

## ON EXTENSION OF GABOR TRANSFORM TO BOEHMIANS

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**Abstract.** On the theory of windowed Fourier transform proposed in the article “Wavelet transforms for integrable Boehmians, *J. Math. Anal. Appl.* 296 (2004) 473–478” many conceptual mistakes are pointed out, and the windowed Fourier transform (Gabor transform) on  $L^2(R)$  is extended to a suitable Boehmian space. The properties of the extended Gabor transform are also established.

### 1. Introduction

The Nobel laureate in physics D. Gabor [2], first introduced an integral transform in 1964, which provides the joint time-frequency representation of a given signal. This integral transform is called windowed Fourier transform [4], short time Fourier transform [3] and Gabor transform [11]. We prefer to use the name Gabor transform. It is well known that it has various applications in signal processing.

On the other hand, starting from the works [12, 13] of J. Mikusiński and P. Mikusiński, many Boehmian spaces have been constructed and various integral transforms have been extended on them. The complete list of references on Boehmians is available in <http://math.ucf.edu/~piotr/Boehmians.pdf>.

Though the title of the article [1] is “Wavelet transforms for integrable Boehmians”, actually windowed Fourier transform is proposed to extend to the space of integrable Boehmians [14]. The definition of windowed Fourier transform of a Boehmian [1] and its properties have so many uncorrectable errors on fundamentals of mathematics, which are discussed in Section 5.

For the theory of wavelet transform in the context of Boehmians, we refer the reader to [6, 7, 8, 9]. While thinking how the windowed Fourier transform can be extended to the context of Boehmians, we arrived at this article, in which, we recall the preliminaries in Section 2, prove the required auxiliary results to construct a Boehmian space in Section 3 and extend the Gabor transform in Section 4.

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## 2. Preliminaries

We denote the space of all infinitely differentiable functions on the set  $\mathbb{R}$  of all reals with compact supports and the Hilbert space of all square integrable functions on  $\Omega$ , respectively by,  $\mathcal{D}(\mathbb{R})$  and  $L^2(\Omega)$ , where  $\Omega = \mathbb{R}$  or  $\mathbb{R}^2$ . Throughout the article, we fix  $0 \neq g \in L^2(\mathbb{R})$ .

DEFINITION 2.1. [4] For  $f \in L^2(\mathbb{R})$ , the Gabor transform  $\mathcal{G}_g f$  of  $f$  is defined by

$$(\mathcal{G}_g f)(b, \xi) = \int_{-\infty}^{\infty} f(x) \overline{g(x-b)} e^{-2\pi i x \xi} dx, \quad \forall (b, \xi) \in \mathbb{R}^2. \quad (1)$$

It is known from the literature [4, 11] that the Gabor transform has the following properties.

LEMMA 2.2. If  $f \in L^2(\mathbb{R})$ , then  $\|f\|_2 = \frac{1}{\|g\|_2} \|\mathcal{G}_g f\|_2$ , where  $\|\cdot\|_2$  and  $\|\cdot\|_2$  are the norms on  $L^2(\mathbb{R})$  and  $L^2(\mathbb{R}^2)$ , respectively.

THEOREM 2.3. The Gabor transform  $\mathcal{G}_g$  is a continuous mapping from  $L^2(\mathbb{R})$  into  $L^2(\mathbb{R}^2)$ .

THEOREM 2.4. The Gabor transform  $\mathcal{G}_g : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}^2)$  is linear.

THEOREM 2.5. The Gabor transform  $\mathcal{G}_g : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}^2)$  is one-to-one. Indeed, the inversion formula of  $\mathcal{G}_g$  is given by

$$f(x) = \frac{1}{\|g\|_2} \int_{\mathbb{R}^2} (\mathcal{G}_g f)(b, \xi) g(x-b) e^{2\pi i x \xi} d(b, \xi), \quad \forall x \in \mathbb{R}. \quad (2)$$

THEOREM 2.6. The range of Gabor transform  $\mathcal{G}_g : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}^2)$  is characterized by the subspace of all functions  $h \in L^2(\mathbb{R}^2)$  satisfying the following condition.

$$h(b', \xi') = \int_{\mathbb{R}^2} h(b, \xi) \int_{-\infty}^{\infty} e^{-2\pi i (\xi' - \xi)x} \overline{g(x-b')} g(x-b) dx d(b, \xi), \quad (3)$$

for every  $(b', \xi') \in \mathbb{R}^2$ .

THEOREM 2.7. Let  $f \in L^2(\mathbb{R})$  and let  $a \in \mathbb{R}$ . If  $(\tau_a f)(x) = f(x-a)$ , and  $(e_a f)(x) = e^{iax} f(x)$ ,  $\forall x \in \mathbb{R}$ , then  $(\mathcal{G}_g(\tau_a f))(b, \xi) = e^{2\pi i \xi a} (\mathcal{G}_g f)(b-a, \xi)$ , and  $(\mathcal{G}_g(e_{2\pi a} f))(b, \xi) = (\mathcal{G}_g f)(b, \xi-a)$ ,  $\forall (b, \xi) \in \mathbb{R}^2$ .

Next, we recall the abstract construction of Boehmian space and two notions of convergence on Boehmians from [13, 16]. Let  $\Gamma$  be a topological vector space,  $(S, \odot)$  be a commutative semi-group and  $\otimes : \Gamma \times S \rightarrow \Gamma$  with the following conditions.

- (1) If  $f, g \in \Gamma$  and  $\psi \in S$ , then  $(f+g) \otimes \psi = (f \otimes \psi) + (g \otimes \psi)$ ,
- (2) If  $f \in \Gamma$ ,  $\phi \in S$  and  $\alpha \in \mathbb{C}$ , then  $(\alpha f) \otimes \phi = \alpha(f \otimes \phi)$ ,
- (3) If  $f \in \Gamma$  and  $\varphi, \psi \in S$ , then  $(f \otimes \varphi) \otimes \psi = f \otimes (\varphi \odot \psi)$ .

Let  $\Delta$  be a collection of sequences from  $S$  with the following properties.

- (1) If  $f_n \rightarrow f$  as  $n \rightarrow \infty$  in  $\Gamma$  and  $(\varphi_n) \in \Delta$ ,  $f_n \otimes \varphi_n \rightarrow f$  as  $n \rightarrow \infty$ ,
- (2) If  $(\varphi_n), (\psi_n) \in \Delta$ , then  $(\varphi_n \odot \psi_n) \in \Delta$ .

