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SPLIT FEASIBILITY PROBLEM FOR COUNTABLE FAMILY OF MULTI-VALUED NONLINEAR MAPPINGS

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Abstract. Based on the recent important result of S. S. Chang, H. W. Joseph Lee, C. K. Chan, L. Wang, L.J. Qin [Appl. Math. Comput. 219 (2013) 10416–10424], we study in this article the split feasibility problem for a countable family of multi-valued $\kappa-$strictly pseudo-contractive mappings and total asymptotically strict pseudo-contractive mappings in infinite dimensional Hilbert spaces. The main results presented in this paper improve and extend the aforementioned result.

1. Introduction

In this work, H will be used to denote a real Hilbert space and K will denote a subset of H. A mapping $T : K \to K$ is said to be $(\kappa, \{\mu_n\}, \{\varepsilon_n\}, \phi)$-total asymptotically strict pseudo contractive, if there exists a constant $\kappa \in [0, 1)$ and sequences $\{\mu_n\} \subset [0,\infty)$, $\{\varepsilon_n\} \subset [0,\infty)$ with $\mu_n \to 0$ and $\varepsilon_n \to 0$ as $n \to \infty$, and a continuous and strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that for all $n \geq 1, x, y \in K$,

$$\|T^n x - T^n y\|^2 \leq \|x - y\|^2 + k\|x - y - (T^n x - T^n y)\|^2 + \mu_n \phi(\|x - y\|) + \varepsilon_n \tag{1}$$

Let (X, d) be a metric space and $CB(X)$ be the family of all closed and bounded subsets of X. Let $\mathcal{H}$ denote the Hausdorff metric induced by the metric d, then for all $A, B \in CB(X)$, $\mathcal{H}(A,B) = \max\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(b,A)\}$, where $d(a,B) := \inf\limits_{b\in B} d(a,b)$.

Let $T : D(T) \subseteq H \to CB(H)$ be a multi-valued mapping, a point $x \in D(T)$ is called a fixed point of T if $x \in Tx$.

A multi-valued mapping T is said to be *L-Lipschitzian* if there exists $L > 0$ such that $\mathcal{H}(Tx, Ty) \leq L||x - y||,\ x, y \in D(T)$. If $L \in (0, 1)$, then T is called contraction, while T is called nonexpansive for $L = 1$.

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T is said to be

(i) *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $\mathcal{H}(Tx, Ty) \leq ||x - y||$, $\forall x \in D(T)$, $y \in F(T)$,

(ii) $\kappa-$*strictly pseudocontractive* (see e.g. [11]) if there exists $\kappa \in (0,1)$ such that $\forall x, y \in D(T)$, $\mathcal{H}^2(Tx, Ty) \leq ||x-y||^2 + k||x - u - (y - v)||^2$, $\forall u \in Tx,\ v \in Ty$.

In 1994, Censor and Elfving [4] introduced, in finite dimensional Hilbert spaces, the split feasibility problem for modeling inverse problems which arise from phase retrievals and in medical image reconstruction. It is now known that split feasibility problems can be used in various disciplines, such as image restoration, computer tomograph and radiation therapy treatment planning (see [2, 3, 5, 6]).

Let H_1 and H_2 be two real Hilbert spaces, K and Q be nonempty closed convex subsets of H_1 and H_2, respectively. The *split feasibility problem* is formulated as follows: find a point $q \in H_1$ such that

$$q \in K \text{ and } Aq \in Q, \tag{2}$$

where $A : H_1 \to H_2$ is a bounded linear operator. If (2) has solutions, it can be shown that $x \in K$ solves (2) if and only if it solves the following fixed point equation:

$$x = P_K((I - \gamma A^*(I - P_Q)A)x), x \in K, \tag{3}$$

where P_K and P_Q are the projections onto K and Q, respectively, γ is a positive constant, and A^* denotes the adjoint of A. When K and Q in (2) are the sets of fixed points of two nonlinear mappings, and K and Q are nonempty closed convex subsets of H_1 and H_2, respectively, then the split feasibility problem (2) is also called *split common fixed point problem or multiple-set split feasibility problem* (see [9, 16]).

