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ADDITION FORMULA FOR COMPLEMENTARY ERROR FUNCTION WITH ASSOCIATED INTEGRAL REPRESENTATIONS

Leila Moslehi and Alireza Ansari

Abstract. In this paper, using the Mellin transform of parabolic cylinder functions we present an addition formula for the complementary error function in terms of the Gaussian functions. Also, some inverse Laplace transforms of the complementary error functions are shown and new integral representations for the exponential integral and Bessel functions are given. Moreover, the solution of diffusion equation in finite domain is presented in terms of the theta functions.

1. Introduction

In the literature, the error function (probability integral) [9]

$$\mathrm{Erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-x^2} dx,$$

along with its complement (complementary error function)

$$\mathrm{Erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-x^2} dx,$$

have many applications in various branches of applied mathematics, such as probability, theory of errors, heat conduction and different branches of mathematical physics [9]. In this paper, we intend to develop the literature review on complementary error function and find some new integral representations for this function and other special functions. For this purpose, first by using the definition of parabolic cylinder function [10, p. 328]

$$D_{-\nu}(z) = \frac{e^{-z^2/4}}{\Gamma(\nu)} \int_0^\infty e^{-zt-t^2/2} t^{\nu-1} dt, \quad \Re(\nu) > 0,$$

and considering the complementary error function as a special case of it

$$D_{-1}(z) = e^{z^2/4} \sqrt{\frac{\pi}{2}}\, \mathrm{Erfc}\Big(\frac{z}{\sqrt{2}}\Big), \tag{1.1}$$

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we get an addition formula for the complementary error function. This formula is obtained by applying the one and two dimensional Mellin transforms [10, p. 397]

$$\mathcal{M}\{f(x);s\} = F(s) = \int_0^\infty x^{s-1} f(x)\,dx,$$
$$\mathcal{M}_2\{f(x,y);s,t\} = \int_0^\infty \int_0^\infty f(x,y) x^{s-1} y^{t-1}\,dx\,dy,$$

and associated properties

$$\mathcal{M}\{f'(x);s\} = -(s-1)F(s-1),$$
$$\mathcal{M}\{f(ax);s\} = \frac{1}{a^s}F(s), \quad a>0.$$

2. Main theorems

Before we state our results for the complementary error function, let us recall the Mellin transform of parabolic cylinder function $D_{-\nu}$ [10, p. 330]

$$\mathcal{M}\{e^{-x^2/4} D_{-\nu}(x);s\} = \pi^{1/2} 2^{-(s+\nu)/2} \frac{\Gamma(s)}{\Gamma\big((s+\nu+1)/2\big)}, \quad \Re(s)>0, \tag{2.1}$$

and find an integral addition formula for the parabolic cylinder function $D_{-\nu}(a+b)$ in special case of ν. In this sense, we use the approach of papers [1, 2, 13] for obtaining our result. Also, see the papers [6, 7] for other addition formulas in terms of the series representations for $\nu = n \in \mathbb{N}$.

Theorem 2.1. *For $|\arg(a)| < \pi/2$ and $|\arg(b)| < \pi/2$, the following addition formula holds for the complementary error function*

$$\mathrm{Erfc}(a+b) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} e^{-\csc^2(\theta)a^2 - \sec^2(\theta)b^2} d\theta. \tag{2.2}$$

Proof. We employ the function $e^{-(a+b)^2/4}D_{-\nu}(a+b)$ and set $a = r\cos^2(\phi)$ and $b = r\sin^2(\phi)$ to get the two dimensional Mellin transform of this function as follows

$$\begin{aligned}\int_0^\infty\!\!\int_0^\infty e^{-(a+b)^2/4} D_{-\nu}(a+b) a^{\alpha-1} b^{\beta-1}\,da\,db &= \mathrm{B}(\alpha,\beta)\mathcal{M}\{e^{-r^2/4}D_{-\nu}(r);\alpha+\beta\} \\ &= \frac{\Gamma(\alpha)\Gamma(\beta)\pi^{1/2}2^{-(\alpha+\beta+\nu)/2}}{\Gamma\big((\alpha+\beta+\nu+1)/2\big)},\end{aligned} \tag{2.3}$$