Let an equivalence relation  $\sim$  on the collection of all quotients

$$\mathcal{A} = \{((f_n), (\varphi_n)) : f_n \in \Gamma, \forall n \in \mathbb{N}, (\varphi_n) \in \Delta, f_n \otimes \varphi_m = f_m \otimes \varphi_n, \forall n, m \in \mathbb{N}\}$$

be defined by

$$((f_n), (\varphi_n)) \sim ((g_n), (\psi_n)) \text{ if } f_n \otimes \psi_m = g_m \otimes \varphi_n, \forall n, m \in \mathbb{N}$$

and the collection of all equivalence classes induced by  $\sim$  on  $\mathcal{A}$  is called the Boehmian space  $\mathcal{B} = \mathcal{B}(\Gamma, (S, \odot), \otimes, \Delta)$  and a typical element of  $\mathcal{B}$  is denoted by  $X = \left[ \frac{(f_n)}{(\varphi_n)} \right]$ . We identify  $\Gamma$  as a subset of  $\mathcal{B}$ , through the identification  $f \mapsto \left[ \frac{(f \otimes \varphi_n)}{(\varphi_n)} \right]$ , where  $(\varphi_n) \in \Delta$  is arbitrary. We also extend addition, scalar multiplication and the operation  $\otimes$  to the context of Boehmians by  $X + Y = \left[ \frac{(f_n \otimes \psi_n + g_n \otimes \varphi_n)}{(\varphi_n \odot \psi_n)} \right]$ ,  $aX = \left[ \frac{(af_n)}{(\varphi_n)} \right]$  and  $X \otimes \eta = \left[ \frac{(f_n \otimes \eta)}{(\varphi_n)} \right]$ , where  $X = \left[ \frac{(f_n)}{(\varphi_n)} \right]$ ,  $Y = \left[ \frac{(g_n)}{(\psi_n)} \right] \in \mathcal{B}$ ,  $a \in \mathbb{C}$  and  $\eta \in S$ .

LEMMA 2.8. *If  $X = \left[ \frac{(f_n)}{(\varphi_n)} \right] \in \mathcal{B}$ , then  $X \otimes \varphi_k = f_k \in \Gamma$  for all  $k \in \mathbb{N}$ .*

DEFINITION 2.9. A sequence  $(X_n)$  of Boehmians is said to  $\delta$ -converge to  $X$  in  $\mathcal{B}$ , (denoted by  $X_n \xrightarrow{\delta} X$  as  $n \rightarrow \infty$ ) if there exists  $(\delta_n) \in \Delta$  such that  $X_n \otimes \delta_k, X \otimes \delta_k \in \Gamma, \forall n, k \in \mathbb{N}$  and for each  $k \in \mathbb{N}, X_n \otimes \delta_k \rightarrow X \otimes \delta_k$  as  $n \rightarrow \infty$  in  $\Gamma$ .

THEOREM 2.10.  *$X_n \xrightarrow{\delta} X$  as  $n \rightarrow \infty$  if and only if there exist  $f_{n,k}, f_k \in \Gamma$  and  $(\delta_n) \in \Delta$  such that  $X_n = \left[ \frac{(f_{n,k})}{(\delta_k)} \right], X = \left[ \frac{(f_k)}{(\delta_k)} \right]$  and  $f_{n,k} \rightarrow f_k$  as  $n \rightarrow \infty$  in  $\Gamma, \forall k \in \mathbb{N}$ .*

DEFINITION 2.11. A sequence  $(X_n)$  of Boehmians is said to  $\Delta$ -converge to  $X$  in  $\mathcal{B}$  (denoted by  $X_n \xrightarrow{\Delta} X$  as  $n \rightarrow \infty$ ) if there exists  $(\delta_n) \in \Delta$  such that  $(X_n - X) \otimes \delta_n \in \Gamma, \forall n \in \mathbb{N}$  and  $(X_n - X) \otimes \delta_n \rightarrow 0$  as  $n \rightarrow \infty$  in  $\Gamma$ .

We recall the Boehmian space  $\mathcal{B}(L^2(\mathbb{R}), (\mathcal{D}(\mathbb{R}), *), *, \Delta_0)$  from [5], where  $*$  is the usual convolution defined by

$$(\phi * \psi)(x) = \int_{-\infty}^{\infty} \phi(x-t)\psi(t)dt, \quad x \in \mathbb{R}$$

and  $\Delta_0$  is the set of all sequences  $(\phi_n)$  from  $\mathcal{D}(\mathbb{R})$  satisfying

- (1)  $\int_{-\infty}^{\infty} \phi_n(x)dx = 1, \forall n \in \mathbb{N}$ ,
- (2)  $\int_{-\infty}^{\infty} |\phi_n(x)|dx \leq M, \forall n \in \mathbb{N}$ , for some  $M > 0$ ,
- (3)  $\text{supp}(\phi_n) \rightarrow 0$  as  $n \rightarrow \infty$ , where  $\text{supp}(\phi_n) = \overline{\{x : x \in \mathbb{R}, \phi_n(x) \neq 0\}}$ .

We denote the Boehmian space  $\mathcal{B}(L^2(\mathbb{R}), (\mathcal{D}(\mathbb{R}), *), *, \Delta_0)$  by  $\mathcal{B}_{\mathbb{R}}^2$ . It is well known that  $\mathcal{B}_{\mathbb{R}}^2$  contains the following spaces:  $L^2(\mathbb{R})$ , the space  $\mathcal{E}'(\mathbb{R})$  of compactly

supported distributions and  $\mathcal{D}'_{L^2}(\mathbb{R})$  [17]. Since the Boehmian not representing any distribution, introduced in [13], is obviously a member of  $\mathcal{B}^2_{\mathbb{R}}$ , we obtain that  $\mathcal{B}^2_{\mathbb{R}}$  is properly larger than all of these spaces.

We recall the translation of a Boehmian and multiplication of a Boehmian by the function  $e_a$ , from [15].

DEFINITION 2.12. Let  $X = \left[ \begin{smallmatrix} (f_n) \\ (\phi_n) \end{smallmatrix} \right]$  and let  $a \in \mathbb{R}$ . Using the notations  $\tau_a$  and  $e_a$  introduced in Theorem 2.7, we define

$$\tau_a X = \left[ \begin{smallmatrix} (\tau_a f_n) \\ (\phi_n) \end{smallmatrix} \right] \text{ and } e_a X = \left[ \begin{smallmatrix} (\lambda_n e_a f_n) \\ (\lambda_n e_a \phi_n) \end{smallmatrix} \right] \text{ where } \lambda_n = \left( \int_{-\infty}^{\infty} e_a(t) \phi_n(t) dt \right)^{-1}.$$

We point out that though the above operations are defined on  $C^\infty$ -Boehmians in [15], it is easy to observe that these two operations can also be defined on Boehmians in  $\mathcal{B}^2_{\mathbb{R}}$ .

### 3. Auxiliary results

In this section, we prove the auxiliary results required to construct the Boehmian space  $\mathcal{B}^2_{\mathbb{R}^2} = \mathcal{B}(L^2(\mathbb{R}^2), \star, (\mathcal{D}(\mathbb{R}), \ast), \Delta_0)$ , where  $\star$  is defined as follows.