Split common fixed point problems for nonlinear mappings in the setting of two Hilbert spaces have been studied by many authors; (see, for instance, [7, 8, 10, 12–14]).

Recently, Chang *et.al* [9] obtained strong and weak convergence results for multiple set split feasibility problems for a family of multi-valued mappings and a single valued nonlinear mapping in Hilbert spaces.

Motivated by the results of Chang *et al.*, we introduce and study multiple-set split feasibility problem for a countable family of multi-valued $\kappa-$strictly pseudo-contractive mapping and total asymptotically strict pseudo-contractive mapping in infinite dimensional Hilbert spaces.

2. Preliminaries

In the sequel, we shall denote by $\rightharpoonup$ and $\to$, the weak and strong convergence of a sequence $\{x_n\}$, respectively.

Definition 2.1. A multi-valued mapping $T : D(T) \subseteq H_1 \to CB(H_1)$ is said to be demi-closed at origin if for any sequence $\{x_n\} \subset H_1$ with $x_n \rightharpoonup q$ and $d(x_n, Tx_n) \to 0$, we have $q \in Tq$.

Definition 2.2. A normed linear space, X is said to satisfy Opial's condition if, for any sequence $\{x_n\}$ with $x_n \rightharpoonup p$, we have $\liminf_{n\to\infty} \|x_n - p\| < \liminf_{n\to\infty} \|x_n - q\|$, $\forall\ q \in X$with $q \neq p$.

A multi-valued mapping $T : D(T) \subseteq H_1 \to CB(H_1)$ is said to be hemi-compact if, for any sequence $\{x_n\}$ in H_1 such that $d(x_n, Tx_n) \to 0$ *as* $n \to \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to q \in H_1$.

Definition 2.3. Let $\{x_n\}$ be a sequence in H. A point $x^* \in H$ is called a weak cluster point of the sequence $\{x_n\}$ if there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup x^*$, as $j \to \infty$.

Lemma 2.4. *([11]) Let $T_i : H \to CB(H)$ be a countable family of multi-valued k_i-strictly pseudo-contractive mappings, $k_i \in (0,1)$, $i = 1, 2, \dots$. If $\sup_{i\geq 1} k_i \in (0,1)$, then T_i is uniformly Lipschitz, i.e., there exists $L > 0$ such that $\mathcal{H}(T_i x, T_i y) \leq L\|x - y\|$, $\forall\ x, y \in H$, $i = 1, 2, \dots$.*

Lemma 2.5. *([9, Lemma 2.3]) Let $T : H \to H$ be a uniformly L-Lipschitzian continuous and $(k, \{\mu_n\}, \{\varepsilon_n\}, \phi)$-total asymptotically strict pseudo contractive mapping. Then T is demi-closed at the origin.*

Lemma 2.6. *([11]) Let $\{x_i\}_{i=1}^{\infty} \subset H$ and $\alpha_i \in (0,1), i = 1, 2, \dots$ such that $\sum_{i=1}^{\infty} \alpha_i = 1$. If $\{x_i\}_{i=1}^{\infty}$ is bounded, then $\|\sum_{i=1}^{\infty} \alpha_i x_i\|^2 = \sum_{i=1}^{\infty} \alpha_i \|x_i\|^2 - \sum_{i,j=1, i\neq j}^{\infty} \alpha_i \alpha_j \|x_i - x_j\|^2$.*

Lemma 2.7. *([15]) Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying $a_{n+1} \leq (1 + \delta_n) a_n + b_n\ \forall\ n \geq 1$. If $\sum_{i=1}^{\infty} \delta_n < \infty$ and $\sum_{i=1}^{\infty} b_n < \infty$, then the $\lim_{n\to\infty} a_n$ exists.*

3. Main result

For solving the multiple-set split feasibility problem, we assume that the following conditions are satisfied:

