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ON CAMINA GROUP AND ITS GENERALIZATIONS

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Abstract. In this paper we present some new results on Camina groups. Infinite generalizations of Camina groups and generalized Camina groups are also discussed. We further define and study some new group structures which arise out of inter-relations between conjugacy classes, order classes, and cosets with respect to a normal subgroup.

1. Introduction

Camina groups were introduced by A.R.Camina in [2] and were further extensively studied by Macdonald, Chillag, Mann, Scoppola and Dark. For an exhaustive list of papers on Camina groups, we refer the readers to [4]. Dark and Scoppola classified Camina groups according to the following theorem ([4]).

THEOREM 1.1. *Let G be a Camina group. Then G satisfies one of the following*

1. *G is a p-group for some prime p;*
2. *G is a Frobenius group with Frobenius kernel G' and cyclic Frobenius complements;*
3. *G is a Frobenius group where the Frobenius complements are isomorphic to the quaternion group.*

In Section 2 we prove some results regarding Camina groups and discuss their classification on new lines. In Section 3 we discuss infinite generalization of Camina and Generalized Camina groups. In Section 4 we introduce some new group structures based on relations between conjugacy classes, cosets and order classes. In the following $cl(g)$ will stand for the conjugacy class of an element g in G.

2. New results on Camina groups

THEOREM 2.1. *If G is a Camina group with non-trivial center and all conjugacy classes outside $Z(G)$ are of size p^r, then G is a special p-group.*

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Proof. G has a conjugate type vector $(1, p^r)$. By Ito [7], G is a p-group. Further we must have $|G'| = p^r$ and $Z(G) \subseteq G'$. It follows that there can not reside a conjugacy class of order p^r in G' outside Z(G). Thus, $Z(G) = G'$, and G has class 2. Hence G is special (by [10, Cor. 2.4]).

In particular, for $r = 1$, $|G'| = p$, so G' is cyclic. Thus, G' and G/G' have exponent p. Therefore $\phi(G) = G' = Z(G)$, where $\phi(G)$ is the Frattini subgroup of G. Thus in this case G is extra-special (by [10, Cor. 2.5]). ■

Theorem 2.2. *Let G be a Camina group with $Z(G) = 1$ and G' its minimal normal subgroup. Then G is a Frobenius group with Frobenius kernel G', and a cyclic Frobenius complement.*

Proof. G is not a p-group since $Z(G) = 1$. By Theorem 1.1, let G be a Frobenius group with Frobenius complement isomorphic to the group of Quaternions. Let K be the Frobenius kernel of G. Since G/K is isomorphic to the quaternions, K has index 2 in G'. As $K > 1$, this implies that G' is not minimal normal, which is a contradiction. The only possibility is that G is a Frobenius group with Frobenius kernel G' and a cyclic Frobenius complement. ■

In what follows we attempt to classify Camina groups based on the number of conjugacy classes contained in G'. We start with the following definition.

Definition 2.1. An n-Camina group, where $n \geq 2$, is a Camina group G for which G' is the union of n conjugacy classes.

We note that if G is an n-Camina group then $|Z(G)| \leq n$ with equality only if $G' = Z(G)$. In [3], we have classified 2-Camina and 3-Camina groups by proving the following theorems.

Theorem 2.3. *Let G be a finite group. Then G is a 2-Camina group if and only if (i) G is a Frobenius group with Frobenius kernel Z_p^r and Frobenius complement Z_{p^r-1}, or (ii) G is an extra-special 2-group.*

Theorem 2.4. *A finite group G is a 3-Camina group if and only if (i) G is a Frobenius group with Frobenius kernel Z_p^r and Frobenius complement $Z_{(p^r-1)/2}$, or (ii) G is an extra-special* 3*-group, or (iii) G is Frobenius with Frobenius kernel Z_3^2 and Frobenius complement Q_8.*

The process can be extended further in the form of the following result.