DEFINITION 3.1. For  $F \in L^2(\mathbb{R}^2)$  and  $\phi \in \mathcal{D}(\mathbb{R})$ , define

$$(F \star \phi)(b, \xi) = \int_{-\infty}^{\infty} F(b - y, \xi) \phi(y) e^{-2\pi i y \xi} dy, \quad \forall (b, \xi) \in \mathbb{R}^2.$$

LEMMA 3.2. If  $F \in L^2(\mathbb{R}^2)$  and  $\phi \in \mathcal{D}(\mathbb{R})$ , then  $\|F \star \phi\|_2 \leq C \|F\|_2$ , where  $C = \int_{-\infty}^{\infty} |\phi(y)| dy$  and hence  $F \star \phi \in L^2(\mathbb{R}^2)$ .

*Proof.* The proof follows immediately, if  $\phi = 0$ . Hence, we assume that  $\phi \neq 0$ . Using Jensen's inequality and Fubini's theorem, we get

$$\begin{aligned} \|F \star \phi\|_2^2 &\leq \int_{\mathbb{R}^2} \left( \int_{-\infty}^{\infty} |F(b - y, \xi) \phi(y)| dy \right)^2 d(b, \xi) \\ &\leq C^2 \int_{\mathbb{R}^2} \left( \int_{-\infty}^{\infty} |F(b - y, \xi)|^2 |\phi(y)| \frac{dy}{C} \right) d(b, \xi) \\ &\quad (\text{ since } |\phi(y)| \frac{dy}{C} \text{ is a probability measure on } \mathbb{R}. ) \\ &\leq C \int_{-\infty}^{\infty} |\phi(y)| dy \int_{\mathbb{R}^2} |F(b - y, \xi)|^2 d(b, \xi) \\ &= C^2 \|F\|_2^2. \end{aligned}$$

Hence,  $F \star \phi \in L^2(\mathbb{R}^2)$ . ■

LEMMA 3.3. If  $F, F_1, F_2 \in L^2(\mathbb{R}^2)$ ,  $\phi \in \mathcal{D}(\mathbb{R})$  and  $c \in \mathbb{C}$ , then

- (1)  $(F_1 + F_2) \star \phi = F_1 \star \phi + F_2 \star \phi,$
- (2)  $(cF) \star \phi = c(F \star \phi).$

*Proof.* Proof of this lemma is straightforward. ■

LEMMA 3.4. *If  $F \in L^2(\mathbb{R}^2)$  and  $\phi_1, \phi_2 \in \mathcal{D}(\mathbb{R})$ , then  $F \star (\phi_1 * \phi_2) = (F \star \phi_1) \star \phi_2.$*

*Proof.* Let  $(b, \xi) \in \mathbb{R}^2$  be arbitrary. Then, by applying Fubini's theorem, we get

$$\begin{aligned}
 (F \star (\phi_1 * \phi_2))(b, \xi) &= \int_{-\infty}^{\infty} F(b - y, \xi) e^{-2\pi i y \xi} (\phi_1 * \phi_2)(y) dy \\
 &= \int_{-\infty}^{\infty} F(b - y, \xi) e^{-2\pi i y \xi} \int_{-\infty}^{\infty} \phi_1(y - t) \phi_2(t) dt dy \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(b - y, \xi) e^{-2\pi i y \xi} \phi_1(y - t) dy \phi_2(t) dt \\
 &\quad \text{(by using the change of variable } z = y - t) \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(b - (z + t), \xi) e^{-2\pi i (z+t)\xi} \phi_1(z) dz \phi_2(t) dt \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F((b - t) - z, \xi) e^{-2\pi i z \xi} \phi_1(z) dz e^{-2\pi i t \xi} \phi_2(t) dt \\
 &= \int_{-\infty}^{\infty} (F \star \phi_1)(b - t, \xi) e^{-2\pi i t \xi} \phi_2(t) dt \\
 &= ((F \star \phi_1) \star \phi_2)(b, \xi).
 \end{aligned}$$

Hence, the lemma follows. ■

LEMMA 3.5. *If  $F_n \rightarrow F$  as  $n \rightarrow \infty$  in  $L^2(\mathbb{R}^2)$  and  $\phi \in \mathcal{D}(\mathbb{R})$ , then  $F_n \star \phi \rightarrow F \star \phi$  as  $n \rightarrow \infty$  in  $L^2(\mathbb{R}^2)$ .*

*Proof.* From Lemma 3.2, we have

$$\|F_n \star \phi - F \star \phi\|_2 = \| (F_n - F) \star \phi \|_2 \leq C \|F_n - F\|_2,$$

which tends to zero as  $n \rightarrow \infty$ , where  $C = \int_{-\infty}^{\infty} |\phi(y)| dy$ . ■

LEMMA 3.6. *If  $F \in L^2(\mathbb{R}^2)$  and if  $(\phi_n) \in \Delta_0$ , then  $F \star \phi_n \rightarrow F$  as  $n \rightarrow \infty$  in  $L^2(\mathbb{R}^2)$ .*

*Proof.* Let  $\epsilon > 0$  be given. We choose  $\Phi \in C_c(\mathbb{R}^2)$  such that  $\|F - \Phi\|_2 < \epsilon$ , by using the fact that the space  $C_c(\mathbb{R}^2)$  of all continuous functions on  $\mathbb{R}^2$  with compact supports is dense in  $L^2(\mathbb{R}^2)$ . For each fixed  $(b, \xi) \in \mathbb{R}^2$ , if we define  $h(t) = \Phi(b - t, \xi) e^{-2\pi i t \xi}$ ,  $\forall (t, \xi) \in \mathbb{R}^2$ , then  $h \in C_c(\mathbb{R})$  and hence  $h$  is uniformly

continuous on  $\mathbb{R}$ . Therefore, there exists  $\delta > 0$  such that  $|h(u) - h(v)| < \epsilon$ , whenever  $u, v \in \mathbb{R}$  such that  $|u - v| \leq \delta$ . Since  $\text{supp } \phi_n \rightarrow 0$  as  $n \rightarrow \infty$ , there exists  $N \in \mathbb{N}$  such that  $\text{supp } \phi_n$  is contained in the closed ball in  $\mathbb{R}^2$  with center origin and radius  $\delta$ ,  $\forall n \geq N$ . Now we observe the following.

- If  $\text{supp } \Phi \subset [p, q] \times [r, s]$ , then  $\Phi(b - y, \xi) = 0$ , for every  $(b, \xi) \notin K$  and for every  $y \in [-\delta, \delta]$ , where  $K = [p - \delta, q + \delta] \times [r, s]$ .
- If  $C_n = \int_{-\infty}^{\infty} |\phi_n(x)| dx$ , then from property (2) of  $(\phi_n) \in \Delta_0$ , we have  $C_n \leq M$ ,  $\forall n \in \mathbb{N}$ , for some  $M > 0$ .

