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ON WEAK AND STRONG CONVERGENCE THEOREMS FOR TWO NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we consider an iteration process for approximating common fixed points of two nonexpansive mappings and prove some strong and weak convergence theorems for such mappings in uniformly convex Banach spaces.

1. Introduction

Let C be a non-empty subset of a real normed linear space E. Let $T\colon C \to C$ be a mapping, then we denote the set of all fixed points of T by $F(T)$. The set of common fixed points of two mappings S and T will be denoted by $F = F(S) \cap F(T)$. A mapping $T\colon C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|,$$

for all $x, y \in C$.

For the last thirty years, weak and strong convergence theorems for nonexpansive mappings have been established by many authors (see, e.g. [2], [3], [5], [11], [13], [15]–[17]). In 1995, Xu [20] introduced and studied the Mann and Ishikawa iteration schemes with errors. Since then, these schemes have been further investigated by a number of authors for approximating fixed points of nonlinear mappings. Recently Khan and Fakhar-ud-din [8] studied the following iterative scheme with errors involving two nonexpansive mappings

$$\begin{aligned} &x_1 \in C \\ &x_{n+1} = \alpha_n x_n + \beta_n S y_n + \gamma_n u_n \\ &y_n = \alpha'_n x_n + \beta'_n T x_n + \gamma'_n v_n, \end{aligned} \tag{1.1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}$, and $\{\gamma'_n\}$ are real sequences in [0,1] such that

$$\begin{aligned} &\alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n \\ &\sum_{n=1}^{\infty} \gamma_n < \infty, \quad \text{and} \quad \sum_{n=1}^{\infty} \gamma'_n < \infty, \end{aligned} \tag{1.2}$$

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and $\{u_n\}$ and $\{v_n\}$ are bounded sequences in C and obtained some weak and strong convergence theorems. Moreover, recently Shahzad and AL-Dubiban [16] studied the following iterative scheme without error terms to prove some weak and strong convergence results.

$$\begin{aligned} x_1 &\in C \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n S y_n \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, \end{aligned} \tag{1.3}$$

where $\{\alpha_n\}, \{\beta_n\}$ are real sequences in [0,1].

Motivated and inspired by the above work we will study the following iterative scheme without error terms.

$$\begin{aligned} x_1 &\in C \\ x_{n+1} &= (1-\alpha_n)Tx_n + \alpha_n S y_n \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, \end{aligned} \tag{1.4}$$

for all $n \in N$, where $\{\alpha_n\}, \{\beta_n\}$ are real sequences in [0,1].

REMARK 1.1. The process (1.4) is independent of (1.3); neither of them reduces to other. Following the method of Agarwal et al. [1], it can be shown that (1.4) converges faster than (1.3) for contractions.

We remark that once a convergence theorem has been proved for an iteration scheme without errors, such as (1.4), it is not always difficult to establish the corresponding result for the case with errors such as the results of Khan and Fakhar-ud-din [8] and Kim et al. [9] under the conditions (1.3). As pointed out by Chidume [3], if error terms satisfying (1.2) is introduced in either the Mann or the Ishikawa iterative scheme, the proofs of the results are basically unnecessary repetitions of the proofs when no error terms are added. Usually, we are interested, in mathematics, in simpler algorithms, unless the better rate of convergence or some other advantage is gained.

The purpose of this paper is use the iteration process (1.4) for approximating the common fixed point of two nonexpansive mappings (when such common fixed point exists)and to prove some strong and weak convergence theorems for such maps. Our results improve and extends the results of Khan and Fakhar-ud-din [8], Shahzad and AL-Dubiban [16] and many known results in the literatures.

2. Preliminaries

DEFINITION 2.1. Let E be a real Banach space. Then E is said to have the Kadec-Klee property if for every sequence $\{x_n\}$ in E, $x_n \to x$ weakly and $\|x_n\| \to \|x\|$ strongly together imply $\|x_n - x\| \to 0$.

