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GENERALIZED RAYCHAUDHURI'S EQUATION FOR NULL HYPERSURFACES

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Abstract. Black hole kinematics and laws governing their event horizons in spacetimes are usually based on the expansion properties of families of null geodesics which generate such horizons. Raychaudhuri's equation is one of the most important tools in investigating the evolution of such geodesics. In this paper, we use the so-called Newton transformations to give a generalized vorticity-free Raychaudhuri's equation (Theorem 3.1), with a corresponding null global splitting theorem (Theorem 3.5) for null hypersurfaces in Lorentzian spacetimes. Two supporting physical models are also given.

1. Introduction

Expansion properties of families of null geodesics play a crucial role, both in the singularity theorems of general relativity – in which, for example, the so-called trapped surfaces are characterized by negative expansions for both ingoing and outgoing families of light rays, and in the study of black holes and the laws governing the evolution of their event horizons – with focus on the null geodesic congruences generating the horizons. More precisely, in the latter case Raychaudhuri's equation is very important in the proof of the following laws of black hole mechanics:

(a) *For a black hole of mass* $\mathbf{M}$*, with angular momentum* $\mathbf{J}$ *and area* $\mathbf{A}$*, the changes* $\delta\mathbf{M}$*,* $\delta\mathbf{J}$ *and* $\delta\mathbf{A}$ *in the above quantities are related by*
$$\delta\mathbf{M} = \frac{\kappa}{8\pi}\delta\mathbf{A} + \Omega\delta\mathbf{J},$$
where κ *is the surface gravity and* Ω *is the angular velocity.*

(b) *If the null energy condition is satisfied, then the surface area of a black hole cannot decrease,* $\delta\mathbf{A} > 0$ *(Stephen W. Hawking, 1971).*

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(c) *The surface gravity κ of a black hole cannot be reduced to zero within a finite advanced time.*

More details on these laws and their proofs can be found in [3, 8] and references therein.

In most cases, totally geodesic null hypersurfaces have played as models of time-independent event and isolated black hole horizons [4]. It is quite obvious that the geodesics in these horizons have zero expansions. If, in addition, the corresponding spacetimes are shear-free, then we deduce that these horizons are stationary [10]. In fact, the event horizons of Schwarzschild, Reissener, Kerr and the Cauchy horizon of Taub-NUT spacetimes are all stationary semi-Riemannian hypersurfaces (see [10] for more details). However, due to the fact that black holes are surrounded by a local mass distribution, then there is a significant difference in the structure of the surrounding region (horizon) of isolated black holes. Hence, in a recent paper [4] by K. L. Duggal, he used metric conformal symmetry to study a family of totally umbilical null hypersurfaces M as models of these horizons, in which the following first order vorticity-free Raychaudhuri's equation

$$E(\Theta_1) = -Ric(E,E) - \operatorname{tr}(\sigma^2) - \frac{\Theta_1^2}{2}, \tag{1}$$

where E is a null vector field, Θ_1 is the expansion, σ is the shear tensor and Ric denotes the Ricci tensor of M, played part.

In [6], the authors gave a null version of Raychaudhuri's equation and found integrability conditions for some specific distinguished structures.

In this paper, we generalize (1) and give two of its applications to null hypersurfaces in flat spacetimes. Null geometry of submanifolds has been studied by many researchers, including but not limited to the following: [5, 7, 9–14]. The paper is organized as follows. In Section 2 we give the basic preliminaries on null hypersurfaces as well as Newton transformations, needed for the rest of the paper. In Section 3 we derive a generalized Raychaudhuri's equation for null hypersurfaces in Lorentzian spacetimes. Finally, in Section 4 we give two physical models of null hypersurfaces in flat spacetimes to support our results in the previous section.

2. Preliminaries

Let (M^{n+1}, g) be a null hypersurface of a Lorentzian manifold $(\overline{M}^{n+2}, \overline{g})$, endowed with the following distributions: the radical distribution $\operatorname{Rad}(TM) = TM \cap TM^{\perp} = TM^{\perp}$ and a screen distribution $S(TM)$. Then the following decomposition of TM is well-known [5]: $TM = TM^{\perp} \perp S(TM)$, where TM and $TM^{\perp}$ denote the tangent and normal bundles of M respectively, and $\perp$ is the orthogonal direct sum. We will denote by $\Gamma(\Xi)$ the set of smooth sections of the vector bundle Ξ over M.