Hence, applying Jensen's inequality, for  $n \geq N$ , we get

$$\begin{aligned} \|\Phi \star \phi_n - \Phi\|_2^2 &= \int_K \left( \int_{-\infty}^{\infty} [\Phi(b - y, \xi) e^{-2\pi i y \xi} - \Phi(b, \xi)] \phi_n(y) dy \right)^2 d(b, \xi) \\ &= \int_K \left( \int_{-\infty}^{\infty} [h(y) - h(0)] \phi_n(y) dy \right)^2 d(b, \xi), \\ &\leq C_n \int_K \int_{-\delta}^{\delta} |h(y) - h(0)|^2 |\phi_n(y)| dy d(b, \xi) \\ &\leq M^2 \epsilon^2 \int_K d(b, \xi) = M^2 m(K) \epsilon^2, \end{aligned}$$

where  $m(K)$  is the Lebesgue measure of  $K$ . Using Lemma 3.2 and the estimate obtained just above, we get

$$\begin{aligned} \|\|F \star \phi_n - F\|_2 &\leq \||(F - \Phi) \star \phi_n|\|_2 + \|\|\Phi \star \phi_n - \Phi\|_2 + \|\|\Phi - F\|_2 \\ &< ([1 + \sqrt{m(K)}]M + 1)\epsilon. \end{aligned}$$

This completes the proof of the lemma. ■

**LEMMA 3.7.** *If  $F_n \rightarrow F$  as  $n \rightarrow \infty$  in  $L^2(\mathbb{R}^2)$  and  $(\phi_n) \in \Delta_0$ , then  $F \star \phi_n \rightarrow F$  as  $n \rightarrow \infty$  in  $L^2(\mathbb{R}^2)$ .*

*Proof.* Let  $M > 0$  be as in property (2) of  $(\phi_n) \in \Delta_0$ . Applying Lemma 3.3, we get

$$\|\|F_n \star \phi_n - F\|_2 \leq \||(F_n - F) \star \phi_n|\|_2 + \|\|F \star \phi_n - F\|_2. \quad (4)$$

In the right hand side of (4), from Lemma 3.2, the first term is dominated by  $M\|F_n - F\|_2$ , which tends to zero as  $n \rightarrow \infty$  and by applying Lemma 3.6, the second term tends to zero. Hence the lemma follows. ■

Thus the Boehmian space  $\mathcal{B}_{\mathbb{R}^2}^2$  is constructed. We denote a typical element of  $\mathcal{B}_{\mathbb{R}^2}^2$  by  $[(F_n)/(\phi_n)]$ .

#### 4. Extended Gabor transform

In this section, we first prove a convolution theorem for Gabor transform, using which, we shall define the extended Gabor transform.

**THEOREM 4.1.** [Convolution theorem for Gabor transform] *If  $f \in L^2(\mathbb{R})$  and  $\phi \in \mathcal{D}(\mathbb{R})$ , then  $\mathcal{G}_g(f * \phi) = \mathcal{G}_g f \star \phi$ .*

*Proof.* Let  $(b, \xi) \in \mathbb{R}^2$  be arbitrary. By using Fubini's theorem, we get

$$\begin{aligned} \mathcal{G}_g(f * \phi)(b, \xi) &= \int_{-\infty}^{\infty} e^{-2\pi i x \xi} (f * \phi)(x) \overline{g(x - b)} dx \\ &= \int_{-\infty}^{\infty} e^{-2\pi i x \xi} \int_{-\infty}^{\infty} f(x - y) \phi(y) dy \overline{g(x - b)} dx \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi i x \xi} f(x - y) \overline{g(x - b)} dx \phi(y) dy \\ &\quad (\text{by putting } x - y = z) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi i(y+z)\xi} f(z) \overline{g(y + z - b)} dz \phi(y) dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi i z \xi} f(z) \overline{g(z - (b - y))} dz e^{-2\pi i y \xi} \phi(y) dy \\ &= \int_{-\infty}^{\infty} (\mathcal{G}_g f)(b - y, \xi) e^{-2\pi i y \xi} \phi(y) dy \\ &= (\mathcal{G}_g f \star \phi)(b, \xi). \end{aligned}$$

Thus the theorem follows. ■

**DEFINITION 4.2.** The extended Gabor transform  $\mathcal{G}_g : \mathcal{B}_{\mathbb{R}}^2 \rightarrow \mathcal{B}_{\mathbb{R}^2}^2$  is defined by

$$\mathcal{G}_g \left( \left[ \begin{array}{c} (f_n) \\ (\phi_n) \end{array} \right] \right) = [(\mathcal{G}_g f_n) / (\phi_n)].$$

**LEMMA 4.3.** *The above notion is well defined.*

*Proof.* Let  $\left[ \begin{array}{c} (f_n) \\ (\phi_n) \end{array} \right] \in \mathcal{B}_{\mathbb{R}}^2$ . Then  $f_n \in L^2(\mathbb{R})$  and  $(\phi_n) \in \Delta_0$ . This implies that  $\mathcal{G}_g f_n \in L^2(\mathbb{R}^2)$  and

$$f_n * \phi_m = f_m * \phi_n, \quad \forall m, n \in \mathbb{N}.$$

Applying Theorem 4.1, we get

$$\mathcal{G}_g f_n \star \phi_m = \mathcal{G}_g(f_n * \phi_m) = \mathcal{G}_g(f_m * \phi_n) = \mathcal{G}_g f_m \star \phi_n, \quad \forall m, n \in \mathbb{N}.$$

Therefore,  $((\mathcal{G}_g f_n) / (\phi_n))$  is a quotient in the context of  $\mathcal{B}_{\mathbb{R}}^2$  and hence it represents a Boehmian in  $\mathcal{B}_{\mathbb{R}^2}^2$ . Next we show that the definition of  $\mathcal{G}_g$  is independent of the choice of the representatives of the Boehmians. If  $\left[ \begin{array}{c} (f_n) \\ (\phi_n) \end{array} \right] = \left[ \begin{array}{c} (g_n) \\ (\psi_n) \end{array} \right]$  in  $\mathcal{B}_{\mathbb{R}}^2$ . Then we have

$$f_n * \psi_m = g_m * \phi_n, \quad \forall m, n \in \mathbb{N}.$$

By applying Gabor transform and by using Theorem 4.1, we obtain

$$\mathcal{G}_g f_n \star \psi_m = \mathcal{G}_g g_m \star \phi_n, \quad \forall m, n \in \mathbb{N}.$$

Hence,  $\mathcal{G}_g \left( \left[ \begin{array}{c} (f_n) \\ (\phi_n) \end{array} \right] \right) = \mathcal{G}_g \left( \left[ \begin{array}{c} (g_n) \\ (\psi_n) \end{array} \right] \right)$  in  $\mathcal{B}_{\mathbb{R}^2}^2$ . ■