DEFINITION 2.2. Two mappings $S, T\colon C \to C$, where C is a subset of a normed space E, are said to satisfy condition (B) if there exists a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that either $\|x - Sx\| \geq f(d(x,F))$ or $\|x - Tx\| \geq f(d(x,F))$ for all $x \in C$ where

$$d(x,F) = \inf\{\|x-p\| : p \in F\}.$$

Remark 2.3. Note that when $S = I$, the identity map, or $S = T$, Condition (B) reduces to condition (I) of Sentor and Dotson [13]. Our Condition (B) also contains Condition (A') of Khan and Fakhar-ud-din [8]. We further note that when $S = I$, Condition (A') of Khan and Fakhar-ud-din [8] does not reduces to Condition (I) of Senter and Dotson [13].

Definition 2.4. A mapping $T: C \to C$ is called (1) demicompact if any bounded sequence $\{x_n\}$ in C such that $\{x_n - Tx_n\}$ converges has a convergent subsequence; (2) semi-compact (or hemi-compact) if any bounded sequence $\{x_n\}$ in C satisfying $\|x_n - Tx_n\| \to 0$ as $n \to \infty$ has a convergent subsequence.

Remark 2.5. Every demicompact mapping is semi-compact but the converse is not true in general. It is known [12] that if $T: C \to C$ is nonexpansive and demicompact, then T satisfies Condition (I).

The following lemmas are needed in the sequel.

Lemma 2.6. (see, e.g. [18]) *Let $\{\lambda_n\}$ and $\{\sigma_n\}$ be sequences of non-negative real numbers such that $\lambda_{n+1} \le \lambda_n + \sigma_n$, for all $n \ge 1$ and $\sum_{n=1}^{\infty} \sigma_n < \infty$. Then $\lim_{n\to\infty} \lambda_n$ exists. Moreover, if there exists a subsequence $\{\lambda_{n_j}\}$ of $\{\lambda_n\}$ such that $\lambda_{n_j} \to 0$ as $j \to \infty$, then $\lambda_n \to 0$ as $n \to \infty$.*

Lemma 2.7. (see, e.g. [6]) *Let E be a real reflexive Banach space such that its dual E^* has the Kadec-Klee property. Let $\{x_n\}$ be a bounded sequence in E and $x^*, y^* \in \omega_w(x_n)$; here $\omega_w(x_n)$ denotes the w-limit set of $\{x_n\}$. Suppose $\lim_{n\to\infty} \|tx_n + (1-t)x^* - y^*\|$ exists for all $t \in [0,1]$. Then $x^* = y^*$.*

Lemma 2.8. (see, e.g. [9]) *Let K be a nonempty closed convex subset of a Banach space E. Let $S, T: C \to C$ be two nonexpansive mappings with $x^* \in F = F(S) \cap F(T)$. Suppose that $\{x_n\}$ is defined by (1.2) and that for every given n, a mapping $T_n: C \to C$ is defined by*

$$T_n x = \alpha_n x + \beta_n S[\alpha'_n x + \beta'_n Tx + \gamma'_n x] + \gamma_n x,$$

for all $x \in C$. If there are $\alpha_n, \alpha'_n \in [a,b]$ for some $a, b \in \mathbb{R}$ with $0 < a \le b < 1$, then $\{T_n T_{n-1} \cdots T_1 - x_{n+1}\}$ converges strongly to 0 as $n \to \infty$.

Lemma 2.9. (see, e.g. [14]) *Let E be a uniformly convex Banach space and $\{\alpha_n\}$ a sequence in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. Suppose $\{x_n\}$ and $\{y_n\}$ are sequences in E such that $\limsup_{n\to\infty} \|x_n\| \le r$, $\limsup_{n\to\infty} \|y_n\| \le r$ and $\limsup_{n\to\infty} \|\alpha_n x_n + (1-\alpha_n) y_n\| = r$ hold for some $r \ge 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

3. Main results

Lemma 3.1. *Let E be a real normed space and C a nonempty closed convex subset of E. Let $S, T: C \to C$ be two nonexpansive mappings with $x^* \in F = F(S) \cap F(T)$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$. For arbitrary $x_1 \in C$ define the sequence $\{x_n\}$ by (1.4). Then $\lim_{n\to\infty} \|x_n - x^*\|$ exists.*