It is well-known from [5, p. 79] that for any null section E of $TM^{\perp}$, there exists a unique null section N of the *transversal vector bundle* $\operatorname{tr}(TM)$ on a coordinate neighbourhood $\mathcal{U} \subset M$ such that $g(E,N) = 1$ and $\overline{g}(N,N) = \overline{g}(N,Z) = 0$, for any

$Z \in \Gamma(S(TM))$. Then we have the following decomposition of $T\overline{M}$:

$$T\overline{M} = S(TM) \perp \{TM^{\perp} \oplus \mathrm{tr}(TM)\}.$$

It is important to note that the distribution $S(TM)$ is not unique, and is canonically isomorphic to the factor vector bundle $TM/TM^{\perp}$ by Kupeli [10]. Let P be the projection of TM on to $S(TM)$. Then the local Gauss-Weingarten equations of M are the following:

$$\begin{aligned} &\overline{\nabla}_X Y = \nabla_X Y + B(X,Y)N, \quad \overline{\nabla}_X N = -A_N X + \tau(X)N, \\ &\nabla_X PY = \nabla^*_X PY + C(X,PY)E, \quad \nabla_X E = -A^*_E X - \tau(X)E, \end{aligned}$$

for all $E \in \Gamma(TM^{\perp})$ and $N \in \Gamma(\mathrm{tr}(TM))$. ∇ and ∇^* are induced linear connections on TM and $S(TM)$, respectively, B is the local second fundamental form of M and C is the local second fundamental form on $S(TM)$. Furthermore, A_N and A^*_E are the shape operators on TM and $S(TM)$ respectively, and τ is a differential 1-form on TM. Note that ∇^* is a metric connection on $S(TM)$ while ∇ is generally not a metric connection. In fact, ∇ satisfies the following relation

$$(\nabla_X g)(Y,Z) = B(X,Y)\lambda(Z) + B(X,Z)\lambda(Y), \tag{2}$$

for all $X,Y,Z \in \Gamma(TM)$, where λ is a 1-form on TM given as $\lambda(\cdot) = \overline{g}(\cdot, N)$. It is well-known from [5] and [7] that B is independent of the choice of $S(TM)$ and it satisfies $B(X,E) = 0$, $\forall\, X \in \Gamma(TM)$.

The local second fundamental forms B and C are related to their shape operators by the following equations

$$\begin{aligned} g(A^*_E X, Y) &= B(X,Y), & \overline{g}(A^*_E X, N) &= 0, \\ g(A_N X, PY) &= C(X,PY), & \overline{g}(A_N X, N) &= 0, \end{aligned} \tag{3}$$

for all $X,Y \in \Gamma(TM)$. From equations (3) we deduce that A^*_E is $S(TM)$-valued, self-adjoint operator and satisfies $A^*_E E = 0$. Let $\overline{R}$ denote the curvature tensor of $\overline{M}$; then

$$\begin{aligned} \overline{g}(\overline{R}(X,Y)Z,E) = {} & (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \\ & + \tau(X)B(Y,Z) - \tau(Y)B(X,Z), \end{aligned} \tag{4}$$

for all $X,Y,Z \in \Gamma(TM)$.

Let $\{Z_i\}$, for $i = 1, \cdots, n$, be an orthonormal frame field of $S(TM)$ which diagonalizes A^*_E. Let us suppose that $k_1, \cdots, k_n$ are the respective eigenvalues (or principal curvatures). Then the r-th mean curvature S^*_r is given by [15]:

$$S^*_0 = 1, \quad \text{and} \quad S^*_r = \sigma_r(k_1, \cdots, k_n) = \sum_{1 \le i_1 < \cdots < i_r \le n} k_{i_1} \cdots k_{i_r}.$$

The characteristic polynomial of A^*_E is given by $\det(A^*_E - t\mathbb{I}) = \sum_{\alpha=0}^{n} (-1)^{\alpha} S^*_r t^{n-\alpha}$, where $\mathbb{I}$ is the identity in $\Gamma(TM)$. The normalized r-th mean curvature H^*_r of M is defined by $\binom{n}{r} H^*_r = S^*_r$ and $H^*_0 = 1$. Furthermore, M will be called r-maximal if

$$H^*_r = 0, \quad \forall\, r \in \{1, \cdots, n\}. \tag{5}$$

Let $\Theta_r = (-1)^r S^*_r$ be the *generalized null expansion*. The Newton transformations $T_r : \Gamma(S(TM)) \to \Gamma(S(TM))$, of A^*_E are given by the inductive formula (see [1,2] for more details)

$$T_0 = \mathbb{I}, \quad T_r = \Theta_r \mathbb{I} + A^*_E \circ T_{r-1}, \quad 1 \leq r \leq n. \tag{6}$$