1. H_1 and H_2 are two real Hilbert spaces, $A : H_1 \to H_2$ is a bounded linear operator and $A^* : H_2 \to H_1$ is the adjoint of A.
2. $T_i : H_1 \to H_1$, $i = 1, 2, \dots$ is a countable family of multi-valued k_i-strictly pseudo-contractive mappings and for each $i \geq 1$, T_i is demi-closed at the origin.
3. $T : H_2 \to H_2$ is a uniformly L-Lipschitzian continuous and $(k, \{\mu_n\}, \{\varepsilon_n\}, \phi)$-total asymptotically strict pseudo-contractive mapping satisfying the following conditions:
   (i) $\sum_{i=1}^{\infty} \mu_n < \infty$; $\sum_{i=1}^{\infty} \varepsilon_n < \infty$.
   (ii) there exist constants $M > 0, M^* > 0$ such that $\phi(\lambda) \leq M^* \lambda^2, \forall\ \lambda \geq M$.

4. $K := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $Q := F(T) \neq \emptyset$.

5. For each $q \in K$, $T_i q = \{q\},\ for\ each\ i \geq 1$.

The set of solutions of the multiple set split feasibility problem will be denoted by

$$\mathcal{F}, \text{ i.e. } \mathcal{F} = \{x \in K : Ax \in Q\} = K \cap A^{-1}(Q).$$

THEOREM 3.1. *Let $H_1, H_2, A, A^*, T_i, T, K, Q, \kappa, \{\mu_n\}, \{\varepsilon_n\}, \phi$ and L satisfy conditions 1–5 above. Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_1 \in H_1 \text{ chosen arbitrary}, \\ x_{n+1} = \alpha_{0,n} y_n + \sum_{i=1}^{\infty} \alpha_{i,n} w_{i,n}, w_{i,n} \in T_i y_n, \\ y_n = x_n + \beta A^*(T^n - I)Ax_n,\ \forall\ n \geq 1, \end{cases} \tag{4}$$

where $\{\alpha_{i,n}\} \subset (k, 1)$ with $k = \sup_{i \geq 1} k_i \in (0,1)$ and $\beta > 0$ satisfy the following conditions:

(a) $\sum_{i=0}^{\infty} \alpha_{i,n} = 1$, *for each* $n \geq 1$,

(b) for each $i \geq 1, \liminf_{n\to\infty} \alpha_{0,n}\alpha_{i,n} \geq 0$,

(c) $\beta \in (0, \frac{1-k}{\|A\|^2})$.

If $\mathcal{F}$ is nonempty, then

(A) both $\{x_n\}$ and $\{y_n\}$ converge weakly to some point $q \in \mathcal{F}$.

(B) In addition, if there exists some positive integer m such that $T_m \in \{T_i\}_{i=1}^{\infty}$ is hemi-compact, then both $\{x_n\}$ and $\{y_n\}$ converge strongly to $q \in \mathcal{F}$.

Proof. We observe that by condition (3)(ii), and the fact that ϕ is a continuous and strictly increasing function, there exist constants $M > 0, M^* > 0$ such that

$$\phi(\lambda) \leq \phi(M) + M^*\lambda^2,\ \forall\ \lambda \geq 0. \tag{5}$$

To establish conclusion [A], we divide the proof into six steps.

Step 1: We prove that all the sequences $\{x_n\}, \{y_n\}$ *and* $\{w_{i,n}\}$ are bounded. and for each $q \in \mathcal{F}$, we also show that $\lim_{n\to\infty} \|x_n - q\| = \lim_{n\to\infty} \|y_n - q\|$.

