it follows that $\sum_{i=1}^{m} \frac{1}{q_i} = \frac{1}{q}$.

Next, let $0 < \epsilon < \min\{\frac{dp_1}{q_1}, \dots, \frac{dp_m}{q_m}\}$ and define $g_{\epsilon,K}(x) := \chi_{\{0\le|x|<1\}}(x) + \sum_{j=1}^{K} \chi_{\{j\le|x|\le j+j^{-\epsilon}\}}(x)$, for arbitrary $K \in \mathbb{N}$. For $i = 1, \dots, m$, let $f_i := g_{\epsilon,K}$. We observe that

$$\left\|\prod_{i=1}^{m} f_i\right\|_{w\mathcal{M}^p_q} \ge \frac{1}{2}|B(0,K+K^{-\epsilon})|^{\frac{1}{q}-\frac{1}{p}} \left|\left\{x \in B(a,r) : |f_i(x)| > \frac{1}{2}\right\}\right|^{\frac{1}{p}}$$

$$\ge C(K+K^{-\epsilon})^{\frac{d}{q}-\frac{d}{p}}(K+K^{-\epsilon})^{\frac{d}{p}-\frac{\epsilon}{p}} = C(K+K^{-\epsilon})^{\frac{d}{q}-\frac{\epsilon}{p}}.$$

Meanwhile, by Lemma 1.2 and the Morrey-norm estimate for f_i, we obtain $\|f_i\|_{w\mathcal{M}^{p_i}_{q_i}} \le \|f_i\|_{\mathcal{M}^{p_i}_{q_i}} \le C(K+K^{-\epsilon})^{\frac{d}{q_i}-\frac{\epsilon}{p_i}}$ for $i = 1, \dots, m$. Since $\sum_{i=1}^{m} \frac{1}{q_i} = \frac{1}{q}$ and $\left\|\prod\limits_{i=1}^{m} f_i\right\|_{w\mathcal{M}^p_q} \le m \prod\limits_{i=1}^{m} \|f_i\|_{w\mathcal{M}^{p_i}_{q_i}}$, we have $(K+K^{-\epsilon})^{-\frac{\epsilon}{p}+\sum_{i=1}^{m}\frac{\epsilon}{p_i}} \le C$.

Since it holds for every $K \in \mathbb{N}$, we must have $\sum_{i=1}^{m} \frac{1}{p_i} \le \frac{1}{p}$. $\square$

3.3 The proof of Theorem 2.3

Proof. (1) Suppose that $\prod_{i=1}^{m} \phi_i(r) \le \phi(r)$ for every $r > 0$. Take an arbitrary $B := B(a,R) \subseteq \mathbb{R}^d$ and $f_i \in \mathcal{M}^{p_i}_{\phi_i}(\mathbb{R}^d)$, where $i = 1, \ldots, m$. Putting $\frac{1}{p^*} := \sum_{i=1}^{m} \frac{1}{p_i}$, it follows from the generalized Hölder's inequality in Lebesgue spaces that

$$\frac{1}{\phi(R)} \left(\frac{1}{|B|} \int_B \prod_{i=1}^{m} |f_i(x)|^p dx \right)^{\frac{1}{p}} \le \frac{1}{\phi(R)} \left(\frac{1}{|B|} \int_B \prod_{i=1}^{m} |f_i(x)|^{p^*} dx \right)^{\frac{1}{p^*}}$$
$$\le \prod_{i=1}^{m} \frac{1}{\phi_i(R)} \left(\frac{1}{|B|} \int_B |f_i(x)|^{p_i} dx \right)^{\frac{1}{p_i}}.$$

We can now take the supremum over B to obtain $\left\| \prod_{i=1}^{m} f_i \right\|_{\mathcal{M}^p_\phi} \le \prod_{i=1}^{m} \|f_i\|_{\mathcal{M}^{p_i}_{\phi_i}}$.