LEMMA 4.4. *The extended Gabor transform  $\mathcal{G}_g : \mathcal{B}_{\mathbb{R}}^2 \rightarrow \mathcal{B}_{\mathbb{R}^2}^2$  is consistent with the classical Gabor transform  $\mathcal{G}_g : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}^2)$ .*

*Proof.* Let  $f \in L^2(\mathbb{R})$ . Then  $f$  is represented by the Boehmian  $\left[ \frac{(f * \phi_n)}{(\phi_n)} \right]$  in  $\mathcal{B}_{\mathbb{R}}^2$ , where  $(\phi_n) \in \Delta_0$ . By definition, it is clear that  $\mathcal{G}_g \left[ \frac{(f * \phi_n)}{(\phi_n)} \right] = [(\mathcal{G}_g(f * \phi_n))/(\phi_n)] = [(\mathcal{G}_g f * \phi_n)/(\phi_n)]$ , which is the Boehmian representing the function  $\mathcal{G}_g f$  in  $\mathcal{B}_{\mathbb{R}^2}^2$ . Thus the consistency follows. ■

THEOREM 4.5. *The extended Gabor transform  $\mathcal{G}_g : \mathcal{B}_{\mathbb{R}}^2 \rightarrow \mathcal{B}_{\mathbb{R}^2}^2$  is linear.*

*Proof.* Proof of this theorem is straight forward by using Theorem 4.1 and the linearity of the Gabor transform  $\mathcal{G}_g : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}^2)$ . ■

THEOREM 4.6. *The extended Gabor transform  $\mathcal{G}_g : \mathcal{B}_{\mathbb{R}}^2 \rightarrow \mathcal{B}_{\mathbb{R}^2}^2$  is one-to-one.*

*Proof.* Let  $\beta_1, \beta_2 \in \mathcal{B}_{\mathbb{R}}^2$  be such that  $\mathcal{G}_g \beta_1 = \mathcal{G}_g \beta_2$ . If  $\beta_1 = \left[ \frac{(f_n)}{(\phi_n)} \right]$  and  $\beta_2 = \left[ \frac{(g_n)}{(\psi_n)} \right]$ , then by assumption, we have  $[(\mathcal{G}_g f_n)/(\phi_n)] = [(\mathcal{G}_g g_n)/(\psi_n)]$ , and hence

$$\mathcal{G}_g f_n * \psi_m = \mathcal{G}_g g_m * \phi_n, \quad \forall m, n \in \mathbb{N}. \tag{5}$$

Then, by applying Theorem 4.1 in (5), we get

$$\mathcal{G}_g(f_n * \psi_m) = \mathcal{G}_g(g_m * \phi_n), \quad \forall m, n \in \mathbb{N}.$$

Since  $\mathcal{G}_g : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}^2)$  is one-to-one, we get

$$f_n * \psi_m = g_m * \phi_n, \quad \forall m, n \in \mathbb{N}.$$

So,  $\beta_1 = \beta_2$ . Thus the theorem follows. ■

THEOREM 4.7. *The range of extended Gabor transform  $\mathcal{G}_g : \mathcal{B}_{\mathbb{R}}^2 \rightarrow \mathcal{B}_{\mathbb{R}^2}^2$  is*

$$\{\gamma \in \mathcal{B}_{\mathbb{R}^2}^2 : \gamma \text{ has a representation } [(F_n)/(\psi_n)] \text{ with } F_n \in \mathcal{G}_g(L^2(\mathbb{R})), \forall n \in \mathbb{N}\}.$$

*Proof.* By definition, if  $\gamma \in \mathcal{G}_g(\mathcal{B}_{\mathbb{R}}^2)$ , then there exists  $\left[ \frac{(f_n)}{(\phi_n)} \right] \in \mathcal{B}_{\mathbb{R}}^2$  such that  $\mathcal{G}_g \left( \left[ \frac{(f_n)}{(\phi_n)} \right] \right) = \gamma$ . Obviously,  $[(\mathcal{G}_g f_n)/(\phi_n)]$  itself is a required representation of  $\gamma$ . Conversely, let  $[(F_n)/(\phi_n)] \in \mathcal{B}_{\mathbb{R}^2}^2$  be such that  $F_n \in \mathcal{G}_g(L^2(\mathbb{R}))$ ,  $\forall n \in \mathbb{N}$ . Then, for each  $n \in \mathbb{N}$ , there exists  $f_n \in L^2(\mathbb{R})$  such that  $\mathcal{G}_g f_n = F_n$ . We claim that  $((f_n), (\phi_n))$  is a quotient in the context of  $\mathcal{B}_{\mathbb{R}}^2$ . From  $[(F_n)/(\phi_n)] \in \mathcal{B}_{\mathbb{R}^2}^2$ , we have

$$F_n * \phi_m = F_m * \phi_n, \quad \forall m, n \in \mathbb{N}.$$

This implies that

$$\mathcal{G}_g(f_n * \phi_m) = \mathcal{G}_g(f_m * \phi_n), \quad \forall m, n \in \mathbb{N}.$$

By invoking the injectivity of  $\mathcal{G}_g : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}^2)$ , we obtain that our claim holds.

Then,  $\left[ \frac{(f_n)}{(\phi_n)} \right] \in \mathcal{B}_{\mathbb{R}}^2$  and  $\mathcal{G}_g \left( \left[ \frac{(f_n)}{(\phi_n)} \right] \right) = [(\mathcal{G}_g f_n)/(\phi_n)] = [(F_n)/(\phi_n)]$ . ■

**THEOREM 4.8.** *If  $\beta \in \mathcal{B}_{\mathbb{R}}^2$  and  $\phi \in \mathcal{D}(\mathbb{R})$ , then  $\mathcal{G}_g(\beta * \phi) = \mathcal{G}_g(\beta) \star \phi$ .*

*Proof.* Let  $\beta = \left[ \frac{(f_n)}{(\phi_n)} \right]$ . Then, applying Theorem 4.1, we obtain that  $\mathcal{G}_g(\beta * \phi) = \mathcal{G}_g \left( \left[ \frac{(f_n * \phi)}{(\phi_n)} \right] \right) = [(\mathcal{G}_g(f_n * \phi))/(\phi_n)] = [(\mathcal{G}_g f_n \star \phi)/(\phi_n)] = [(\mathcal{G}_g f_n)/(\phi_n)] \star \phi = \mathcal{G}_g(\beta) \star \phi$ . ■

**THEOREM 4.9.** *The extended Gabor transform  $\mathcal{G}_g : \mathcal{B}_{\mathbb{R}}^2 \rightarrow \mathcal{B}_{\mathbb{R}^2}^2$  is continuous with respect to  $\delta$ -convergence as well as  $\Delta$ -convergence.*