For any given $q \in \mathcal{F}$, we have $q \in K$ and $Aq \in Q := F(T)$. By the assumption that for each $i \geq 1, T_i$ is a multi-valued strictly pseudo-contractive mapping, the fixed point set $F(T_i)$ is closed, and so is $K := \bigcap_{i=1}^{\infty} F(T_i)$. From equation (4), condition (5) and using Lemma 2.4, we have that for each $n \geq 1$ and $q \in \mathcal{F}$,

$$\begin{aligned} \|x_{n+1} - q\|^2 =& \|\alpha_{0,n}(y_n - q) + \sum_{i=1}^{\infty} \alpha_{i,n}(w_{i,n} - q)\|^2 \\ =& \alpha_{0,n}||(y_n - q)||^2 + \sum_{i=1}^{\infty} \alpha_{i,n}||w_{i,n} - q||^2 - \sum_{i=1}^{\infty} \alpha_{i,n}\alpha_{0,n}||y_n - w_{i,n}||^2 \\ &- \sum_{i,j=1, i\neq j}^{\infty} \alpha_{i,n}\alpha_{j,n}||w_{i,n} - w_{j,n}||^2 \end{aligned} \tag{6}$$

$$\begin{aligned}
&\le \alpha_{0,n}||(y_n-q)||^2+\sum_{i=1}^{\infty}\alpha_{i,n}||w_{i,n}-q||^2-\sum_{i=1}^{\infty}\alpha_{i,n}\alpha_{0,n}||y_n-w_{i,n}||^2\\
&\le \alpha_{0,n}||(y_n-q)||^2+\sum_{i=1}^{\infty}\alpha_{i,n}D\big(T_iy_n-T_iq\big)^2\\
&\quad-\sum_{i=1}^{\infty}\alpha_{i,n}\alpha_{0,n}||y_n-w_{i,n}||^2\\
&\le \alpha_{0,n}||(y_n-q)||^2+\sum_{i=1}^{\infty}\alpha_{i,n}\big(||y_n-q||^2+k||y_n-w_{i,n}||^2\big)\\
&\quad-\sum_{i=1}^{\infty}\alpha_{i,n}\alpha_{0,n}||y_n-w_{i,n}||^2\\
&=\sum_{i=0}^{\infty}\alpha_{i,n}||y_n-q||^2-\sum_{i=1}^{\infty}\alpha_{i,n}(\alpha_{0,n}-k)||y_n-w_{i,n}||^2\\
&=||y_n-q||^2-\sum_{i=1}^{\infty}\alpha_{i,n}(\alpha_{0,n}-k)||y_n-w_{i,n}||^2\le \|y_n-q\|^2 \quad \text{and}
\end{aligned}$$

$$\|y_n-q\|^2=\|x_n-q\|^2+2\beta\langle x_n-q,A^*(T^n-I)Ax_n\rangle+\beta^2\|A^*(T^n-I)Ax_n\|^2. \tag{7}$$

Notice that

$$\beta^2\|A^*(T^n-I)Ax_n\|^2\le\beta^2\|A\|^2\|(T^n-I)Ax_n\|^2. \tag{8}$$

Since $Aq\in F(T)$, and T is a $(k,\{\mu_n\},\{\varepsilon_n\},\emptyset)$-total asymptotically strict pseudo contractive mapping, we have

$$\begin{aligned}
&\langle x_n-q,A^*(T^n-I)Ax_n\rangle=\langle A(x_n-q),(T^n-I)Ax_n\rangle\\
&\quad=\langle A(x_n-q)+(T^n-I)Ax_n-(T^n-I)Ax_n,(T^n-I)Ax_n\rangle\\
&\quad=\langle (T^nAx_n-Aq),(T^n-I)Ax_n\rangle-||(T^n-I)Ax_n||^2\\
&\quad=\frac{1}{2}\big\{||T^nAx_n-Aq||^2+||(T^n-I)Ax_n||^2-||Ax_n-Aq||^2-||(T^n-I)Ax_n||^2\big\}\\
&\quad\le\frac{1}{2}\big\{||Ax_n-Aq||^2+k||(T^n-I)Ax_n||^2+\mu_n\phi\big(||Ax_n-Aq||\big)+\varepsilon_n\big\}\\
&\qquad+\frac{1}{2}\big\{||(T^n-I)Ax_n||^2-||Ax_n-Aq||^2\big\}-||(T^n-I)Ax_n||^2\\
&\quad=\frac{k-1}{2}||(T^n-I)Ax_n||^2+\frac{1}{2}\big\{\mu_n\phi\big(||Ax_n-Aq||\big)+\varepsilon_n\big\}\\
&\quad\le\frac{k-1}{2}\|(T^n-I)Ax_n\|^2+\frac{\mu_n}{2}\{M^*\|A\|^2\|x_n-q\|^2+\phi(M)\}+\frac{1}{2}\varepsilon_n. \qquad (9)
\end{aligned}$$