where B is the beta function given by $\mathrm{B}(\alpha,\beta) = \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$. Now, we consider the following integral and intend to get the two dimensional Mellin transform of it

$$I = \int_0^1 x^{\nu/2-1}(1-x)^{\nu/2-1} \frac{d}{d\xi}\big(e^{-\xi^2/4}D_{-\nu}(\xi)\big)_{\xi\to\frac{a}{x^{1/2}}} \frac{d}{d\xi}\big(e^{-\xi^2/4}D_{-\nu}(\xi)\big)_{\xi\to\frac{b}{(1-x)^{1/2}}}\,dx.$$

For this purpose, by using the definition of two dimensional Mellin transform we have

$$\begin{aligned}
&\int_0^1 x^{\nu/2-1}(1-x)^{\nu/2-1}\Big\{\int_0^\infty \frac{d}{d\xi}\big(e^{-\xi^2/4}D_{-\nu}(\xi)\big)_{\xi\to\frac{a}{x^{1/2}}} a^{\alpha-1}\,da\Big\}\\
&\qquad\times\Big\{\int_0^\infty \frac{d}{d\xi}\big(e^{-\xi^2/4}D_{-\nu}(\xi)\big)_{\xi\to\frac{b}{(1-x)^{1/2}}} b^{\beta-1}\,db\Big\}\,dx\\
&\qquad=\frac{\Gamma(\alpha)\Gamma(\beta)\pi 2^{-\alpha/2-\beta/2-\nu+1}}{\Gamma(\alpha/2+\nu/2)\Gamma(\beta/2+\nu/2)}\int_0^1 x^{\alpha/2+\nu/2-1}(1-x)^{\beta/2+\nu/2-1}\,dx\\
&\qquad=\frac{\Gamma(\alpha)\Gamma(\beta)\pi 2^{-\alpha/2-\beta/2-\nu+1}}{\Gamma(\alpha/2+\nu/2)\Gamma(\beta/2+\nu/2)}\mathrm{B}(\alpha/2+\nu/2,\beta/2+\nu/2)\\
&\qquad=\frac{\pi\Gamma(\alpha)\Gamma(\beta)2^{-\alpha/2-\beta/2-\nu+1}}{\Gamma(\alpha/2+\beta/2+\nu)}. \qquad (2.4)
\end{aligned}$$

At this point, in view of the relations (2.4) and (2.3) in special case $\nu = 1$, we get

$$e^{-(a+b)^2/4}D_{-1}(a+b) =$$

$$\frac{1}{\sqrt{2\pi}}\int_0^1 x^{-1/2}(1-x)^{-1/2}\frac{d}{d\xi}\big(e^{-\xi^2/4}D_{-1}(\xi)\big)_{\xi\to\frac{a}{x^{1/2}}}\frac{d}{d\xi}\big(e^{-\xi^2/4}D_{-1}(\xi)\big)_{\xi\to\frac{b}{(1-x)^{1/2}}}dx,$$

which by using the relation (1.1) leads to

$$\operatorname{Erfc}(a+b) = \frac{1}{\pi}\int_0^1 e^{-\frac{a^2}{x}-\frac{b^2}{1-x}}\frac{dx}{x^{1/2}(1-x)^{1/2}},$$

or equivalently, by setting $x = \sin^2(\theta)$ we have

$$\operatorname{Erfc}(a+b) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} e^{-\csc^2(\theta)a^2-\sec^2(\theta)b^2}d\theta. \quad ■$$

Remark 2.2. Throughout this paper, we denote the function $\kappa(\theta)$ as $\kappa(\theta) = \sec^2(\theta)$ or $\kappa(\theta) = \csc^2(\theta)$.

Using the above notation for $\kappa(\theta)$, we simplify the relation (2.2) for $a = 0$ or $b = 0$, and derive an integral representation for the complementary error function in the next corollary.