Theorem 2.5. *If G is an n-Camina group where neither G' nor G/G' is a p-group for any prime p, then $n \geq 16$.*

Proof. If G is an n-Camina group where neither G' nor G/G' is a p-group, then G is Frobenius with cyclic Frobenius complement H and Frobenius kernel $N = G'$. It is clear that $|H| \geq 6$. As N is nilpotent $N = P \times Q$ where P is a nontrivial p-group and Q is a nontrivial p-complement. As H acts Frobeniusly on both P

and Q, $|P| = a|H| + 1$ and $|Q| = b|H| + 1$ where a and b are distinct nonnegative integers. We have $ab \geq 2$ and $a + b \geq 3$. Thus,

$$\frac{|N|-1}{|H|} = \frac{|P||Q|-1}{|H|} = \frac{(a|H|+1)(b|H|+1)-1}{|H|}$$
$$= \frac{ab|H|^2 + a|H| + b|H|}{|H|} = ab|H| + a + b.$$

As the number of conjugacy classes in G' is $n = \frac{|N|-1}{|H|} + 1$, we have $n = ab|H| + a + b + 1 \geq 2 \cdot 6 + 3 + 1 = 16$. ∎

This bound can be attained by considering a Frobenius group of order 546 with Frobenius kerenel of order 91 and a cyclic Frobenius complement of order 6.

Theorem 2.6. *There do not exist 4-Camina groups which are Frobenius with Frobenius complement isomorphic to quaternions.*

Proof. Let G be a 4-Camina group with Frobenius complement Q_8 and Frobenius kernel N. As the number of conjugacy classes in $G' = \frac{|N|-1}{8} + 2$, we have $|N| = 17$. Thus automorphism group of N is cyclic contradicting the fact that Q_8 is the Frobenius complement of N. ∎

However, there exists a 5-Camina group $C_5^2 \rtimes_f Q_8$ which is Frobenius with a quaternion complement. This example suggests that for a 5-Camina group, G' may not be a p-group.

We give below a few examples of n-Camina groups with $n \leq 8$. Examples 1 to 4 are 4-Camina groups, 5 to 7 are 5-Camina groups, 8 is a 6-Camina group, 9 is a 7-Camina group, and 10 is a 8-Camina group.

1. $C_{13} \rtimes_f C_4$
2. $C_2^4 \rtimes_f C_5$
3. $C_{19} \rtimes_f C_6$
4. $P_3 \rtimes_f C_7$ where P_3 is a 2-group of type sz(8)
5. D_{18}
6. $C_3^2 \rtimes_f C_2$
7. $C_5^2 \rtimes_f Q_8$
8. $C_2^4 \rtimes C_3$
9. $C_{19} \rtimes_f C_3$
10. D_{30}

Note: It is worth noting that for every positive integer n, D_{4n+2} is an $(n+1)$-Camina group.

2.1. Groups satisfying an arithmetic condition on conjugacy class order

Definition 2.2. A vector in the form $(1, n_1, n_2, n_3, \ldots, n_r)$ where $1 \leq n_1 \leq n_2 \leq n_3 \leq \cdots \leq n_r$ are the orders of all the conjugacy classes of the elements of the group G is called a multiple conjugate type vector of the group G.

Let G be an r-Camina group. Then it is clear that G' is a union of r conjugacy classes and the size of conjugacy classes outside G' is equal to

$|G'|$. Equivalently, the multiple conjugate type vector of G must be of the form $(1, n_1, n_2, n_3, \dots, n_{r-1}, n_r, n_r, \dots, n_r)$, where $1 + n_1 + n_2 + n_3 \dots + n_{r-1} = n_r = |G'|$. Thus r-Camina groups have multiple conjugate type vector of the form $(1, n_1, n_2, n_3, \dots, n_{r-1}, n_r, n_r, \dots, n_r)$, where $1 + n_1 + n_2 + n_3 + \dots + n_{r-1} = n_r = |G'|$.

However, just having multiple conjugate type vector of the form $(1, n_1, n_2, n_3, \dots, n_{r-1}, n_r, n_r, \dots, n_r)$ where $1 + n_1 + n_2 + n_3 + \dots + n_{r-1} = n_r$ is not sufficient for the group to be an r-Camina group. There exists a 3-group of order 243 having multiple conjugate type vector $(1, 1, 1, 1, 1, 1, 1, 1, 1, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9)$ for which G' is of order 27.