Using inequalities (8) and (9) in equation (7), we obtain

$$\begin{aligned}
\|y_n-q\|^2\le&(1+\beta\mu_nM^*\|A\|^2)\|x_n-q\|^2\\
&-\beta(1-k-\beta\|A\|^2)\|(T^n-I)Ax_n\|^2+\beta\{\mu_n\phi(M)+\varepsilon_n\}
\end{aligned}$$

By condition (c), $(1-k-\beta\|A\|^2)>0$, therefore we have

$$\|y_n-q\|^2 \leq (1+\beta\mu_n M^*\|A\|^2)\|x_n-q\|^2+\beta\{\mu_n\phi(M)+\varepsilon_n\}. \quad (10)$$

Substituting inequality (10) into equation (6), we have $a_{n+1}\leq(1+\delta_n)a_n+b_n$, where $a_n=\|x_n-q\|^2, \delta_n=\beta\mu_n M^*\|A\|^2$ and $b_n=\beta\{\mu_n\phi(M)+\varepsilon_n\}$. Applying condition (3) (i), we get that $\sum_{n=1}^{\infty}\delta_n<\infty$ and $\sum_{n=1}^{\infty}b_n<\infty$. So, from Lemma 2.7, we get $\lim_{n\to\infty}\|x_n-q\|$ exists. From equation (6) and inequality (10), we know that $\lim_{n\to\infty}\|y_n-q\|$ exists. Also, $\lim_{n\to\infty}\|x_n-q\|=\lim_{n\to\infty}\|y_n-q\| \ \forall\ q\in\mathcal{F}$. Hence, $\{x_n\}$ and $\{y_n\}$ are bounded. Furthermore, since for each i, T_i is a multi-valued k_i- strictly pseudo contractive, we have

$$\|w_{i,n}-q\|^2 \leq \big(\mathcal{H}(T_iy_n,T_iq)\big)^2 \leq \|y_n-q\|^2+k||y_n-w_{i,n}||^2$$

implying $\|w_{i,n}-q\| \leq \|y_n-q\|+\sqrt{k}||y_n-w_{i,n}||$. (11)

So, from (11) and the fact that $\{y_n\}$ is bounded, we get that $\{w_{i,n}\}$ is also bounded.

Step 2: Now we prove that for any $i\geq 1$, the following conditions hold:

$$\lim_{n\to\infty} d(y_n,T_iy_n)=0, \text{ and } \lim_{n\to\infty}\|T^nAx_n-Ax_n\|=0.$$

For any $q\in\mathcal{F}$, it follows from (3.1), (10) and Lemma 2.6, that for any $n\geq 1$ we have

$$\begin{aligned}\|x_{n+1}-q\|^2 \leq&(1+\delta_n)\|x_n-q\|+b_n-\beta(1-k-\beta\|A\|^2)\|(T^n-I)Ax_n\|^2\\ &+k\sum_{i=1}^{\infty}\alpha_{i,n}\|y_n-w_{i,n}\|^2-\sum_{i=1}^{\infty}\alpha_{i,n}\alpha_{0,n}\|y_n-w_{i,n}\|^2.\end{aligned}$$

So,

$$\begin{aligned}&\beta(1-k-\beta\|A\|^2)\|(T^n-I)Ax_n\|^2-k\sum_{i=1}^{\infty}\alpha_{i,n}\|y_n-w_{i,n}\|^2+\sum_{i=1}^{\infty}\alpha_{i,n}\alpha_{0,n}\|y_n-w_{i,n}\|^2\\ &\leq(1+\delta_n)\|x_n-q\|+b_n-\|x_{n+1}-q\|\to 0 \text{ as } n\to\infty. \quad (12)\end{aligned}$$