*Proof.* Let  $\beta_n \xrightarrow{\delta} \beta$  as  $n \rightarrow \infty$  in  $\mathcal{B}_{\mathbb{R}}^2$ . Then by Lemma 2.4 of [13], there exists  $f_{n,k}, f_k \in L^2(\mathbb{R})$  and  $(\phi_k) \in \Delta_0$  such that  $\beta_n = \left[ \frac{(f_{n,k})}{(\phi_k)} \right], \beta = \left[ \frac{(f_k)}{(\phi_k)} \right]$  and

$$\text{for each } k \in \mathbb{N}, f_{n,k} \rightarrow f_k \text{ as } n \rightarrow \infty \text{ in } L^2(\mathbb{R}).$$

Using the continuity of  $\mathcal{G}_g : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}^2)$ , we obtain that

$$\mathcal{G}_g f_{n,k} \rightarrow \mathcal{G}_g f_k \text{ as } n \rightarrow \infty \text{ in } L^2(\mathbb{R}^2).$$

Since  $\mathcal{G}_g \beta_n = [(\mathcal{G}_g f_{n,k})/(\phi_k)], \mathcal{G}_g \beta = [(\mathcal{G}_g f_k)/(\phi_k)]$ , we get  $\mathcal{G}_g \beta_n \xrightarrow{\delta} \mathcal{G}_g \beta$  as  $n \rightarrow \infty$  in  $\mathcal{B}_{\mathbb{R}^2}^2$ .

Next, let  $\beta_n \xrightarrow{\Delta} \beta$  as  $n \rightarrow \infty$  in  $\mathcal{B}_{\mathbb{R}}^2$ . Then, by definition, there exists  $(\phi_n) \in \Delta_0$  such that  $(\beta_n - \beta) * \phi_n \in L^2(\mathbb{R}), \forall n \in \mathbb{N}$  and  $(\beta_n - \beta) * \phi_n \rightarrow 0$  as  $n \rightarrow \infty$  in  $L^2(\mathbb{R})$ . This means that there exist  $h_n \in L^2(\mathbb{R})$  such that  $(\beta_n - \beta) * \phi_n = \left[ \frac{(h_n * \phi_k)}{(\phi_k)} \right], \forall n \in \mathbb{N}$  and  $h_n \rightarrow 0$  as  $n \rightarrow 0$  in  $L^2(\mathbb{R})$ . Since the Gabor transform  $\mathcal{G}_g : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}^2)$  is continuous,  $\mathcal{G}_g h_n \rightarrow 0$  as  $n \rightarrow 0$  in  $L^2(\mathbb{R}^2)$ . Using Theorem 4.5 and Theorem 4.8, we get

$$\begin{aligned} (\mathcal{G}_g \beta_n - \mathcal{G}_g \beta) \star \phi_n &= \mathcal{G}_g(\beta_n - \beta) \star \phi_n = \mathcal{G}_g((\beta_n - \beta) * \phi_n) \\ &= \mathcal{G}_g \left[ \frac{(h_n * \phi_k)}{(\phi_k)} \right] = [(\mathcal{G}_g(h_n * \phi_k))/(\phi_k)] \\ &= [(\mathcal{G}_g h_n \star \phi_k)/(\phi_k)], \forall n \in \mathbb{N}. \end{aligned}$$

Therefore, it follows that  $\mathcal{G}_g \beta_n \xrightarrow{\Delta} \mathcal{G}_g \beta$  as  $n \rightarrow \infty$  in  $\mathcal{B}_{\mathbb{R}^2}^2$ . Hence, the theorem follows. ■

Now, we define three operations on  $\mathcal{B}_{\mathbb{R}^2}^2$  to discuss the properties of the extended Gabor transform.

**DEFINITION 4.10.** For  $\gamma = [(F_n)/(\phi_n)] \in \mathcal{B}_{\mathbb{R}^2}^2$  and  $a \in \mathbb{R}$ , we define

- $T_{1,a} \gamma = [(T_{1,a} F_n)/(\phi_n)]$ , where  $(T_{1,a} F_n)(b, \xi) = F_n(b - a, \xi), \forall (b, \xi) \in \mathbb{R}^2$ .
- $T_{2,a} \gamma = [(\lambda_n T_{2,a} F_n)/(\lambda_n e_{2\pi a} \phi_n)]$ , where  $(T_{2,a} F_n)(b, \xi) = F_n(b, \xi - a), \forall (b, \xi) \in \mathbb{R}^2$  and  $\lambda_n = \left( \int_{-\infty}^{\infty} e_{2\pi a}(t) \phi_n(t) dt \right)^{-1}$ .
- $E_{2\pi a} \cdot \gamma = [(E_{2\pi a} \cdot F_n)/(\phi_n)]$ , where  $(E_{2\pi a} \cdot F_n)(b, \xi) = e_{2\pi i a \xi} F_n(b, \xi), \forall (b, \xi) \in \mathbb{R}^2$ .

LEMMA 4.11. *The operations given in the previous definition are well defined.*

*Proof.* First, we prove the following identities, which will be required to prove the lemma. If  $F \in L^2(\mathbb{R}^2)$  and  $\phi \in \mathcal{D}(\mathbb{R})$ , then

$$T_{1,a}(F \star \phi) = (T_{1,a}F) \star \phi, \quad (6a)$$

$$T_{2,a}(F \star \phi) = (T_{2,a}F) \star (e_{2\pi a}\phi), \quad (6b)$$

$$E_{2\pi a} \cdot (F \star \phi) = (E_{2\pi a} \cdot F) \star \phi. \quad (6c)$$

$$\begin{aligned} (T_{1,a}(F \star \phi))(b, \xi) &= (F \star \phi)(b - a, \xi) \\ &= \int_{-\infty}^{\infty} F(b - a - y, \xi) \phi(y) e^{-2\pi i y \xi} dy \\ &= ((T_{1,a}F) \star \phi)(b, \xi). \\ (T_{2,a}(F \star \phi))(b, \xi) &= (F \star \phi)(b, \xi - a) \\ &= \int_{-\infty}^{\infty} F(b - y, (\xi - a)) \phi(y) e^{-2\pi i y (\xi - a)} dy \\ &= \int_{-\infty}^{\infty} F(b - y, (\xi - a)) (e^{2\pi i a y} \phi(y)) e^{-2\pi i y \xi} dy \\ &= ((T_{2,a}F) \star (e_{2\pi a}\phi))(b, \xi). \\ (E_{2\pi a} \cdot (F \star \phi))(b, \xi) &= e^{2\pi i a \xi} \int_{-\infty}^{\infty} F(b - y, \xi) \phi(y) e^{-2\pi i y \xi} dy \\ &= \int_{-\infty}^{\infty} (e^{2\pi i a \xi} F(b - y, \xi)) \phi(y) e^{-2\pi i y \xi} dy \\ &= ((E_{2\pi a} \cdot F) \star \phi)(b, \xi). \end{aligned}$$

If  $\gamma = [(F_n)/(\phi_n)] \in \mathcal{B}_{\mathbb{R}^2}^2$ , then we have

$$F_n \star \phi_m = F_m \star \phi_n, \quad \forall m, n \in \mathbb{N}. \quad (7)$$

Since  $T_{1,a}$  satisfies (6a), applying  $T_{1,a}$  on both sides of (7), we get

$$(T_{1,a}F_n) \star \phi_m = (T_{1,a}F_m) \star \phi_n, \quad \forall m, n \in \mathbb{N}.$$

Since,  $T_{1,a}F_n \in L^2(\mathbb{R}^2)$ ,  $\forall n \in \mathbb{N}$ ,  $[(T_{1,a}F_n)/(\phi_n)] \in \mathcal{B}_{\mathbb{R}^2}^2$ .

