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SOME REMARKS ON PARAMEDIAL SEMIGROUPS

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Abstract. Semigroups satisfying some type of generalized commutativity were considered in quite a number of papers. S. Lajos, A. Nagy and M. Yamada dealed with externally commutative semigroups. N. Stevanović and P. V. Protić in [Structure of weakly externally commutative semigroups, Algebra Colloq. 13:3 (2006) 441-446], introduced the notion of weakly externally commutative semigroup and gave a structural description for some subclasses of this class of semigroups. In this paper we consider a class which is a generalization of the class of externally commutative semigroups.

1. Introduction

A semigroup S in which the following holds

$$(\forall x, y, z \in S) \quad xyz = zyx$$

is an *externally commutative semigroup* [4]. The class of externally commutative semigroups appears as a natural generalization of the class of commutative semigroups.

Now we are going to introduce the concept of *paramedial semigroups* as a generalization of externally commutative semigroups.

DEFINITION 1. A semigroup S is a *paramedial semigroup* if the paramedial law

$$(\forall a, b, c, d \in S) \quad abcd = dbca$$

holds in S.

If S is an externally commutative semigroup, then $abcd = dbca$ for all $a, b, c, d \in S$, thus S is a paramedial semigroup. Hence, the class of externally commutative semigroups is a subclass of the class of paramedial semigroups.

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Let S be a paramedial semigroup, $a, c \in S$ and $b \in S^2$; then $abc = cba$. It follows by above that if S is a paramedial semigroup, then S^2 is an externally commutative subsemigroup of S.

A semigroup S is a *universal* (*global idempotent*) semigroup if it satisfies $S^2 = S$. Therefore a universal paramedial semigroup S is an externally commutative semigroup. Also, by [8, Proposition 1.1], an universal externally commutative semigroup is commutative. Hence, each universal paramedial semigroup is commutative.

The concept of weakly externally commutative semigroup was introduce in [7].

Definition 2. [7] A semigroup S in which the following holds

$$(\exists a \in S)(\forall x, y \in S) \quad xay = yax,$$

is a *weakly externally commutative semigroup.*

It is clear from the definition that the class of paramedial semigroups is included in the class of weakly externally commutative semigroups.

2. Some general properties of paramedial semigroups

Lemma 1. *Let S be a simple paramedial semigroup. Then $E(S) \neq \emptyset$.*

Proof. Since S is a simple semigroup, it follows that $S = SaS$ for all $a \in S$. Now, for $x \in S$, from $x^2 \in Sx^4S$ it follows that $x^2 = ux^4v$ for some $u, v \in S$. Consequently,

$$\begin{aligned}(ux^2v)^2 &= ux^2vux^2v = uxxvux^2v = uuxvxxxv \\ &= uuxxxxvv = u(ux^4v)v = ux^2v.\end{aligned}$$

Hence, $ux^2v \in E(S)$. ∎

If S is a semigroup, then $C(S) = \{a \in S \mid (\forall x \in S)\ xa = ax\}$ is the well known *center* of S.

Lemma 2. *Let S be a paramedial semigroup. If $E(S) \neq \emptyset$ then $E(S)$ is a semilattice and $E(S) \subseteq C(S)$.*

Proof. Let $e, f \in E(S)$ be arbitrary elements. Then $ef = eeef = feee = fe$ and so

$$(ef)^2 = efef = eeff = ef.$$

Consequently, $E(S)$ is a commutative subsemigroup of S.