Corollary 2.3. *For complementary error function, the following integral representation holds*

$$\operatorname{Erfc}(a) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} e^{-\kappa(\theta)a^2}d\theta, \quad |\arg(a)| < \pi/2. \qquad (2.5)$$

Theorem 2.4. *For $|\arg(a)| < \pi/2$, the following representations hold for the products of error functions*

$$\operatorname{Erfc}^2(a) = \frac{4}{\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^{-\kappa(\theta)a^2}d\theta, \qquad (2.6)$$

$$\operatorname{Erf}(a)\operatorname{Erfc}(a) = \frac{2}{\pi}\int_0^{\frac{\pi}{4}} e^{-\kappa(\theta)a^2}d\theta - \frac{2}{\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^{-\kappa(\theta)a^2}d\theta. \qquad (2.7)$$

Proof. First, we represent the error function $\mathrm{Erf}^2(a)$ as a double integral on the region $0 \le s \le a$, $0 \le t \le a$

$$\mathrm{Erf}^2(a) = \frac{4}{\pi}\int_0^a\int_0^a e^{-s^2-t^2}\,ds\,dt,$$

and use the polar coordinates $s = r\sin^2(\theta)$ and $t = r\cos^2(\theta)$ to obtain

$$\begin{aligned}\mathrm{Erf}^2(a) &= \frac{4}{\pi}\int_0^{\frac{\pi}{4}}\int_0^{\frac{a}{\cos(\theta)}} re^{-r^2}dr\,d\theta + \frac{4}{\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{\frac{a}{\sin(\theta)}} re^{-r^2}dr\,d\theta\\ &= 1 - \frac{4}{\pi}\int_0^{\frac{\pi}{4}} e^{-\kappa(\theta)a^2}d\theta.\end{aligned}$$

Now, by applying the relation (2.2) and using the identity $\mathrm{Erfc}^2(a) = [1-\mathrm{Erf}(a)]^2$, we get the relation (2.6). Also, the relation (2.7) is obtained after a simple computation on identity $[\mathrm{Erfc}(a)+\mathrm{Erf}(a)]^2 = 1$. ∎

THEOREM 2.5. *For $|\arg(a)| < \pi/2$, the following representation holds for the cubic products of error functions*

$$\mathrm{Erf}^3(a) + \mathrm{Erfc}^3(a) = 1 - \frac{6}{\pi}\int_0^{\frac{\pi}{4}} e^{-\kappa(\theta)a^2}d\theta + \frac{6}{\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} e^{-\kappa(\theta)a^2}d\theta.$$

Proof. By expanding the identity $[\mathrm{Erfc}(a)+\mathrm{Erf}(a)]^3 = 1$ and applying the relation (2.7), we get the result. ∎

3. Identities involving the inverse Laplace transforms

In this section, using the results obtained in previous section we improve the existing tables for the inverse Laplace transforms of the complementary error functions. For example, see the handbooks [4, 12], [16, Sec. 3.7].

THEOREM 3.1. *The following representation holds for the inverse Laplace transform of complementary error function*

$$\begin{aligned}&\mathcal{L}^{-1}\Big\{\frac{1}{s}\,\mathrm{Erfc}\,\big(\sqrt{s+\sqrt{s}-a}+\sqrt{s+\sqrt{s}-b}\big);x\Big\}\\ &\quad= \frac{2}{\pi}\int_{\sin^{-1}(2/\sqrt{x})/2}^{\frac{\pi}{2}} e^{a\sec^2(\theta)+b\csc^2(\theta)}\,\mathrm{Erfc}\,\Big(\frac{2[\sec^2(\theta)+\csc^2(\theta)]}{\sqrt{x-\sec^2(\theta)-\csc^2(\theta)}}\Big)\,d\theta. \qquad (3.1)\end{aligned}$$