(2) Suppose that $\left\| \prod_{i=1}^{m} f_i \right\|_{\mathcal{M}^p_\phi} \le \prod_{i=1}^{m} \|f_i\|_{\mathcal{M}^{p_i}_{\phi_i}}$ for every $f_i \in \mathcal{M}^{p_i}_{\phi_i}(\mathbb{R}^d)$, $i = 1, \ldots, m$. Take an arbitrary $R > 0$ and define $f_i := \chi_{B(0,R)}$ for $i = 1, \ldots, m$. Then there exists $C > 0$ (independent of R) such that $\frac{1}{\phi(R)} \le \|\chi_{B(0,R)}\|_{\mathcal{M}^p_\phi} \le \prod_{i=1}^{m} \|\chi_{B(0,R)}\|_{\mathcal{M}^{p_i}_{\phi_i}} \le \prod_{i=1}^{m} \frac{C}{\phi_i(R)}$. Thus $\prod_{i=1}^{m} \phi_i(R) \le C\, \phi(R)$, as desired. □

3.4 The proof of Theorem 2.4

Proof. (1)⇒(2) Suppose that $\sum_{i=1}^{m} \frac{1}{p_i} \le \frac{1}{p}$. As before, one may easily observe that $\left\| \prod_{i=1}^{m} f_i \right\|_{\mathcal{M}^p_\phi} \le \prod_{i=1}^{m} \|f_i\|_{\mathcal{M}^{p_i}_{\phi_i}}$ for every $f_i \in \mathcal{M}^{p_i}_{\phi_i}(\mathbb{R}^d)$, $i = 1, \ldots, m$.

(2)⇒(1) Let $\left\| \prod_{i=1}^{m} f_i \right\|_{\mathcal{M}^p_\phi} \le \prod_{i=1}^{m} \|f_i\|_{\mathcal{M}^{p_i}_{\phi_i}}$ for every $f_i \in \mathcal{M}^{p_i}_{\phi_i}(\mathbb{R}^d)$, where $i = 1, \ldots, m$. For arbitrary $K \in \mathbb{N}$, we define $g_{\epsilon,K}(x) := \chi_{\{0 \le |x| < 1\}}(x) + \sum_{j=1}^{K} \chi_{\{j \le |x| \le j + j^{-\epsilon}\}}(x)$, where $\epsilon > 0$ satisfies the hypothesis (which forces us to have $\epsilon < d$). For $i = 1, \ldots, m$, let $f_i := g_{\epsilon,K}$. It is easy to check that

$$\left\| \prod_{i=1}^{m} f_i \right\|_{\mathcal{M}^p_\phi} \ge \frac{1}{\phi\left(K + K^{-\epsilon}\right)} \left(\frac{1}{|B(0, K + K^{-\epsilon})|} \int_{B(0,K+K^{-\epsilon})} g_{\epsilon,K}(x)\, dx \right)^{\frac{1}{p}}$$
$$\ge \frac{C}{(K + K^{-\epsilon})^{\frac{d}{p}} \phi(K + K^{-\epsilon})} \left(\sum_{j=1}^{K} j^{d-\epsilon-1} \right)^{\frac{1}{p}} \ge \frac{C}{(K + K^{-\epsilon})^{\frac{\epsilon}{p}} \phi(K + K^{-\epsilon})}.$$

Meanwhile, for $i = 1, \ldots, m$, by using similar arguments as in the proof of Theorem

2.1 one may observe that for $2 \le L \le K+1$,

$$\|f_i\|_{\mathcal{M}^{p_i}_{\phi_i}} \le \frac{C}{\phi_i(L)}\left(\frac{1}{|B(0,L)|}\int\limits_{B(0,L)} g_{\epsilon,K}(x)\,dx\right)^{\frac{1}{p_i}} \le \frac{C}{(K+K^{-\epsilon})^{\frac{\epsilon}{p_i}}\phi_i(K+K^{-\epsilon})}.$$

Since $\left\|\prod_{i=1}^m f_i\right\|_{\mathcal{M}^p_\phi} \le \prod_{i=1}^m \|f_i\|_{\mathcal{M}^{p_i}_{\phi_i}}$ and $\prod_{i=1}^m \phi_i(r) = \phi(r)$ for every $r>0$, then for arbitrary $K \in \mathbb{N}$ it holds $(K+K^{-\epsilon})^{-\frac{\epsilon}{p}+\sum_{i=1}^m \frac{\epsilon}{p_i}} \le C$. Hence, $\sum_{i=1}^m \frac{1}{p_i} \le \frac{1}{p}$. □