Let $e \in E(S)$ and $x \in S$ be arbitrary elements, then $ex = eeex = xeee = xe$, hence $E(S) \subseteq C(S)$. ∎

Lemma 3. *Let S be a paramedial semigroup. Then S^3 is a commutative semigroup.*

Proof. Let $a, b \in S^3$. Then there exist elements $x, y, z, u, v, w \in S$ such that $a = xyz, b = uvw$. Now it follows that

$$ab = xyzuvw = uvyzxw = uyzxvw = uvzxyw = uvwxyz = ba. \quad \blacksquare$$

Lemma 4. *Let S be a paramedial semigroup and $x, y \in S$ be its arbitrary elements. Then $(xy)^2 = y^2x^2$ and for $n \in N$ and $n \geq 3$ it follows that*

$$(xy)^n = x^ny^n = y^nx^n = (yx)^n. \tag{1}$$

Proof. Let $x, y \in S$. Then

$$(xy)^2 = xyxy = yyxx = y^2x^2.$$

We are going to prove the second part of lemma by induction. For $n = 3$ it follows

$$(xy)^3 = xyxyxy = xy(xyxy) = xyyyxx = yyyxxx = y^3x^3.$$

By Lemma 3, it follows that $x^3y^3 = y^3x^3$. Hence, $(xy)^3 = x^3y^3 = y^3x^3 = (yx)^3$.

Let $(xy)^n = x^ny^n = y^nx^n = (yx)^n$. Now we get

$$(xy)^{n+1} = (xy)^nxy = x^ny^nxy = yy^nxx^n = y^{n+1}x^{n+1}.$$

Since S^n, $n \geq 3$, is a commutative semigroup, it follows that $x^{n+1}y^{n+1} = y^{n+1}x^{n+1}$, which gives $(xy)^{n+1} = x^{n+1}y^{n+1} = y^{n+1}x^{n+1} = (yx)^{n+1}$ and the lemma is proved. $\blacksquare$

By above, if S is a paramedial semigroup, $m, n \in N$, $x_1, x_2, \ldots, x_m \in S$, then

$$(x_1x_2\cdots x_m)^n = (x_{p(1)}x_{p(2)}\cdots x_{p(m)})^n = x^n_{p(1)}x^n_{p(2)}\cdots x^n_{p(m)},$$

where $\{p(1), p(2), \ldots, p(n)\}$ is a permutation of $\{1, 2, \ldots, n\}$.

A semigroup S is a (well known) E-m-semigroup if $(xy)^m = x^my^m$, $m \geq 2$ holds for some $m \in N$ and for all $x, y \in S$.

By the above lemma, every paramedial semigroup is an E-m-semigroup for all $m \geq 3$.

3. Semilattice decomposition of paramedial semigroups

Theorem 1. *Let S be a paramedial semigroup. Then the relation ρ defined on S by*

$$a\rho b \iff (\forall x, y \in S)(\exists m, n \in N)\ xa^my \in xbyS,\ xb^ny \in xayS$$

is a semilattice congruence.

Proof. Let $a, x, y \in S$. Since S is a paramedial semigroup, then

$$xaty = xaaaay = xayaaa \in xayS$$

and so $a\rho a$. Clearly, ρ is a symmetric relation. Let $a, b, c \in S$ and

$$a\rho b \iff (\forall x, y \in S)(\exists m, n \in N)\ xa^m y \in xbyS,\ xb^n y \in xayS,$$
$$b\rho c \iff (\forall x, y \in S)(\exists p, q \in N)\ xb^p y \in xcyS,\ xc^q y \in xbyS.$$

Now

$$\begin{aligned} xa^{m(p+1)}y &= x\underbrace{a^m \dots a^m}_{p+1} y = xa^m y \underbrace{a^m \dots a^m}_{p} \subseteq xbyS \underbrace{a^m \dots a^m}_{p} \\ &= xa^m yS \underbrace{a^m \dots a^m}_{p-1} b \subseteq xbySS \underbrace{a^m \dots a^m}_{p-1} b = xa^m ySS \underbrace{a^m \dots a^m}_{p-2} b^2 \\ &\subseteq xbySSS \underbrace{a^m \dots a^m}_{p-2} b^3 = \dots = xa^m \underbrace{SS \dots S}_{p} b^p \subseteq xby \underbrace{SS \dots S}_{p} b^p \\ &= xb^p y \underbrace{SS \dots S}_{p} b \subseteq xb^p yS \subseteq xcyS. \end{aligned}$$

Similarly, $xc^{q(n+1)} \subseteq xayS$. Hence, $a\rho c$ and so the relation ρ is transitive. It follows that ρ is an equivalence relation.