Proof. Using the complex inversion formula for the Laplace transform and applying the addition formula (2.2) for the complementary error function, we have

$$\begin{aligned}&\mathcal{L}^{-1}\Big\{\frac{1}{s}\,\mathrm{Erfc}\,\big(\sqrt{s+\sqrt{s}-a}+\sqrt{s+\sqrt{s}-b}\big);s\Big\}\\ &\quad= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Big[\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{1}{s}e^{-\sec^2(\theta)(s+\sqrt{s}-a)-\csc^2(\theta)(s+\sqrt{s}-b)+sx}\Big]\,ds\end{aligned}$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} e^{a\sec^2(\theta)+b\csc^2(\theta)} \times \Big[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{s} e^{-\left(\sec^2(\theta)+\csc^2(\theta)\right)\sqrt{s}+s\left(x-\sec^2(\theta)-\csc^2(\theta)\right)}\Big]\, ds\, d\theta.$$

At this point, by using the fact that $\mathcal{L}^{-1}\{e^{-a\sqrt{s}}/s;x\} = \mathrm{Erfc}(2a/\sqrt{x})$, we simplify the above relation as follows

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} e^{a\sec^2(\theta)+b\csc^2(\theta)}\,\mathrm{Erfc}\Big(\frac{2(\sec^2(\theta)+\csc^2(\theta))}{\sqrt{x-\sec^2(\theta)-\csc^2(\theta)}}\Big) H\big(x-\sec^2(\theta)-\csc^2(\theta)\big)\, d\theta,$$

where H is the Heaviside unite step function. Finally, the relation (3.1) can be easily derived after a simple computation. ∎

Theorem 3.2. *The following relation holds for the inverse Laplace transform of complementary error function*

$$\mathcal{L}^{-1}\{\mathrm{Erfc}\,\big((s-a)^{\alpha/2}\big);x\} = \frac{2e^{ax}}{\pi x}\int_0^{\frac{\pi}{2}} W\Big(-\alpha,0;-\frac{\kappa(\theta)}{x^{\alpha}}\Big)\, d\theta, \quad 0<\alpha<1, \tag{3.2}$$

where W is the Wright function given by [11]

$$W(\gamma,\beta;x) = \sum_{n=0}^{\infty}\frac{x^n}{n!\,\Gamma(\gamma n+\beta)}, \quad \gamma>-1, \beta\in\mathbb{C}. \tag{3.3}$$

Proof. By applying the inverse Laplace transform on the function Erfc $\big((s-a)^{\alpha/2}\big)$ and taking the relation (2.2) into account, we have

$$\mathcal{L}^{-1}\big\{\,\mathrm{Erfc}\,\big((s-a)^{\alpha/2}\big);x\big\} = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \mathcal{L}^{-1}\{e^{-\kappa(\theta)(s-a)^{\alpha}};x\}\, d\theta.$$

Now, if we apply the inverse Laplace transform of function $e^{-\kappa(\theta)(s-a)^{\alpha}}$ in terms of the Wright function

$$\mathcal{L}^{-1}\{e^{-\kappa(\theta)s^{\alpha}}\} = \frac{1}{x}W\Big(-\alpha,0;-\frac{\kappa(\theta)}{x^{\alpha}}\Big),$$

then we get

$$\mathcal{L}^{-1}\big\{\,\mathrm{Erfc}\,\big((s-a)^{\alpha/2}\big);x\big\} = \frac{2e^{ax}}{\pi x}\int_0^{\frac{\pi}{2}} W\Big(-\alpha,0;-\frac{\kappa(\theta)}{x^{\alpha}}\Big)\, d\theta. \quad ∎$$