3.5 The proof of Theorem 2.6

Proof. (1) Suppose that $\prod_{i=1}^m \phi_i(r) \le \phi(r)$ for every $r>0$. Let $f_i \in w\mathcal{M}^{p_i}_{\phi_i}(\mathbb{R}^d)$, where $i=1,\ldots,m$. For an arbitrary $B := B(a,R) \subseteq \mathbb{R}^d$ and $\gamma>0$, let

$$A(B,\gamma) := \left[\frac{1}{\phi^p(R)|B|}\gamma^p\left|\left\{x\in B : \prod_{i=1}^m\left|\frac{f_i(x)}{\|f_i\|_{w\mathcal{M}^{p_i}_{\phi_i}}}\right| > \gamma\right\}\right|\right]^{\frac{1}{p}}.$$

Putting $\frac{1}{p^*} := \sum_{i=1}^m \frac{1}{p_i}$, we observe that

$$\begin{aligned} A(B,\gamma) &\le \left[\frac{1}{|B|\left(\prod_{i=1}^m \phi_i(R)\right)^{p^*}}\gamma^{p^*}\left|\left\{x\in B : \prod_{i=1}^m\left|\frac{f_i(x)}{\|f_i\|_{w\mathcal{M}^{p_i}_{\phi_i}}}\right| > \gamma\right\}\right|\right]^{\frac{1}{p^*}} \\ &= \left[\frac{1}{|B|}\gamma_0^{p^*}\left|\left\{x\in B : \prod_{i=1}^m\left|\frac{f_i(x)}{\phi_i(R)\|f_i\|_{w\mathcal{M}^{p_i}_{\phi_i}}}\right| > \gamma_0\right\}\right|\right]^{\frac{1}{p^*}}, \end{aligned}$$

where $\gamma_0 := \frac{\gamma}{\prod_{i=1}^m \phi_i(R)}$. Furthermore, by using Young's inequality for products, we have

$$\begin{aligned} A(B,\gamma) &\le \left[\frac{1}{|B|}\gamma_0^{p^*}\left|\left\{x\in B : \prod_{i=1}^m\left|\frac{f_i(x)}{\phi_i(R)\|f_i\|_{w\mathcal{M}^{p_i}_{\phi_i}}}\right| > \gamma_0\right\}\right|\right]^{\frac{1}{p^*}} \\ &\le \left[\frac{1}{|B|}\gamma_0^{p^*}\left|\left\{x\in B : \sum_{i=1}^m \frac{p^*}{p_i}\left|\frac{f_i(x)}{\phi_i(R)\|f_i\|_{w\mathcal{M}^{p_i}_{\phi_i}}}\right|^{\frac{p_i}{p^*}} > \gamma_0\right\}\right|\right]^{\frac{1}{p^*}} \\ &\le \left[\sum_{i=1}^m \frac{1}{|B|}\gamma_0^{p^*}\left|\left\{x\in B : \frac{p^*}{p_i}\left|\frac{f_i(x)}{\phi_i(R)\|f_i\|_{w\mathcal{M}^{p_i}_{\phi_i}}}\right|^{\frac{p_i}{p^*}} > \frac{\gamma_0}{m}\right\}\right|\right]^{\frac{1}{p^*}}. \end{aligned}$$

Since $\frac{p^*}{p_i}\Big|\frac{f_i(x)}{\phi_i(R)\|f_i\|_{w\mathcal{M}^{p_i}_{\phi_i}}}\Big|^{\frac{p_i}{p^*}} > \frac{\gamma_0}{m}$ is equivalent to $|f_i(x)| > \big(\frac{\gamma_0 p_i}{mp^*}\big)^{\frac{p^*}{p_i}}\phi_i(R)\|f_i\|_{w\mathcal{M}^{p_i}_{\phi_i}} =: \gamma_i$, we obtain