By Cayley-Hamiliton theorem, we have $T_n = 0$. It is easy to know that T_r is also a self-adjoint linear operator with respect to A^*_E; besides, A^*_E and T_r can be simultaneously diagonalized. Also, it is easy to show that T_r satisfies the following properties:

$$\begin{aligned}
\mathrm{tr}(T_r) &= (n-r)\Theta_r, \\
\mathrm{tr}(A^*_E \circ T_r) &= -(r+1)\Theta_{r+1}, \\
\mathrm{tr}(A^{*2}_E \circ T_r) &= \Theta_1 \Theta_{r+1} - (r+2)\Theta_{r+2}, \\
\mathrm{tr}(T_r \circ \nabla_X A^*_E) &= -X(\Theta_{r+1}),
\end{aligned}$$

for all $X \in \Gamma(TM)$.

It is important to note that the operators T_r depend on the choice of the transversal bundle $\mathrm{tr}(TM)$ and the screen distribution $S(TM)$. Suppose a screen distribution $S(TM)$ changes to another screen $S(TM)'$. The following are some of the local transformation equations due to this change (see [5, p. 87] for more details):

$$\begin{aligned}
W'_i &= \sum_{j=1}^{n} W_i^j \left(W_j - \epsilon_j c_j E\right), \\
N'(X) &= N - \frac{1}{2} g(W,W)E + W, \\
A'^*_E X &= A^*_E X + B(X, N - N')E, \\
N'(X)\nabla'_X Y &= \nabla_X Y + B(X,Y)\{\frac{1}{2} g(W,W)E - W\},
\end{aligned} \tag{7}$$

for any X, $Y \in \Gamma(TM|_{\mathcal{U}})$, where $W = \sum_{i=1}^{n} c_i W_i$, $\{W_i\}$ and $\{W'_i\}$ are the local orthonormal bases of $S(TM)$ and $S(TM)'$ with respective transversal sections N and N' for the same null section B. Here c_i and W_i^j are smooth functions on $\mathcal{U}$ and $\{\epsilon_1, \cdots, \epsilon_n\}$ is the signature of the basis $\{W_1, \cdots, W_n\}$. Denote by ω the dual 1-form of W, characteristic vector field of the screen change, with respect to the induced metric $g = \overline{g}|_M$, that is, $\omega(X) = g(X,W)$, $\forall\ X \in \Gamma(TM)$. Consider an orthogonal basis $\{Z_i\}$, for $i \in \{1, \cdots, n\}$, which diagonalizes A'^*_E and A^*_E. Let k'_i and k_i be the eigenvalues corresponding to eigenvector Z_i. Then, from (7) we have $(k'_i - k_i)Z_i = -B(Z_i, W)E$, which shows that the eigenvalues change under the change of the screen distribution. Since the generalized expansion Θ_r depends on the eigenvalues k_i, i.e. $\Theta_r = (-1)^r S^*_r = (-1)^r \sigma_r(k_1, \cdots, k_n)$, then a change of N will cause a change in it. Now, let $\{\Theta, T_r\}$ and $\{\Theta', T'_r\}$ be two sets of the above objects under a change in N. Applying recurrence relation (6) and the fact that $T_r Z_i = (-1)^r S^{*i}_r Z_i$, we have

$$T'_r Z_i = \Theta'_r \mathbb{I} + (-1)^{r-1} S^{*i'}_{r-1} A^{*'}_E Z_i, \quad T_r Z_i = \Theta_r \mathbb{I} + (-1)^{r-1} S^{*i}_{r-1} A^*_E Z_i. \tag{8}$$

Subtracting the second relation in (8) from the first and using relation (7) with $X = Z_i$, we deduce that the operators T_r and T'_r are related by the following equation:

$$T'_r = T_r + (\Theta'_r - \Theta_r)\mathbb{I} + \theta_r A^*_E + B(T'_{r-1}, N - N')E,$$

where $\theta_r := (-1)^{r-1}(S^{*i'}_{r-1} - S^{*i}_{r-1})$.