Proof. Note that

$$\begin{aligned}\|x_{n+1}-x^*\| &= \|(1-\alpha_n)Tx_n+\alpha_n Sy_n - x^*\| \\ &\le (1-\alpha_n)\|Tx_n - x^*\| + \alpha_n\|Sy_n - x^*\| \\ &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|(1-\beta_n)x_n+\beta_n Tx_n - x^*\| \\ &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n(1-\beta_n)\|x_n - x^*\| + \alpha_n\beta_n\|x_n - x^*\| \\ &= \|x_n - x^*\|.\end{aligned}$$

Hence $\lim_{n\to\infty}\|x_n - x^*\|$ exists and so $\{x_n\}$ is bounded. This completes the proof of the lemma. ∎

Lemma 3.2. *Let E be a real uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let $S, T\colon C \to C$ be two nonexpansive self mappings of C with $F = F(S)\cap F(T) \neq \phi$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. For arbitrary $x_1 \in C$ define the sequence $\{x_n\}$ by (1.4). Then*

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0 = \lim_{n\to\infty}\|x_n - Sx_n\|.$$

Proof. By Lemma 3.1, $\lim_{n\to\infty}\|x_n - x^*\|$ exists. Assume that $\lim_{n\to\infty}\|x_n - x^*\| = c$. If $c = 0$, then conclusion is obvious. Let $c > 0$. Now,

$$\begin{aligned}\|y_n - x^*\| &= \|(1-\beta_n)x_n + \beta_n Tx_n - x^*\| \\ &\le (1-\beta_n)\|x_n - x^*\| + \beta_n\|Tx_n - x^*\| \\ &= \|x_n - x^*\|,\end{aligned}$$

implies that

$$\limsup_{n\to\infty}\|y_n - x^*\| \le c. \tag{3.1}$$

Since T is nonexpansive, we have $\|Tx_n - x^*\| \le \|x_n - x^*\|$. Taking $\limsup$ on both sides, we obtain

$$\limsup_{n\to\infty}\|Tx_n - x^*\| \le c. \tag{3.2}$$

In a similar way, we have $\|Sy_n - x^*\| \le \|y_n - x^*\|$. By using (3.1), we obtain

$$\limsup_{n\to\infty}\|Sy_n - x^*\| \le c.$$

Also, it follows from $c = \lim_{n\to\infty}\|x_{n+1} - x^*\| = \lim_{n\to\infty}\|(1-\alpha_n)(Tx_n - x^*) + \alpha_n(Sy_n - x^*)\|$ and Lemma 2.9 that

$$\lim_{n\to\infty}\|Tx_n - Sy_n\| = 0.$$

Now

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|(1-\alpha_n)Tx_n + \alpha_n Sy_n - x^*\| \\ &= \|(Tx_n - x^*) + \alpha_n(Sy_n - Tx_n)\| \\ &\le \|Tx_n - x^*\| + \alpha_n\|Sy_n - Tx_n\|,\end{aligned}$$

yields that $c \le \liminf_{n\to\infty}\|Tx_n - x^*\|$. So that (3.2) gives $\lim_{n\to\infty}\|Tx_n - x^*\| = c$.

On the other hand,

$$\begin{aligned}\|Tx_n - x^*\| &\le \|Tx_n - Sy_n\| + \|Sy_n - x^*\| \\ &\le \|Tx_n - Sy_n\| + \|y_n - x^*\|.\end{aligned}$$

So we have

$$c \le \liminf_{n\to\infty} \|y_n - x^*\|. \tag{3.3}$$

By using (3.1) and (3.3), we get $\lim_{n\to\infty} \|y_n - x^*\| = c$. Thus $c = \lim_{n\to\infty} \|y_n - x^*\| = \lim_{n\to\infty} \|(1-\beta_n)(x_n - x^*) + \beta_n(Tx_n - x^*)\|$ gives by Lemma 2.9 that