By conditions (b) and (c), and conclusion (12), we have

$$\lim_{n\to\infty}\|(T^n-I)Ax_n\|=0, \quad \lim_{n\to\infty}\|y_n-w_{i,n}\|=0 \ \forall\ w_{i,n}\in T_iy_n. \quad (13)$$

Hence, we have

$$\lim_{n\to\infty} d(y_n,T_iy_n)\leq \mathop{lim}_{n\to\infty}\|(y_n-w_{i,n})\|=0. \quad (14)$$

Step 3: We prove that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

From (14), we know that $\|w_{i,n}-y_n\|\to 0$ and $\|y_n-x_n\|=\beta\|A^*(T^n-I)Ax_n\|\to 0$, $n\to\infty$. Using convexity of $||\cdot||^2$, we have from (4)

$$\begin{aligned}||x_{n+1}-x_n||^2 &= ||\alpha_{0,n}(y_n-x_n)+\sum_{i=1}^{\infty}\alpha_{i,n}(w_{i,n}-x_n)||^2\\ &= ||\alpha_{0,n}\gamma(A^*(T^n-I)Ax_n)+\sum_{i=1}^{\infty}\alpha_{i,n}(w_{i,n}-x_n)||^2\\ &\leq \alpha_{0,n}||\gamma(A^*(T^n-I)Ax_n)||^2+\sum_{i=1}^{\infty}\alpha_{i,n}(||w_{i,n}-y_n||+||y_n-x_n||)^2. \quad (15)\end{aligned}$$

Hence from (15), we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{16}$$

Step 4: We prove that $\lim_{n\to\infty} \|TAx_n - Ax_n\| = 0$.

Since T is uniformly L-Lipschitzian continuous, (13) and (16) above hold, and we obtain

$$\begin{aligned} ||TAx_n - Ax_n|| \leq& ||TAx_n - T^{n+1}Ax_n|| + ||T^{n+1}Ax_n - T^{n+1}Ax_{n+1}|| \\ &+ ||T^{n+1}Ax_{n+1} - Ax_{n+1}|| + ||Ax_{n+1} - Ax_n|| \\ \leq& L||Ax_n - T^n Ax_n|| + (L+1)||A||||x_{n+1} - x_n|| \\ &+ ||T^{n+1}Ax_{n+1} - Ax_{n+1}|| \to 0 \text{ as } n \to \infty. \end{aligned} \tag{17}$$

Step 5: We prove that every weak-cluster point of the sequence $\{x_n\}$ and $\{y_n\}$, $q \in \mathcal{F}$.

Since $\{y_n\}$ is bounded and H_1 is reflexive, there exists a subsequence $\{y_{n_k}\} \subset \{y_n\}$ such that $y_{n_k} \rightharpoonup q \in H_1$. Hence from **Step 2**, $\lim_{k\to\infty} d(y_{n_k}, T_i y_{n_k}) = 0$. Using the fact that T_i is demi-closed at the origin, we get that $q \in F(T_i)$. Since $i \geq 1$ is arbitrary, we have $q \in K := \bigcap_{i=1}^{\infty} F(T_i)$. On the other hand, it follows from (4) and (13) that $x_{n_k} = y_{n_k} - \beta A^*(T^{n_k} - I)Ax_{n_k} \rightharpoonup q$. Since A is a bounded linear operator, this means that $Ax_{n_k} \rightharpoonup Aq$. Notice that by **Step 4**, we have $\lim_{k\to\infty} \|TAx_{n_k} - Ax_{n_k}\| = 0$ and by Lemma 2.5, T is demi-closed at the origin and so, $Aq \in F(T) = Q$. Hence, $q \in \mathcal{F}$.

Step 6: We prove that $x_n \rightharpoonup q$ and $y_n \rightharpoonup q$.