First, we note that  $T_{2,a}F_n \in L^2(\mathbb{R}^2)$ ,  $\forall n \in \mathbb{N}$  and  $(\lambda_n e_{2\pi a}\phi_n) \in \Delta_0$ . Then, applying the operator  $T_{2,a}$  and multiplying by  $\lambda_n \lambda_m$  on both sides of (7), we get

$$(\lambda_n T_{2,a}F_n) \star (\lambda_m e_{2\pi a}\phi_m) = (\lambda_m T_{2,a}F_m) \star (\lambda_n e_{2\pi a}\phi_n), \quad \forall m, n \in \mathbb{N}.$$

Hence,  $[(\lambda_n T_{2,a}F_n)/(\lambda_n e_{2\pi a}\phi_n)] \in \mathcal{B}_{\mathbb{R}^2}^2$ .

Finally, using (6c) in (7), we get

$$(E_{2\pi a}F_n) \star \phi_m = (E_{2\pi a}F_m) \star \phi_n, \quad \forall m, n \in \mathbb{N},$$

and hence,  $[(E_a \cdot F_n)/(\phi_n)] \in \mathcal{B}_{\mathbb{R}^2}^2$ . Further, by applying the same technique (as in the proof of Lemma 4.3, we can easily prove that these definitions are independent of the representatives of  $\gamma$ . ■

LEMMA 4.12. *The operations defined in Definition 6 are consistent with the corresponding operations on  $L^2(\mathbb{R}^2)$ .*

*Proof.* Let  $F \in L^2(\mathbb{R}^2)$  and  $a \in \mathbb{R}$  be arbitrary. Then the Boehmian representing  $F$  in  $\mathcal{B}_{\mathbb{R}^2}^2$  is  $[(F \star \phi_n)/(\phi_n)]$ , where  $(\phi_n) \in \Delta$  is arbitrary. Now

$$T_{1,a}[(F \star \phi_n)/(\phi_n)] = [(T_{1,a}(F \star \phi_n))/(\phi_n)] = [((T_{1,a}F) \star \phi_n)/(\phi_n)],$$

which is the Boehmian representing  $T_{1,a}F$  in  $\mathcal{B}_{\mathbb{R}^2}^2$ . Hence, the operation  $T_{1,a}$  on  $\mathcal{B}_{\mathbb{R}^2}^2$  is consistent with  $T_{1,a}$  on  $L^2(\mathbb{R}^2)$ . Similarly, we can prove that the other two operations on  $\mathcal{B}_{\mathbb{R}^2}^2$  are also consistent with the corresponding operations on  $L^2(\mathbb{R}^2)$ . ■

Now we are ready to present the properties of extended Gabor transform in the context of Boehmians.

THEOREM 4.13. *If  $\beta \in \mathcal{B}_{\mathbb{R}}^2$  and  $a \in \mathbb{R}$ , then*

- (1)  $\mathcal{G}_g(\tau_a\beta) = E_{2\pi a}(T_{1,a}(\mathcal{G}_g\beta))$ .
- (2)  $\mathcal{G}_g(e_{2\pi a}\beta) = T_{2,a}(\mathcal{G}_g\beta)$ .

*Proof.* Let  $\beta = \left[ \frac{(f_n)}{(\phi_n)} \right]$ . By using Theorem 2.7, we get

$$\begin{aligned} \mathcal{G}_g(\tau_a\beta) &= \mathcal{G}_g \left( \left[ \frac{(\tau_a f_n)}{(\phi_n)} \right] \right) &&= [(\mathcal{G}_g(\tau_a f_n))/(\phi_n)] \\ &= [(E_{2\pi a}(T_{1,a}(\mathcal{G}_g f_n)))/(\phi_n)] &&= E_{2\pi a}(T_{1,a}([( \mathcal{G}_g f_n )/(\phi_n)])) \\ &= E_{2\pi a}(T_{1,a}(\mathcal{G}_g\beta)) \end{aligned}$$

and

$$\begin{aligned} \mathcal{G}_g(e_{2\pi a}\beta) &= \mathcal{G}_g \left( \left[ \frac{(\lambda_n e_{2\pi a} f_n)}{(\lambda_n e_{2\pi a} \phi_n)} \right] \right) &&= [(\mathcal{G}_g(\lambda_n e_{2\pi a} f_n))/(\lambda_n e_{2\pi a} \phi_n)] \\ &= [(\lambda_n T_{2,a}(\mathcal{G}_g f_n))/(\lambda_n e_{2\pi a} \phi_n)] &&= T_{2,a}([( \mathcal{G}_g f_n )/(\phi_n)])) \\ &= T_{2,a}(\mathcal{G}_g\beta). \end{aligned}$$

Hence, the theorem follows. ■

### 5. Flaws in the work [1]

In this section, we use the notations as used in [1]. We first point out that the title of the article [1] and the definition of wavelet transform recalled in equation (2.1) are misleading in the sense that the wavelet transform defined in [1, (2.1)] is going to be extended in [1]. Actually, the article deals with the windowed Fourier transform  $G[f](\nu, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\tau)\bar{g}(\tau - t)e^{i\nu t} d\tau$  (see [1, (2.6)]). Further, wavelet

transform and windowed Fourier transform are treated as identical in [1] by quoting the reference [10, p. 688]. However, from the same reference, one can see that these transforms are not identical.

Next, we discuss the major conceptual errors in the article [1]. We find the following statement in [1, p. 476]. “*The windowed Fourier transform of an integrable  $F = [(f_t)_n/\delta_n]$  can be defined as the limit of  $(\hat{f}_n)$  in the space of continuous functions on  $\mathbf{R}$ .*”. The mistakes in this statement are:

- In the notation of the integrable Boehmian, it is not clear what  $t$  is. It might be a typographical error. So,  $[(f_t)_n/\delta_n]$  should be replaced by  $[f_n/\delta_n]$ .
- In the article [1], the notation  $\hat{f}_n$  is used to denote the Fourier transform of  $f_n$ . If we assume the definition as it is, then the limit of  $(\hat{f}_n)$  is the Fourier transform of  $F = [f_n/\delta_n]$ , which is already dealt in [14].
- If we replace it by “*limit of windowed Fourier transform  $G(f_n)$  of  $f_n$ ”*, then the limit must be a function on  $\mathbb{R}^2$  (if it exists). But, according to [1], windowed Fourier transform of an integrable Boehmian is defined as a continuous function on  $\mathbb{R}$ .
- Since windowed Fourier transform of a function is a function on  $\mathbb{R}^2$  and windowed Fourier transform of the Boehmian representing the same function is a function on  $\mathbb{R}$ , consistency does not follow.