Corollary 3.3. *The following identities hold for the inverse Laplace transform of complementary error function*

$$\mathcal{L}^{-1}\big\{\,\mathrm{Erfc}\,\big((s-a)^{1/3}\big);x\big\} = \frac{2e^{ax}}{\pi}\int_0^{\frac{\pi}{2}} \frac{2\kappa(\theta)}{3x^{5/3}} e^{-(2/27)\kappa^3(\theta)x^{-2}} \times \Big[3^{-1/3}\kappa(\theta)x^{-2/3}\mathrm{Ai}\Big(\frac{\kappa^2(\theta)x^{-4/3}}{3^{4/3}}\Big) - 3^{1/3}\mathrm{Ai}'\Big(\frac{\kappa^2(\theta)x^{-4/3}}{3^{4/3}}\Big)\Big]\, d\theta, \tag{3.4}$$

$$\mathcal{L}^{-1}\Big\{ \operatorname{Erfc}\big((s-a)^{1/4}\big); x\Big\} = \frac{e^{ax}}{\pi^{3/2}x^{3/2}} \int_0^{\frac{\pi}{2}} \kappa(\theta) e^{-\kappa^2(\theta)/(4x)}\, d\theta, \tag{3.5}$$

$$\mathcal{L}^{-1}\Big\{ \operatorname{Erfc}\big((s-a)^{1/6}\big); x\Big\} = \frac{2e^{ax}}{\pi} \int_0^{\frac{\pi}{2}} \frac{\kappa(\theta)}{3^{1/3}x^{4/3}} \operatorname{Ai}\Big(\frac{\kappa(\theta)x^{-1/3}}{3^{1/3}}\Big)\, d\theta, \tag{3.6}$$

where Ai(x) *is the Airy functions of first kind such that* [17]

$$\operatorname{Ai}(x) = \frac{1}{\pi}\int_0^{\infty} \cos(xt + t^3/3)\, dt.$$

Proof. If we use the following relation for the Wright function (3.3)

$$\frac{1}{z} W(-\alpha, 0; -z) = \alpha W(-\alpha, 1-\alpha; -z),$$

and apply the following identities for the special cases of this function [3, 11]

$$W(-1/2, 1/2; -z) = \frac{1}{\sqrt{\pi}} e^{-\frac{z^2}{4}},$$
$$W(-1/3, 2/3; -z) = 3^{2/3}\operatorname{Ai}(3^{-1/3}z),$$
$$W(-2/3, 1/3; -z) = e^{-(2/27)z^3}[3^{-1/3}z\operatorname{Ai}(3^{-4/3}z^2) - 3^{1/3}\operatorname{Ai}'(3^{-4/3}z^2)],$$

the relation (3.2) is simplified to the identities (3.4)–(3.6) for $\alpha = 2/3, 1/2, 1/3$, respectively. ∎

Function	Laplace transform
$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{x^{\nu\kappa(\theta)-1}}{\Gamma(\nu\kappa(\theta))}\, d\theta$	$\operatorname{Erfc}\big(\sqrt{\nu\ln(s)}\big),\quad \Re(\nu) > 0$
$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \big(\frac{x}{a\kappa(\theta)}\big)^{(\nu-1)/2} I_{\nu-1}(2\sqrt{a\kappa(\theta)x})\, d\theta$	$\frac{1}{s^\nu}\operatorname{Erfc}\big(\frac{a}{\sqrt{s}}\big),\quad \Re(\nu) > 0$
$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} [\frac{1}{\sqrt{\pi x}} e^{-\kappa^2(\theta)/(4x)} - be^{b\kappa(\theta)+b^2x}\operatorname{Erfc}(\frac{\kappa(\theta)}{2\sqrt{x}} + b\sqrt{x})]\, d\theta$	$\frac{1}{\sqrt{s}+b}\operatorname{Erfc}(\sqrt[4]{s})$
$\frac{x^{-\nu-1}}{2^{\nu-1/2}\pi^{3/2}}\int_0^{\frac{\pi}{2}} e^{-\kappa^2(\theta)/(8x)} D_{2\nu+1}(\frac{\kappa(\theta)}{\sqrt{2x}})\, d\theta$	$s^\nu \operatorname{Erfc}(\sqrt[4]{s})$
$\frac{2}{\pi}\int_{\cos^{-1}(1/\sqrt{x})}^{\frac{\pi}{2}} J_0\big(2\csc(\theta)\sqrt{x}\big)\, d\theta$	$\frac{1}{s}\operatorname{Erfc}(\sqrt{s} + \frac{1}{\sqrt{s}})$

Table 1. The inverse Laplace transform of complementary error function

Using the relations (2.6) and (2.7) for the quadratic products complementary error functions, all obtained results in this section can be rewritten for the functions $\operatorname{Erfc}^2(x)$ and $\operatorname{Erf}(x)\operatorname{Erfc}(x)$. It suffices to replace the interval $(0, \pi/2)$ with $(\pi/4, \pi/2)$ or $(0, \pi/4) \cup (\pi/4, \pi/2)$. For example, we can show the following corollary.