$$A(B,\gamma) \leq \left[\sum_{i=1}^{m}\Big(\frac{\gamma_i(mp^*)^{p^*/p_i}}{\phi_i(R)p_i^{p^*/p_i}\|f_i\|_{w\mathcal{M}^{p_i}_{\phi_i}}}\Big)^{p_i}\frac{|\{x\in B:\,|f_i(x)|>\gamma_i\}|}{|B|}\right]^{\frac{1}{p^*}}$$

$$= m\left[\sum_{i=1}^{m}\Big(\frac{p^*}{p_i}\Big)^{p^*}\frac{\gamma_i^{p_i}|\{x\in B:\,|f_i(x)|>\gamma_i\}|}{\phi_i(R)^{p_i}|B|\|f_i\|^{p_i}_{w\mathcal{M}^{p_i}_{\phi_i}}}\right]^{\frac{1}{p^*}} \leq m\left[\sum_{i=1}^{m}\Big(\frac{p^*}{p_i}\Big)^{p^*}\right]^{\frac{1}{p^*}} \leq m\left[\sum_{i=1}^{m}\frac{p^*}{p_i}\right]^{\frac{1}{p^*}} = m,$$

because $1\leq p^*\leq p_i$ for $i=1,\dots,m$. We then take the supremum of $A(B,\gamma)$ over $B:=B(a,R)\subseteq\mathbb{R}^d$ and $\gamma>0$ to obtain $\Big\|\prod_{i=1}^{m} f_i\Big\|_{w\mathcal{M}^p_\phi}$.

(2) Let $\Big\|\prod_{i=1}^{m} f_i\Big\|_{w\mathcal{M}^p_\phi} \leq m\prod_{i=1}^{m}\|f_i\|_{w\mathcal{M}^{p_i}_{\phi_i}}$ for every $f_i\in w\mathcal{M}^{p_i}_{\phi_i}(\mathbb{R}^d)$, $i=1,\dots,m$. Take an arbitrary $R>0$ and define $f_i:=\chi_{B(0,R)}$ for $i=1,\dots,m$. By the hypothesis, we have $\|\chi_{B(0,R)}\|_{w\mathcal{M}^p_\phi}\leq m\prod_{i=1}^{m}\|\chi_{B(0,R)}\|_{w\mathcal{M}^{p_i}_{\phi_i}}$. It thus follows from Lemma 3.1 that there exists $C>0$ (independent of R) such that $\prod_{i=1}^{m}\phi_i(R)\leq C\,\phi(R)$. □

3.6 The proof of Theorem 2.7

Proof. (1)⇒(2) Suppose that $\sum_{i=1}^m\frac{1}{p_i}\leq\frac{1}{p}$. As before, for every $f_i\in w\mathcal{M}^{p_i}_{\phi_i}(\mathbb{R}^d)$, $i=1,\dots,m$, we obtain $\Big\|\prod_{i=1}^{m} f_i\Big\|_{w\mathcal{M}^p_\phi}\leq m\prod_{i=1}^{m}\|f_i\|_{w\mathcal{M}^{p_i}_{\phi_i}}$.

(2)⇒(1) Suppose that $\Big\|\prod_{i=1}^{m} f_i\Big\|_{w\mathcal{M}^p_\phi}\leq m\prod_{i=1}^{m}\|f_i\|_{w\mathcal{M}^{p_i}_{\phi_i}}$ for every $f_i\in w\mathcal{M}^{p_i}_{\phi_i}(\mathbb{R}^d)$, where $i=1,\dots,m$. Define $g_{\epsilon,K}(x):=\chi_{\{0\leq|x|<1\}}(x)+\sum_{j=1}^K\chi_{\{j\leq|x|\leq j+j^{-\epsilon}\}}(x)$, for arbitrary $K\in\mathbb{N}$ (where $\epsilon>0$ satisfies the hypothesis), and for $i=1,\dots,m$ put $f_i:=g_{\epsilon,K}$. By using the same arguments as in the proof of Theorem 2.2, we have

$$\Big\|\prod_{i=1}^{m} f_i\Big\|_{w\mathcal{M}^p_\phi}\geq\frac{C}{(K+K^{-\epsilon})^{\frac{\epsilon}{p}}\phi(K+K^{-\epsilon})}.$$