To justify the existence of the limit, [1, Lemma 1] is used, as follows.

“*If  $[f_n/\delta_n] \in B_{L_1}$ , then  $[\hat{f}_t]_n(\nu) = \int_{-\infty}^{\infty} f_n(\tau)\bar{g}(\tau-t)e^{i\nu t}d\tau$  converges uniformly on each compact subset in  $\mathbb{R}$ .*”, where  $[\hat{f}_t]_n(\nu)$  is the windowed Fourier transform of  $f_n$  evaluated at  $(\nu, t)$  (according to [1]). The objections to the lemma are:

- The notation  $[\hat{f}_t]_n$  is misleading because  $f_n$  is the  $n^{\text{th}}$  term of a sequence of functions. To be reasonable, it is better to replace  $[\hat{f}_t]_n$  by  $(\widehat{f_n})_t$ .
- For  $[f_n/\delta_n] \in B_{L_1}$ , it is not possible to find  $(\widehat{f_n})_t$  (windowed Fourier transform of  $f_n$ ), because it is well known that if  $[f_n/\delta_n] \in B_{L_1}$ , then  $f_n \in L_1, \forall n \in \mathbb{N}$  but  $L_1 \not\subseteq L_2$  and windowed Fourier transform is defined on  $L_2$  (see [1, pp. 475-476]).
- In the proof of Lemma 1, the following equation is presented (see [1, (2.11)]). “ $[\hat{f}_t]_n = (\hat{f}_t)_n \left( \frac{\hat{\delta}_k}{\delta_k} \right) = \frac{((f_t)_n * \hat{\delta})}{\hat{\delta}_k} = \frac{(f_t)_k * \hat{\delta}_n}{\hat{\delta}_k}$ .” After correcting the typographical errors in the equation, it would be “ $(\widehat{f_n})_t = (\widehat{f_n})_t \left( \frac{\hat{\delta}_k}{\delta_k} \right) = \frac{((f_n)_t * \hat{\delta}_k)}{\hat{\delta}_k} = \frac{(\widehat{f_k})_t * \hat{\delta}_n}{\hat{\delta}_k}$ ”. Here, the last equality in the above equation, must be obtained by using  $(f_n)_t * \delta_k = (f_k)_t * \delta_n, \forall n, k \in \mathbb{N}$  and convolution theorem for Fourier transform. But, using  $[f_n/\delta_n] \in B_{L_1}$ , one could obtain that  $f_n * \delta_k = f_k * \delta_n, \forall n, k \in \mathbb{N}$ . From this,  $(f_n)_t * \delta_k = (f_k)_t * \delta_n, \forall n, k \in \mathbb{N}$  does not follow. (At this juncture, we point out that the notation  $f_t$  is introduced nowhere in this article. But from the theory of windowed Fourier transform, we can observe that  $f_t(\tau) = f(\tau)g(\tau - t), \forall \tau \in \mathbb{R}$ , where  $t \in \mathbb{R}$  is arbitrarily fixed.)

In Theorem 1, where properties of windowed Fourier transform of a Boehmian are proposed, there are lots of inconsistencies and mistakes as follow.

- In Theorem 1, the notation  $\hat{F}$  is used, without introducing it earlier. For a function  $f$ , the notation  $\hat{f}$  was used to denote its Fourier transform. Since it is a paper on windowed Fourier transform, we have to assume that  $\hat{F}$  might be used to denote the windowed Fourier transform of  $F$ . So, in Theorem 1, we read  $\hat{F}$  by  $\tilde{F}_g$ .
- The statement (ii) of Theorem 1 is  $(F * G)^\wedge = \hat{F}\hat{G}$ . It might be written as  $(F * G)_g^\wedge = \tilde{F}_g\tilde{G}_g$ , but obviously it is not true.
- In the statements (iii) and (iv) of Theorem 1, the usage of lower case letter  $f$  is also confusing. If we correct  $f$  by  $F$ , then the windowed transform of a Boehmian is treated as a function on  $\mathbb{R}^2$ , in statement (iii). It is not consistent with the definition of the windowed Fourier transform of a Boehmian in [1], because, according to [1], windowed Fourier transform of a Boehmian is a function on  $\mathbb{R}$ .
- The statement (iv) of Theorem 1 is “ $f^n(i\tau) = (-i\tau)^n = \hat{f}(\tau)$ ”, which is not meaningful.
- The proofs of the statements (i), (ii), (iii), (iv) of Theorem 1 are not given. The proof of the statement (v) of Theorem 1 is obtained from Theorem 2. But Theorem 2 is also not carefully presented. (The reasons follow after a few lines.)
- In the statement (vi) of Theorem 1,  $\Delta$ -convergence is involved, but in its proof,  $\delta$ -convergence is involved. Moreover, proof of the statement (iv) uses the identity “ $\hat{F}_n \cdot \hat{\delta}_k - \hat{F} - \delta_k = ((F_n - F) * \delta_k)^\wedge$ ”. After correcting the typographical errors in this identity, it will be “ $\hat{F}_n \cdot \hat{\delta}_k - \hat{F} \cdot \hat{\delta}_k = ((F_n - F) * \delta_k)^\wedge$ ”, which does not hold for windowed Fourier transform.

In [1, Lemma 2 and Theorem 2], inversion formulae for windowed Fourier transform of a function and a Boehmian are respectively proposed.

- Both are not meaningful, because the left-hand sides are depending on  $n$  and the right-hand sides are independent of  $n$ .
- Lemma 2 is neither proved nor justified by any reference.
- In the statement of Theorem 2, windowed Fourier transform  $\tilde{F}_g(\nu, t)$  of  $F \in B_{L_1}$  is used as a function on  $\mathbb{R}^2$ , whereas according to article [1], definition of windowed Fourier transform of a Boehmian is a function on  $\mathbb{R}$ .
- In the proof of Theorem 2, there is a statement as follows. “by Lemma 1,  $\|f_n * \delta_k - F * \delta_n\| \rightarrow 0$  as  $n \rightarrow \infty$ ”. Here, referring to Lemma 1 is irrelevant and the proof of the statement is left in the air.

Thus, the work on windowed Fourier transform for integrable Boehmians proposed in [1] is not meaningful.

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