Corollary 3.4. *The following identities hold for the inverse Laplace transform of the quadratic products complementary error functions*

$$\mathcal{L}^{-1}\{\operatorname{Erfc}^2(s^{1/4}); x\} = \frac{2}{\pi^{3/2}x^{3/2}} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \kappa(\theta) e^{-\kappa^2(\theta)/(4x)}\, d\theta,$$

$$\mathcal{L}^{-1}\{\operatorname{Erf}(s^{1/4})\operatorname{Erfc}(s^{1/4});x\} = \frac{1}{\pi^{3/2}x^{3/2}}\int_0^{\frac{\pi}{4}}\kappa(\theta)e^{-\kappa^2(\theta)/(4x)}\,d\theta - \frac{1}{\pi^{3/2}x^{3/2}}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\kappa(\theta)e^{-\kappa^2(\theta)/(4x)}\,d\theta.$$

At the end of this section, we conclude that the simple relation (2.2) enables us to obtain various inverse Laplace transforms of complementary error functions with complicated arguments. For example, we draw Table 1 for different forms of complementary error functions and their inverse Laplace transforms. The necessary relations for obtaining the results have been taken from [16]. Also, the notations

$$J_\nu(z) = \frac{(\frac{z}{2})^\nu}{\Gamma(\nu+1/2)\Gamma(1/2)}\int_{-1}^{1}(1-t^2)^{\nu-1/2}\cos(zt)\,dt, \quad \Re(\nu) > -1/2,$$
$$I_\nu(z) = \frac{(\frac{z}{2})^\nu}{\Gamma(\nu+1/2)\Gamma(1/2)}\int_{-1}^{1}(1-t^2)^{\nu-1/2}\cosh(zt)\,dt, \quad \Re(\nu) > -1/2,$$

are the Bessel function of first kind and the modified Bessel function of first kind, respectively.

4. Integral representations for exponential integral and Bessel functions

Definition 4.1. The exponential integral function Ei is defined as [8, p. 883, 8.211.1]

$$-\operatorname{Ei}(-s) = \int_s^\infty \frac{e^{-t}}{t}\,dt, \quad s > 0.$$

Theorem 4.2. *For all* $|\arg(s)| \le \pi/2$*, the following representations hold for the exponential integral function*

$$-\operatorname{Ei}(-s) = \frac{1+s}{s^2}e^{-s} - \frac{8}{\pi s^2}\int_0^{\frac{\pi}{2}} K_1(s\sqrt{\kappa(\theta)})\,d\theta, \tag{4.1}$$

$$-\operatorname{Ei}(-s) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} K_0\big(s\sqrt{\kappa(\theta)}\big)\,d\theta, \tag{4.2}$$

where K_ν *is the modified Bessel function of second kind* [8, *p.* 917, 8.432(1)]

$$K_\nu(z) = \int_0^\infty e^{-z\cosh(t)}\cos(\nu t)\,dt, \quad |\arg(z)| < \frac{\pi}{2} \text{ or } \Re(z) = 0,\ \nu = 0.$$

Proof. By considering the relation (2.2) and using the following integral for the error function [14, p. 106, no. 16]

$$\int_0^\infty x e^{-\frac{s^2}{4x^2}}\operatorname{Erfc}(x)\,dx = \frac{1+s}{4}e^{-s} + \frac{s^2}{4}\operatorname{Ei}(-s), \quad |\arg(s)| \le \pi/2,$$

we get

$$\frac{1+s}{4}e^{-s} + \frac{s^2}{4}\operatorname{Ei}(-s) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\int_0^\infty x e^{-x^2\kappa(\theta)}e^{-\frac{s^2}{4x^2}}\,dx\,d\theta.$$