It is easy to see that the tensor T_r is unique if and only if the null hypersurface M is totally geodesic. For more details on Newton transformations and their properties, we refer the reader to [1, 2] and many more references therein.

3. Generalized Raychaudhuri's equation

Raychaudhuri's equation is central to the study of gravitational focusing and spacetime singularities. The equation was first derived by Amal Kumar Raychaudhuri, in 1955, in order to describe gravitational focusing properties in cosmology.

In this section, a generalized Raychaudhuri's equation for null hypersurfaces is derived with the help of general relativity concepts and Newton transformations. A local null normal section E is called geodesic if $\overline{\nabla}_E E = 0$, for which the integral curves of E are called null geodesic generators [7]. This condition has been shown to have interesting geometrical and physical meanings and it also simplifies the algebraic computations. Moreover, it is well known from [7] that if U is a null normal section on M, then for all $p \in M$ one can scale U to be geodesic on a suitable neighborhood $\mathcal{U}$ of p.

Let us consider the *tidal force operator* (also known as the *Jacobi operator*) $\overline{R}_E : \Gamma(TM_{|\mathcal{U}}) \to \Gamma(TM_{|\mathcal{U}})$ (see [7, p. 103] for details) defined as follows

$$\overline{R}_E(X) = \overline{R}(X, E)E = \overline{\nabla}_{[E,X]}E - \overline{\nabla}_E \overline{\nabla}_X E.$$

It is easy to show that $\overline{R}_E$ is a symmetric operator and $\mathrm{tr}(\overline{R}_E) = \overline{Ric}(E, E)$. Let us define $R_E : \Gamma(TM_{|\mathcal{U}}) \to \Gamma(TM_{|\mathcal{U}})$ in the same way but using the induced connection ∇ instead of $\overline{\nabla}$. Then one can easily verify that $\overline{R}_E = R_E$. This allows us to define the tidal force in terms of the induced objects on a null hypersurface.

Consider, as in [7], the flux of E as a local congruence of null geodesic curves. It is known that the *vorticity tensor* ω is the antisymmetric part of $-A_E^*$ and the *shear tensor* σ is the trace-free of the symmetric part of $-A_E^*$. Since $-A_E^*$ is symmetric, then

$$\omega = 0, \qquad \sigma = -A_E^* - \frac{\Theta_1}{n}\mathbb{I}. \tag{9}$$

Then the first order vorticity-free Raychaudhuri's equation [7] for null hypersurfaces is given by

$$E(\Theta_1) = -Ric(E, E) - \mathrm{tr}(\sigma^2) - \frac{\Theta_1^2}{n}. \tag{10}$$

Note that if the screen distribution $S(TM)$ changes to another screen $S(TM)'$, then the shear tensors σ and σ' associated with $S(TM)$ and $S(TM)'$, respectively, are related by

$$\sigma' = \sigma + B(\cdot, W) \otimes E + \frac{\Theta_1 - \Theta_1'}{n}\mathbb{I}.$$

The uniqueness of the shear tensor holds if the null hypersurface M is totally geodesic.

The following theorem is a generalization of the first order vorticity-free Raychaudhuri's equation (10). It uses the divergence of the operator T_r as the vector field $\mathrm{div}^\nabla(T_r) \in \Gamma(TM)$ which is defined as

$$\mathrm{div}^\nabla(T_r) = \mathrm{tr}(\nabla T_r) = \sum_{\beta=0}^{n} (\nabla_{Z_\beta} T_r) Z_\beta.$$

THEOREM 3.1. *Let (M, g) be a null hypersurface in a Lorentzian manifold $(\overline{M}, \overline{g})$. Let E be a null generator of M, which is parameterized to be a geodesic, with a generalized null expansion Θ_r. Then, the generalized vorticity free Raychaudhuri's equation is given by*

$$E(\Theta_r) = -\mathrm{tr}(T_{r-1} \circ R_E) - \mathrm{tr}(\sigma^2 \circ T_{r-1}) - 2\frac{\Theta_1}{n}\mathrm{tr}(\sigma \circ T_{r-1}) - \frac{\Theta_1^2}{n},$$

for all $r \in \{1, \cdots, n\}$.