Suppose that there exists some subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup p$ with $p \neq q$. Using similar arguments as the ones above, we can prove that $p \in \mathcal{F}$. Hence from the conclusions of **Step 1** and the Opial's property of Hilbert space, we have

$$\begin{aligned} \liminf_{k\to\infty} \|x_{n_k} - q\| &< \liminf_{k\to\infty} \|x_{n_k} - p\| = \lim_{n\to\infty} \|x_n - p\| = \lim_{k\to\infty} \|x_{n_k} - p\| \\ &< \liminf_{k\to\infty} \|x_{n_k} - q\| = \lim_{n\to\infty} \|x_n - q\| = \liminf_{k\to\infty} \|x_{n_k} - q\|. \end{aligned} \tag{18}$$

This is a contradiction. Therefore, $x_n \rightharpoonup q$. By using (4), we have $y_n = x_n + \beta A^*(T^n - I)Ax_n \rightharpoonup q$. This completes the proof of conclusion (A).

Next, we prove conclusion (B). Without loss of generality, we may assume that T_1 is hemi-compact. From **Step 2**, we have that $d(y_n, T_1 y_n) \to 0$ as $n \to \infty$. Therefore, there exists a subsequence of $\{y_{n_k}\} \subset \{y_n\}$ such that $y_{n_k} \to t \in H_1$. Since $y_{n_k} \rightharpoonup q$, we have $q = t$ and so $y_{n_k} \to q \in \mathcal{F}$. By virtue of conclusions of **Step 1**, we have $\lim_{n\to\infty} \|x_n - q\| = \lim_{n\to\infty} \|y_n - q\| = 0$. That is, $\{y_n\}$ and $\{x_n\}$ both converge strongly to the point $q \in \mathcal{F}$. This completes the proof. □

Lemma 3.2. *Let H be a real Hilbert space. The following identity holds*

$$||tx + (1-t)y||^2 = t||x||^2 + (1-t)||y||^2 - t(1-t)||x-y||^2, \forall t \in [0,1], x, y \in H. \tag{19}$$

Remark 3.3. If $\{T_i\}$ is a family of single-valued strictly pseudocontractive self mappings of H_1, then Theorem 3.1 still holds. In fact we have the following result.

Theorem 3.4. *Let H_1 and H_2 be two real Hilbert spaces and $\{T_i\}_{i=1}^{\infty} : H_1 \to H_1$ be a family of single-valued strictly pseudocontractive mappings, and for each $i \geq 1, T_i$*

is demi-closed at 0*. Let* $H_1, H_2, A, A^*, T, K, Q, \kappa, \{\mu_n\}, \{\varepsilon_n\}, \phi$*, and* L *satisfy same conditions as in Theorem 3.1 and* $\{x_n\}$ *be the sequence generated by*

$$\begin{cases} x_1 \in H_1 \text{ chosen arbitrary} \\ x_{n+1} = \alpha_{0,n} y_n + \sum_{i=1}^{\infty} \alpha_{i,n} T_i y_n, \\ y_n = x_n + \beta A^*(T^n - I)Ax_n, \ \forall\, n \geq 1. \end{cases} \tag{20}$$

If $\mathcal{F} \neq \emptyset$*, then the conclusions of Theorem 3.1 still hold.*

Proof. Using Lemma 3.2, one can easily get that $\{x_n\}$, $\{y_n\}$ and $\{T_i y_n\}$ are all bounded. The rest of the proof follows as in the proof of Theorem 3.1. □

4. Application

Let H be a real Hilbert space, $S : H \to H$ be a nonexpansive mapping with $\mathcal{F} := F(S) \neq \emptyset$ and $T : H \to H$ be a nonexpansive mapping. The hierarchical variational inequality problem for a nonexpansive mapping S with respect to the nonexpansive mapping T is to find an $x^* \in \mathcal{F}$ such that

$$\langle x^* - Tx^*, x^* - x\rangle \leq 0 \ \ \forall\, x \in \mathcal{F}. \tag{21}$$

It is known that (21) is equivalent to the following fixed point problem:

$$\text{find } x^* \in \mathcal{F} \text{ such that } x^* = P_{\mathcal{F}}(Tx^*), \tag{22}$$

where $P_{\mathcal{F}}$ is the metric projection from H onto F. Letting $K = \mathcal{F}$ and $Q = F(P_{\mathcal{F}})$ (the fixed point set of $P_{\mathcal{F}}$) and $A = I$ (the identity mapping on H), then problem (22) is equivalent to the following multi-set split feasibility problem: find $x^* \in K$ such that $Ax^* \in Q$.