Now, using the following integral representation for the modified Bessel function of second kind [8, p. 370, 3.478(4)]

$$\int_0^\infty x e^{-\beta x^2-\gamma/x^2}\,dx=\sqrt{\frac{\gamma}{\beta}}\,K_1(2\sqrt{\beta\gamma}),\quad \Re(\beta),\Re(\gamma)>0,$$

we obtain (4.1). For the proof of relation (4.2), we begin with the following identities [14, p. 106, no. 17], [8, p. 370, 3.478(4)]

$$\int_0^\infty e^{-s^2/(4x^2)}\operatorname{Erfc}(x)\,\frac{dx}{x}=-\operatorname{Ei}(-s),\quad \Re(s)>0,$$
$$\int_0^\infty e^{-\beta x^2-\gamma/x^2}\,\frac{dx}{x}=K_0(2\sqrt{\beta\gamma}),\quad \Re(\beta),\Re(\gamma)>0,$$

to get

$$-\operatorname{Ei}(-s)=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\int_0^\infty e^{-\kappa(\theta)x^2-s^2/(4x^2)}\,\frac{dx}{x}=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}K_0\big(\sqrt{\kappa(\theta)}s\big)\,d\theta. \quad ■$$

Theorem 4.3. *For* $|\arg(a)|\le\pi/2$ *and* $|\arg(b)|\le\pi/2$, *the following addition formula holds for the exponential integral function*

$$-\operatorname{Ei}\big(-(a+b)\big)=\frac{4}{\pi^2}\int_0^{\frac{\pi}{2}}\int_0^\infty K_{i\tau}\big(\sqrt{\kappa(\theta)}a\big)K_{i\tau}\big(\sqrt{\kappa(\theta)}b\big)\,d\tau\,d\theta.$$

Proof. Using the following relation for the addition formula of modified bessel function of second kind [8, p. 749, 6.791(4)]

$$\frac{2}{\pi}\int_0^\infty K_{i\tau}(a)K_{i\tau}(b)\,d\tau=K_0(a+b),$$

and applying the identity (4.2), we get the result. ■

Theorem 4.4. *The following relations hold for the Bessel functions*

$$K_{1/4}(s)=2e^{-s}\int_0^{\frac{\pi}{2}}\kappa(\theta)e^{2s\kappa^2(\theta)}\operatorname{Erfc}\big(\sqrt{2s}\kappa(\theta)\big)\,d\theta, \tag{4.3}$$
$$\boldsymbol{H}_0(s)=Y_0(s)+\frac{4}{\pi^2}\int_0^{\frac{\pi}{2}}K_0(s\sqrt{\kappa(\theta)+1})\,d\theta, \tag{4.4}$$

where Y_0 *is the Bessel function of second kind* [8, *p.* 914, 8.415(1)]

$$Y_0(z)=-\frac{2}{\pi}\int_1^\infty\frac{\cos(zt)}{\sqrt{t^2-1}}\,dt,$$

and $\boldsymbol{H}_0(z)$ *is the Struve function of zero order given by* [8, *p.* 942, 8.550(1)]

$$\boldsymbol{H}_0(z)=\sum_{n=0}^\infty\frac{(-1)^n(z/2)^{2n+1}}{\Gamma^2(n+3/2)}.$$