Let

$$a\rho b \iff (\forall x, y \in S)(\exists m, n \in N)\ xa^m y \in xbyS,\ xb^n y \in xayS$$

and $c \in S$ be an arbitrary element. Then, by Lemma 4,

$$\begin{aligned} x(ac)^{m+3}y &= xa^{m+3}c^{m+3} = xa^{m+3}c^{m+1}ccy = xa^{m+3}yc^{m+3} = xa^m aaayc^{m+3} \\ &= xa^m ya^3 c^{m+3} \in xbySa^3c^{m+2}c = xbcSa^3c^{m+2}y = xbcya^3c^{m+2}S \\ &\subseteq xbcyS. \end{aligned}$$

Similarly, $x(bc)^n y \in xacyS$. Hence, $ac\rho bc$ and so ρ is a left congruence on S. Also,

$$\begin{aligned} x(ca)^{n+3}y &= xc^{n+3}a^{n+3}y = xc^{m+1}cca^m a^3 y = xa^m cc^{m+2}a^3 y \\ &= xa^m yc^{m+2}a^3 c \subseteq xbySc^{m+2}a^3c = xcySc^{m+2}a^3b = xcbSc^{m+2}a^3y \\ &= xcbyc^{m+2}a^3S \subseteq xcbyS. \end{aligned}$$

Similarly, $x(cb)^n y \in xcayS$. Hence, $ca\rho cb$ and so ρ is a right congruence on S.

Hence, ρ is a congruence relation on S.

Let $a, x, y \in S$ be arbitrary elements. Then

$$xa^5y = xa^2aaay = xa^2yaaa \in xa^2yS,$$
$$x(a^2)^2y = xaaaay = xaya^3 \in xayS.$$

Now, $a\rho a^2$ and so ρ is a band congruence on S.

Let $a, b, x, y \in S$ be arbitrary elements. Then

$$x(ab)^3y = xaba(ba)by = xbabaaby = xbayaabb \in xbayS.$$

Similarly, $x(ba)^3y \in xabyS$. Hence, $ab\rho ba$ and it follows that ρ is a semilattice congruence on S. ■

Corollary 1. *If S is a paramedial semigroup, then S is a semilattice of Archimedean paramedial subsemigroups on S.*

Proof. Let S be a paramedial semigroup. Then the relation ρ defined as in the above theorem, is a semilattice congruence. We prove that ρ-classes are Archimedean semigroups. Hence, $S = \bigcup_{\alpha\in Y} S_\alpha$, Y is a semilattice and S_α are ρ-clases. Let $a, b \in S_\alpha$. Then

$$a\rho b \iff (\forall x, y \in S)(\exists m, n \in N)\ xa^m y \in xbyS,\ xb^n y \in xayS.$$

For $x = a = y$ it follows that $a^{m+2} \in abaS \subseteq SbS$. Now $a^{m+2} = ubv$ for some $u, v \in S$. Let $u \in S_\beta$, $v \in S_\gamma$. Since $a^{m+2} \in S_\alpha$ and $a^{m+2} = ubv \in S_\beta S_\alpha S_\gamma \subseteq S_{\beta\alpha\gamma}$, we have $\alpha = \beta\alpha\gamma$. Also, since Y is a semilattice, we have $\beta\alpha\gamma\beta = \alpha$, $\gamma\beta\alpha\gamma = \alpha$ and so $ubvu, vubv \in S_\alpha$. Now

$$a^{3(m+2)} = ubv \cdot ubv \cdot ubv = (ubvu)b(vubv) \in S_\alpha b S_\alpha.$$

Hence, S_α is an Archimedean semigroup. ■

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