Proof. Using the recurrence relation (6), we have

$$\begin{aligned} g(\mathrm{div}^\nabla(T_r), X) = PX(\Theta_r) + g((\nabla_E A_E^*)T_{r-1}E, X) + g(\mathrm{div}^\nabla(T_{r-1}), A_E^* X) \\ + \sum_{i=1}^{n} g((\nabla_{Z_i} A_E^*)T_{r-1}Z_i, X), \end{aligned} \tag{11}$$

for any $X \in \Gamma(TM)$. But

$$\begin{aligned} g((\nabla_{Z_i} A_E^*)T_{r-1}Z_i, X) = g(T_{r-1}Z_i, (\nabla_{Z_i} A_E^*)X) + g(\nabla_{Z_i} A_E^*(T_{r-1}Z_i), X) \\ - g(\nabla_{Z_i}(A_E^* X), T_{r-1}Z_i) + g(A_E^*(\nabla_{Z_i} X), T_{r-1}Z_i) \\ - g(A_E^*(\nabla_{Z_i} T_{r-1}Z_i), X), \end{aligned} \tag{12}$$

for all $X \in \Gamma(TM)$. Applying (2) to (12) while considering the fact that A_E^* is screen-valued, we get

$$\begin{aligned} g((\nabla_{Z_i} A_E^*)T_{r-1}Z_i, X) = g(T_{r-1}Z_i, (\nabla_{Z_i} A_E^*)X) \\ - \lambda(X)B(Z_i, A_E^* \circ T_{r-1}Z_i). \end{aligned} \tag{13}$$

Furthermore, using (4), the first term on the right-hand side of (13) reduces to

$$\begin{aligned} g(T_{r-1}Z_i, (\nabla_{Z_i} A_E^*)X) = \overline{g}(\overline{R}(Z_i, X)T_{r-1}Z_i, E) + g((\nabla_X A_N)Z_i, T_{r-1}Z_i) \\ + \tau(X)B(Z_i, T_{r-1}Z_i) - \tau(Z_i)B(X, T_{r-1}Z_i), \end{aligned} \tag{14}$$

for any $X \in \Gamma(TM)$. Replacing (14) in (13) and plunging the resulting equation in (11) we get

$$\begin{aligned} g(\mathrm{div}^\nabla(T_r), X) = -\lambda(X)E(\Theta_r) + g((\nabla_E A_E^*)T_{r-1}E, X) + g(\mathrm{div}^\nabla(T_{r-1}), A_E^* X) \\ + \tau(X)\mathrm{tr}(A_E^* \circ T_{r-1}) - \lambda(X)\mathrm{tr}(A_E^{*2} \circ T_{r-1}) \\ - \sum_{i=1}^{n} \{\overline{g}(\overline{R}(Z_i, X)E, T_{r-1}Z_i) - \tau(Z_i)B(X, T_{r-1}Z_i)\}, \end{aligned} \tag{15}$$

for any $X \in \Gamma(TM)$. Finally, setting $X = E$ in (15), and using $A_E^* = -\sigma - \frac{\Theta_1}{n}\mathbb{I}$ and (9), to the resultant equation, we get the desired result. □

DEFINITION 3.2. Let M be a null hypersurface of $\overline{M}$ and let σ denote the shear tensor of M. Then M is said to be r-stationary if Θ_r vanishes and the operator $(\sigma^2 \circ T_r)$ is trace-free for all $r \in \{0, \cdots, n\}$.

The following result also holds.

Theorem 3.3. *Let (M, g) be a null hypersurface in a Lorentzian manifold $(\overline{M}, \overline{g})$. Let E be a null generator of M, which is parameterized to be a geodesic, with a generalized null expansion Θ_r. If M satisfies the r-th energy condition $\mathrm{tr}(T_{r-1} \circ R_E) \geq 0$ and the operator T_{r-1} has non-negative eigenvalues, then for all $r \in \{1, \cdots, n\}$, the following are equivalent*

(a) Θ_r vanishes along every null generator E of M which is parameterized to be a null geodesic;

(b) M is r-stationary;

(c) M is r-maximal in $\overline{M}$.

Proof. First note that when $\overline{M}$ is Lorentzian then the screen distribution is Riemannian. Since T_{r-1} has non-negative eigenvalues,, hence $\mathrm{tr}(\sigma^2 \circ T_{r-1}) \geq 0$. Now, (a) $\Leftrightarrow$ (b): If $\Theta_r = 0$ then we see from Theorem 3.1 that $\mathrm{tr}(\sigma^2 \circ T_{r-1}) \leq 0$ and therefore, $\mathrm{tr}(\sigma^2 \circ T_{r-1}) = 0$. Hence, M is r-stationary. Then, (b) $\Leftrightarrow$ (c): It follows immediately from (5). □

When $r = 1$ we deduce the following well-known result [10, p. 88].