Next, using Lemma 1.3 and the generalized Morrey-norm estimate for f_i, we have

$$\|f_i\|_{w\mathcal{M}^{p_i}_{\phi_i}}\leq\|f_i\|_{\mathcal{M}^{p_i}_{\phi_i}}\leq\frac{C}{(K+K^{-\epsilon})^{\frac{\epsilon}{p_i}}\phi_i(K+K^{-\epsilon})},$$

for $i=1,\dots,m$. Since $\Big\|\prod_{i=1}^{m} f_i\Big\|_{w\mathcal{M}^p_\phi}\leq m\prod_{i=1}^{m}\|f_i\|_{w\mathcal{M}^{p_i}_{\phi_i}}$ and $\prod_{i=1}^{m}\phi_i(r)=\phi(r)$ for every $r>0$, it follows that $(K+K^{-\epsilon})^{-\frac{\epsilon}{p}+\sum_{i=1}^m\frac{\epsilon}{p_i}}\leq C$. We therefore conclude that $\sum_{i=1}^m\frac{1}{p_i}\leq\frac{1}{p}$. □

4. Concluding remarks

We have shown sufficient and necessary conditions for generalized Hölder's inequality in several spaces, namely Morrey spaces and their weak type versions. From Theorems 2.1 and 2.2, we see that both generalized Hölder's inequality in Morrey spaces and in weak Morrey spaces are equivalent to the same condition, namely $\sum_{i=1}^{m} \frac{1}{p_i} \leq \frac{1}{p}$ and $\sum_{i=1}^{m} \frac{1}{q_i} = \frac{1}{q}$. Accordingly, we have the following corollary.

Corollary 4.1. *For $m \geq 2$, the following statements are equivalent:*

(1) $\left\| \prod_{i=1}^{m} f_i \right\|_{\mathcal{M}_q^p} \leq \prod_{i=1}^{m} \|f_i\|_{\mathcal{M}_{q_i}^{p_i}}$ *for every* $f_i \in \mathcal{M}_{q_i}^{p_i}(\mathbb{R}^d)$, *where* $i = 1, \ldots, m$.

(2) $\left\| \prod_{i=1}^{m} f_i \right\|_{w\mathcal{M}_q^p} \leq m \prod_{i=1}^{m} \|f_i\|_{w\mathcal{M}_{q_i}^{p_i}}$ *for every* $f_i \in w\mathcal{M}_{q_i}^{p_i}(\mathbb{R}^d)$, *where* $i = 1, \ldots, m$.

Similarly, from Theorems 2.4 and 2.7, we have the following corollary about Hölder's inequality in generalized Morrey spaces and in generalized weak Morrey spaces.

Corollary 4.2. *Let $m \geq 2$ and $1 \leq p, p_i < \infty$ for $i = 1, \ldots, m$. If $\phi \in \mathcal{G}_p$ and $\phi_i \in \mathcal{G}_{p_i}$ such that $\prod_{i=1}^{m} \phi_i(r) = \phi(r)$ for every $r > 0$ and there exists $\epsilon > 0$ such that $r^{\frac{\epsilon}{p_i}} \phi_i(r)$ are almost decreasing for $i = 1, \ldots, m$, then the following statements are equivalent:*

(1) $\left\| \prod_{i=1}^{m} f_i \right\|_{\mathcal{M}_\phi^p} \leq \prod_{i=1}^{m} \|f_i\|_{\mathcal{M}_{\phi_i}^{p_i}}$ *for every* $f_i \in \mathcal{M}_{\phi_i}^{p_i}(\mathbb{R}^d)$, *where* $i = 1, \ldots, m$.

(2) $\left\| \prod_{i=1}^{m} f_i \right\|_{w\mathcal{M}_\phi^p} \leq m \prod_{i=1}^{m} \|f_i\|_{w\mathcal{M}_{\phi_i}^{p_i}}$ *for every* $f_i \in w\mathcal{M}_{\phi_i}^{p_i}(\mathbb{R}^d)$, *where* $i = 1, \ldots, m$.

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