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0. \tag{3.4}$$

Now, $\|y_n - x_n\| = \beta\|Tx_n - x_n\|$. Hence by (3.4) $\lim_{n\to\infty} \|y_n - x_n\|$. Also,

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|(1-\alpha_n)Tx_n + \alpha_n Sy_n - x_n\| \\ &\le \|Tx_n - x_n\| + \alpha_n\|Sy_n - Tx_n\| \\ &\to 0 \quad (\text{as } n \to \infty).\end{aligned}$$

So that

$$\|x_{n+1} - y_n\| \le \|x_{n+1} - x_n\| + \|y_n - x_n\| \to 0 \quad (\text{as } n \to \infty).$$

Furthermore,

$$\|x_{n+1} - Sy_n\| \le \|x_{n+1} - x_n\| + \|x_n - Tx_n\| + \|Tx_n - Sy_n\|,$$

implies that $\lim_{n\to\infty} \|x_{n+1} - Sy_n\| = 0$. Now

$$\begin{aligned}\|x_n - Sx_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - Sy_n\| + \|Sy_n - Sx_n\| \\ &\le \|x_n - x_{n+1}\| + \|x_{n+1} - Sy_n\| + \|y_n - x_n\| \\ &\to 0 \quad (\text{as } n \to \infty).\end{aligned}$$

i.e. $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$. This completes the proof of the lemma. ∎

The following result was proved by Shahzad in [15] (using Lemma 2.7), which contains the result of [17] (Theorem 3.3) for the case when E is a uniformly convex space whose norm is Frechet differentiable.

THEOREM 3.3. *Let E be a real uniformly convex Banach space such that its dual E^* has the Kadec-Klee property and C a nonempty closed convex subset of E. Let $S, T\colon C \to C$ be two nonexpansive mappings with $F = F(S) \cap F(T) \neq \phi$.Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. For arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by (1.4). Then $\{x_n\}$ converges weakly to some common fixed point of S and T.*

We remark that once a result has been proved for (1.4), it is not difficult to prove it for the iteration process (1.1). For example, combining Theorem 3.3 and Lemma 2.8, we can obtain the following result which can be applied to the spaces not covered by [8] (Theorem 1) and by [9] (Theorem 3.5).

THEOREM 3.4. *Let E be a real uniformly convex Banach space such that its dual E^* has the Kadec-Klee property and C a nonempty closed convex subset of*

E. Let $S, T\colon C \to C$ be two nonexpansive mappings with $F = F(S) \cap F(T) \neq \phi$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$ and $\{\gamma'_n\}$ be real sequences in $[0,1]$, satisfying (1.2) and $\alpha_n, \alpha'_n \in [\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. For arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by (1.1). Then $\{x_n\}$ converges weakly to some common fixed point of S and T.

Theorem 3.5. *Let E be a real uniformly convex Banach space such that its dual E^* has the Kadec-Klee property and C a nonempty closed convex subset of E. Let $S, T\colon C \to C$ be two nonexpansive mappings with $F = F(S) \cap F(T) \neq \phi$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. For arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by (1.4). Suppose S and T satisfy Condition (B). Then $\{x_n\}$ converges strongly to some common fixed point of S and T.*

Proof. Let $x^* \in F$. Then by Lemma 3.1, $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Also

$$\|x_{n+1} - x^*\| \leq \|x_n - x^*\| \quad \text{(for all } n \geq 1)$$

implies that $d(x_{n+1}, F) \leq d(x_n, F)$ and so, $\lim_{n\to\infty} d(x_n, F)$ exists. Also, by Lemma 3.2

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0 = \lim_{n\to\infty} \|x_n - Tx_n\|.$$

Since S and T satisfy Condition (B), we have

$$\lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty} \|x_n - Tx_n\| = 0$$

Or

$$\lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty} \|x_n - Sx_n\| = 0.$$

Hence $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since $f\colon [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $f(0) = 0, f(r) > 0$ for all $r \in (0,\infty)$, therefore we have $\lim_{n\to\infty} d(x_n, F) = 0$. Thus, we can find a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and a sequence $\{x_j^*\} \subset F$ satisfying

$$\|x_{n_j} - x_j^*\| \leq \frac{1}{2^j}.$$

Put $n_{j+1} = n_j + k$ for some $k \geq 1$. Then

$$\|x_{n_{j+1}} - x_j^*\| \leq \|x_{n_j+k-1} - x_j^*\| \leq \|x_{n_j} - x_j^*\| \leq \frac{1}{2^j}$$

and so we have $\|x_{j+1}^* - x_j^*\| \leq \frac{3}{2^{j+1}}$.