Proof. Similar to the proof of previous theorem, we use the following integral identities [15, p. 174, 3.7.2(17)], [15, p. 30, 2.2.1(15)]

$$\mathcal{L}\{\mathrm{Erfc}(\sqrt[4]{x}); s\} = \frac{1}{s} - \frac{1}{2\sqrt{\pi}s^{3/2}} e^{1/(8s)} K_{1/4}\Big(\frac{1}{8s}\Big),$$
$$\mathcal{L}\{e^{-a\sqrt{x}}; s\} = \frac{1}{s} - \frac{a\sqrt{\pi}}{2s^{3/2}} e^{a^2/(4s)}\,\mathrm{Erfc}\Big(\frac{a}{2\sqrt{s}}\Big),$$

to get the relation (4.3). Also, we use the following relations [15, p. 180, 3.7.4(13)], [15, p. 31, 2.2.2(1)]

$$\mathcal{L}\Big\{\frac{1}{x}e^{-1/x}\,\mathrm{Erfc}(1/\sqrt{x}); s\Big\} = \pi[\boldsymbol{H}_0(2\sqrt{s}) - Y_0(2\sqrt{s})],$$
$$\mathcal{L}\Big\{\frac{1}{x}e^{-a/x}; s\Big\} = 2K_0(2\sqrt{as}),$$

to obtain (4.4). ∎

5. Application to diffusion equations in finite domains

One of the important representations of addition formula for the complementary error function occurs in diffusion equations. We consider the following one-dimensional diffusion equation in a finite medium $0 < x < a$ with the concentration function $u(x,t)$

$$u_t = \kappa u_{xx}, \quad t > 0,\ 0 < x < a, \qquad (5.1)$$
$$u(x,0) = 0, \quad u(a,t) = u_0, \quad u_x(0,t) = 0.$$

By applying the Laplace transform on the both sides of equation (5.1)

$$\bar{u}(x,s) = \int_0^\infty e^{-st} u(x,t)\,dt,$$

and using the boundary conditions we get the solution as [5]

$$\bar{u}(x,s) = \frac{u_0 \cosh(x\sqrt{s/k})}{s\cosh(a\sqrt{s/k})},$$

which can be rewritten in the following form

$$\begin{aligned}\bar{u}(x,s) &= \frac{u_0}{s}\frac{e^{x\sqrt{s/k}} + e^{-x\sqrt{s/k}}}{e^{a\sqrt{s/k}} + e^{-a\sqrt{s/k}}}\\ &= \frac{u_0}{s}\sum_{n=0}^{\infty}(-1)^n\big[e^{-\sqrt{s/k}[(2n+1)a-x]} + e^{-\sqrt{s/k}[(2n+1)a+x]}\big].\end{aligned}$$

Now, by applying the inverse Laplace transform, the final solution is given by

$$u(x,t) = u_0\sum_{n=0}^{\infty}(-1)^n\Big[\,\mathrm{Erfc}\Big(\frac{(2n+1)a-x}{2\sqrt{kt}}\Big) + \mathrm{Erfc}\Big(\frac{(2n+1)a+x}{2\sqrt{kt}}\Big)\Big].$$

At this point, we intend to present another representation for the above solution in terms of the theta functions. We recall the definition of theta function [12]

$$\theta(x,q) = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos(2nx),$$

and employ the addition formula (2.2) to show the solution in a different form as

$$\begin{aligned} u(x,t) &= \frac{2u_0}{\pi}\sum_{n=0}^{\infty}(-1)^n \int_0^{\frac{\pi}{2}} e^{-n^2\csc^2(\theta)/(kt)} \\ &\quad\times\left[e^{-(a-x)^2\sec^2(\theta)/(4kt)} + e^{-(a+x)^2\sec^2(\theta)/(4kt)}\right]d\theta \\ &= \frac{u_0}{\pi}\int_0^{\frac{\pi}{2}}\left[\theta(0, e^{-\csc^2(\phi)/(kt)}) + 1\right] \\ &\quad\times\left[e^{-(a-x)^2\sec^2(\phi)/(4kt)} + e^{-(a+x)^2\sec^2(\phi)/(4kt)}\right]d\phi. \quad \blacksquare \end{aligned}$$

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Department of Applied Mathematics, Faculty of Mathematical Sciences, Shahrekord University, P.O.Box 115, Shahrekord, Iran

E-mail: leilamoslehi@yahoo.com, alireza_1038@yahoo.com