Lemma 4.1. *([1]) Let* K *be a closed and convex subset of a smooth Banach space* E*. Suppose that* $\{T_n\}_{n=1}^{\infty}$ *is a family of* λ*-strictly pseudocontractive mappings from* K *into* E *with* $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ *and* $\{\mu_n\}_{n=1}^{\infty}$ *is a real sequence in* $(0, 1)$ *such that* $\sum_{n=1}^{\infty} \mu_n = 1$*. Then the following conclusions hold:*

1. $G := \sum_{n=1}^{\infty} \mu_n T_n : K \to E$ *is a* $\lambda-$*strictly pseudocontractive mapping;*
2. $F(G) = \bigcap_{n=1}^{\infty} F(T_n)$*.*

Lemma 4.2. *([17]) Let* K *be a nonempty subset of a real* $2-$*uniformly smooth Banach space* E *and* $T : K \to K$ *be a* $\lambda-$*strict pseudo-contraction. For* $\alpha \in (0, 1)$*, define* $T_\alpha x = (1 - \alpha)x + \alpha Tx$*. Then, for* $\lambda \in (0, \frac{\alpha}{K^2})$*,* $T_\alpha : K \to K$ *is nonexpansive and* $F(T_\alpha) = F(T)$*.*

Recall that Hilbert spaces are 2-uniformly smooth, they are also smooth. For $\alpha \in (0, 1)$, let $\Lambda := T_\alpha x = (1-\alpha)x+\alpha Gx$, where G is defined as in Lemma 4.1. Using Lemmas 4.1 and 4.2, and Theorem 3.1, we obtain the following result.

Theorem 4.3. *Let H, S, T, Λ, K, Q be as defined. Let $\{x_n\}, \{y_n\}$ be the sequences defined by*

$$\begin{cases} x_1 \in H_1 \text{ chosen arbitrary} \\ x_{n+1} = \alpha_n y_n + (1-\alpha_n)\Lambda y_n, \\ y_n = x_n + \beta(T^n - I)x_n, \ \forall\ n \geq 1, \end{cases} \tag{23}$$

where $\{\alpha_n\} \subset (0,1)$ and $\beta > 0$ satisfy the following conditions:

(i) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$;

(ii) $\beta \in (0,1)$.

If $K \cap Q \neq \emptyset$, then $\{x_n\}$ converges weakly to a solution of the hierarchical variational inequality problem (21)*. In addition, if the mapping G is semi-compact, then both $\{x_n\}$ and $\{y_n\}$ converge strongly to a solution of the hierarchical variational inequality problem* (21)*.*

Proof. From Theorem 3.1, $F(G) \neq \emptyset$, so Λ is quasi-nonexpansive. Furthermore, since T is nonexpansive, it is uniformly $L-$Lipschitzan continuous and $(\kappa, \{\mu_n\}, \{\varepsilon_n\}, \phi)-$total asymptotically strict pseudocontractive with $L = 1$, $\mu_n \equiv 0$, $\varepsilon_n \equiv 0$, and $\phi = 0$. Therefore, all conditions of Theorem 3.1 are satisfied. Hence, the conclusions of Theorem 4.3 follow from Theorem 3.1. □

Remark 4.4. A look at the proof of Theorem 3.1 shows that the proof carries over to the case of multi-valued quasi nonexpansive maps. Consequently, Theorem 3.1 is an improvement and extension of Theorem 3.1 of Chang *et al* [9], and other important results in this direction of research.

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