Corollary 3.4 ([10]). *Let (M, g) be a null hypersurface in a Lorentzian manifold $(\overline{M}, \overline{g})$. Let E be a null generator of M, which is parameterized to be a geodesic. If M satisfies the energy condition $\mathrm{tr}(R_E) = Ric(E, E) \geq 0$ then the following are equivalent*

(a) Θ_1 vanishes along every null generator E of M which is parameterized to be a null geodesic;

(b) M is stationary;

(c) M is totally geodesic in $\overline{M}$.

Next, we give the generalized global null splitting theorem.

Theorem 3.5. *Let M be a null hypersurface in a spacetime $\overline{M}$ admitting a unique distinguished global structure $(S(TM), E)$ and satisfies the generalized null energy condition $\mathrm{tr}(T_{r-1} \circ R_E) \geq 0$ for every $E \in \Gamma(\mathrm{Rad}(TM))$, and $r = 1, 2, \cdots, n$, and the operator T_{r-1} has non-negative eigenvalues. Then, M is r-maximal if and only if its r-th null mean curvature S_r^* vanishes identically for all $E \in \Gamma(\mathrm{Rad}(TM))$.*

Proof. A proof is straightforward and therefore we leave it out. □

By setting $r = 1$ in Theorem 3.5, we deduce the following well-known result.

Corollary 3.6 ([7]). *Let M be a null hypersurface in a spacetime $\overline{M}$ admitting a unique distinguished global structure $(S(TM), E)$ and satisfies the null energy condition* $\mathrm{tr}(R_E) = Ric(E,E) \geq 0$ *for every* $E \in \Gamma(\mathrm{Rad}(TM))$. *Then, M is totally geodesic if and only if its first order null mean curvature S_1^* vanishes identically for all* $E \in \Gamma(\mathrm{Rad}(TM))$.

4. Physical models

A star which is heavier than a few solar masses has a tendance of collapsing under its gravitational attraction in a process called *gravitational collapse*, which occurs without achieving any equilibrium state. The outcome of such collapse is usually a black hole which covers the resulting space-time singularity and causal message from the singularity cannot reach the external observer at infinity. The Schwarzschild, Reissener, Kerr and Taub-NUT spacetimes [5, 7, 10] are nondegenerate semi-Riemannian manifolds satisfying $Ric(E,E) \geq 0$ for every null $E \in \Gamma(TM)$. The event horizons of Schwarzschild, Reissener, Kerr and the Cauchy holizon of Taub-NUT spacetimes are r-stationally semi-Riemannian hypersurfaces, and thus by Theorem 3.3 they are r-maximal (and particularly, totally geodesic [10]). Other examples are given hereunder.

Example 4.1 (**Slices of Minkowski null cone**). Consider the congruence formed by the generators of the null cone in a flat spacetime $\overline{M}^4$. The geodesics originate from a single point, say O, which we can consider to be at the origin of the Cartesian coordinate system x^α. Then, the corresponding null hypersurface can be represented as $M = \{t - a = 0 : \; a^2 = x^2 + y^2 + z^2\}$. By considering the parametric transformations $t = \rho$, $x = \rho \sin\theta \cos\phi$, $y = \rho \sin\theta \sin\phi$ and $z = \rho\cos\theta$, in which $y^\alpha = (\rho, \theta, \phi)$ are the intrinsic coordinates; ρ is an affine parameter on the null cone's generators, which moves with constant values of $\theta^A = (\theta, \phi)$, $A = 2, 3$. Then the null geodesics can be represented by

$$E^\alpha = -\partial_\alpha u = (1, \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta),$$

and the corresponding transversal vector field N^α lies in (t, a) plane and is given by

$$N^\alpha = \frac{1}{2}\partial_\alpha v = \frac{1}{2}(-1, \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$