Likewise, for any positive integer m we have,

$$\|x_{n_{j+m}} - x_j^*\| \leq \frac{1}{2^j},$$

and consequently,

$$\|x_{j+m}^* - x_j^*\| \leq \frac{1}{2^j} + \frac{1}{2^{j+m}} \leq \frac{3}{2^{j+1}}.$$

Thus $\{x_j^*\}$ is a Cauchy sequence and so there exists $y^* \in C$ such that $x_j^* \to y^*$. Since F is closed, $y^* \in F$. Thus,we have $x_{n_j} \to y^*$ as $n \to \infty$. Since $\lim_{n\to\infty} \|x_n - y^*\|$ exists by Lemma 3.1 the conclusion follows. ∎

Combining Theorem 3.5 and Lemma 2.8, we obtain the following result which contains Theorem 2 of [8] as a special case.

Theorem 3.6. *Let E be a real uniformly convex Banach space and C a nonempty closed convex subset of E. Let $S, T\colon C \to C$ be two nonexpansive mappings with $F = F(S) \cap F(T) \neq \phi$. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}$ and $\{\gamma'_n\}$ be real sequences in $[0,1]$, satisfying (1.2) and $\alpha_n, \alpha'_n \in [\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. For arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by (1.1). Suppose S and T satisfy Condition (B). Then $\{x_n\}$ converges strongly to some common fixed point of S and T.*

Finally we prove the following strong convergence theorem.

Theorem 3.7. *Let E be a real uniformly convex Banach space and C a nonempty closed convex subset of E. Let $S, T\colon C \to C$ be two nonexpansive mappings with $F = F(S) \cap F(T) \neq \phi$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. For arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by (1.4). Suppose one of S and T is semi-compact. Then $\{x_n\}$ converges strongly to some common fixed point of S and T.*

Proof. Assume that T is semi-compact. By Lemma 3.1 $\{x_n\}$ is bounded and by Lemma 3.2

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0 = \lim_{n\to\infty} \|x_n - Tx_n\|.$$

So there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to x^* \in C$ as $j \to \infty$.

Now by Lemma 3.2 $\lim_{j\to\infty} \|x_{n_j} - Sx_{n_j}\| = 0 = \lim_{j\to\infty} \|x_{n_j} - Tx_{n_j}\|$ and so $\|x^* - Tx^*\| = 0 = \|x^* - Sx^*\|$, implies that $x^* \in F$.Since, $\lim_{n\to\infty} d(x_n, F) = 0$, it follows that, as in the proof of Theorem 3.5, that $\{x_n\}$ converges strongly to some common fixed point of S and T. This completes the proof of the theorem. ■

The following proposition was noted in [4] (see [4] for definitions).

Proposition 3.8. *Let E be a uniformly convex Banach space and C be a nonempty closed bounded convex subset of E. Suppose $T\colon C \to C$. Then T is semi-compact if T satisfies any of the following conditions:*

1. *T is either set-condensing or ball-condensing (or compact);*
2. *T is generalized contraction;*
3. *T is uniformly strictly contractive;*
4. *T is strictly semi-contractive;*
5. *T is strictly semi-contractive type;*
6. *T is of strongly semi-contractive type.*

Remark 3.9. 1. Let E be a reflexive Banach space. Then the dual E^* of E has the Kadec-Klee property if and only if E is asymptotically smooth [7].

2. It is possible to replace the semi-compactness assumption in Theorem 3.7. by any of the contractive assumptions 1–6 of Proposition 3.8.

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