The screen distribution $S(TM)$ is the 2-surface spanned by ∂_θ and ∂_ϕ, where

$$\partial_\theta = (0, \rho\cos\theta\cos\phi, \rho\cos\theta\sin\phi, -\rho\sin\theta),$$

and

$$\partial_\phi = (0, -\rho\sin\theta\sin\phi, \rho\sin\theta\cos\phi, 0).$$

Also, one can show that the line element induced on $S(TM)$ is

$$ds^2|_{S(TM)} = \mathrm{diag}(0, 0, a^2, a^2\sin^2\theta).$$

By straightforward calculations, we get $k_1 = k_2 = \frac{1}{a}$. Hence, $\Theta_1 = -\frac{2}{a}$ and $\Theta_2 = \frac{1}{a^2}$ (Gaussian curvature) and the corresponding Newton transformations are given by

$$T_1 = -\frac{2}{a}\mathbb{I} + \frac{1}{a}P \qquad \text{and} \qquad T_2 = \frac{1}{a^2}\mathbb{I} - A^*_E \circ T_1.$$

It is easy to see that both σ and ω are vanishing and hence, the corresponding generalized Raychaudhuris's equation is given by

$$E^\alpha(\Theta_r) + \frac{\Theta_1^2}{2} = 0, \quad r = 1, 2. \tag{16}$$

Note that $\omega = 0$ and $\mathrm{tr}(T_{r-1} \circ R_E) = 0$, for $r = 1, 2$. Furthermore, (16) indicates that $E^\alpha(\Theta_r) < 0$ and hence the generalized null expansion Θ_r decreases (or simply, the geodesics are focused) during the congruence's evolution.

Example 4.2 (**Null geodesics in Schwarzschild spacetime**). Let us consider the radial null geodesics of the Schwarzschild spacetime $\overline{M}^4$. The line element is given by

$$ds^2 = -\psi dt^2 + \psi^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2,$$

where $\psi = 1 - \frac{2m}{r}$ and m denotes the mass of the black hole. For $d\theta = d\phi = 0$, we have that the Schwarzschild line element of the metric reduces to

$$ds^2 = -\psi dt^2 + \psi^{-1} dr^2.$$

In this case, the displacements will be null if $ds^2 = 0$. Let

$$u = t - \overline{r} \quad \text{and} \quad v = t + \overline{r},$$

where

$$\overline{r} = \int \frac{1}{\psi} dr = r + 2m \ln\left|\frac{r}{2m} - 1\right|.$$

We can see that u = constant on outgoing null geodesics and v = constant on the ingoing null geodesics. Hence, $\{u = \text{constant}\}$ and $\{v = \text{constant}\}$ are null subspaces of $\overline{M}$. It is easy to see that the vector fields

$$E^\alpha_{\text{out}} = -\partial_\alpha u \ \text{ and } \ E^\alpha_{\text{in}} = -\partial_\alpha v,$$

are both null and satisfies the null geodesic condition, with $+r$ and $-r$ as the affine parameters for E^α_{out} and E^α_{in}, respectively. Let $M = \{u = \text{constant}\}$ or $M = \{u = \text{constant}\}$. The corresponding transversal vector fields are respectively given by $N^\alpha_{\text{out}} = \frac{1}{2}\partial_\alpha v$ and $N^\alpha_{\text{in}} = \frac{1}{2}\partial_\alpha u$. Also, $S(TM) = \mathrm{span}\{\partial_\theta, \partial_\phi\}$. Following simple calculations, we can see that

$$A^*_{E^\alpha} = \mathrm{diag}(\pm\frac{1}{\overline{r}}, \pm\frac{1}{\overline{r}}),$$

on $S(TM)$. Hence, $\Theta_1 = \pm\frac{2}{\overline{r}}$, where the + (respectively, -) represents the outgoing (respectively, ingoing) congruence, and $\Theta_2 = \frac{1}{\overline{r}^2}$ (Gaussian curvature). The Newton transformations for this system are given by

$$T_1 = \pm\frac{2}{a}\mathbb{I} + \frac{1}{a}P \qquad \text{and} \qquad T_2 = \frac{1}{a^2}\mathbb{I} - A^*_E \circ T_1.$$

Observe that $\sigma = 0$ and $\omega = 0$. Hence, the generalized Raychaudhuri's equation is given by $E^\alpha(\Theta_r) + \frac{\Theta_1^2}{2} = 0$, $r = 1, 2$. As in the previous example, we have $E^\alpha(\Theta_r) < 0$ and hence the generalized null expansion Θ_r decreases during the congruence's evolution. Notice that M is neither r-stationary nor r-maximal. Notice also that on the event horizon $M := \{r = 2m\}$ of the above spacetime, we have $\Theta_r = 0$ and $\sigma = 0$. Hence, M in this case is r-stationary and also r-maximal.

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