It is possible to construct such a p-group with ACCC for odd prime p as follows. Let $G = \langle a, b \rangle$ with relations $a^p = b^p = 1$ and $\gamma_4(G) = 1$. Then G/G' is elementary abelian of order p^2, and $G'' \leq \gamma_4(G) = 1$. It is well known that $G' = \langle [a, b], [a, b, a], [a, b, b] \rangle$ is elementary abelian of order p^3. Hence $|G| = p^5$. Multiple conjugate vector of these groups will have p^2 entries equal to 1 and $p^3 - 1$ entries equal to p^2.

This example led us to study the above condition on multiple conjugate type vector in some more details.

Arithmetic Condition ACCC. Let $r \geq 2$. We call any finite group G having a multiple conjugate type vector of the form $(1, n_1, n_2, \dots, n_{r-1}, n_r, n_r, \dots, n_r)$ where $1 + n_1 + n_2 + \dots + n_{r-1} = n_r$ and union of the conjugacy classes of order $1, n_1, n_2, \dots, n_{r-1}$ forms a normal subgroup N of G, as a group with arithmetic condition on conjugacy classes (or *G has ACCC*).

In the following, whenever G is a group with ACCC, N will always denote the normal subgroup which is the union of the first r conjugacy classes in the multiple conjugate type vector of G.

The following result explicitly gives the relation between Camina groups and groups with ACCC.

Theorem 2.7. *A finite group G is an r-Camina group if and only if it is a group with ACCC and $G' = N$.*

Proof. The direct part follows from the previous argument.

Conversely, if $x \notin G'$, then $|cl(x)| = n_r = |G'| = |xG'|$. Also if $y \in cl(x)$, then $y \in xG'$. So, $cl(x) \subseteq xG'$. But both of these are finite sets with $|xG'| = |cl(x)|$. Thus $xG' = cl(x)$. ∎

Groups with ACCC can further be investigated in the light of the following lemma.

Lemma 2.1. *Let G be a finite group with ACCC. Then $N \leq G'$.*

Proof. As G is a group with ACCC, all conjugacy classes of G outside N have size $|N|$. Put $M = G' \cap N$. Then $G' - M$ is a union of conjugacy classes of size

$|N|$. Hence

$$|N| \text{ divides } |G' - M| = |G'| - |M|. \tag{1}$$

First, let $G'N < G$. Choose $g \in G - G'N$. Then the coset gG' is the union of conjugacy classes of size $|N|$ and hence

$$|N| \text{ divides } |gG'| = |G'|. \tag{2}$$

From (1) and (2), $|N|$ divides $|G'| - (|G'| - |M|) = |M|$. But $M \leq N$. Thus $|N| \leq |M| \leq |N|$. Hence $N = M = G' \cap N$ and so $N \leq G'$ as required.

We may now suppose that $G'N = G$ and $G' < G$. Let $n_0 = |N/M| = |N/(G' \cap N)| = |G'N/G'| = |G/G'|$. It follows from (1) that $n_0|M| = |N|$ and $|N|$ divides $|G'| - |M| = (|G'/M| - 1)|M|$. So, n_0 divides $|G'/M| - 1$. Thus, there is an integer k such that $kn_0 = |G'/M| - 1$. So, $kn_0 + 1 = |G'/M| = |G'/(G' \cap N)| = |G'N/N| = |G/N|$. Choose an element $h \in G - (G' \cup N)$, and put $H = C_G(h)$. Then h is not in N and h has $|N|$ conjugates in G. Therefore $|H| = |G|/|N| = kn_0 + 1$. Hence $|H/(G' \cap H)|$ divides $kn_0 + 1 = |H|$. On the other hand $|H/(G' \cap H)| = |G'H/G'|$ and $|G'H/G'|$ divides $n_0 = |G/G'|$. Therefore $|H/(G' \cap H)|$ divides the highest common factor $(kn_0 + 1, n_0) = 1$. This implies that $H = G' \cap H$, so $H \leq G'$. But $h \in C_G(h) = H$ and h is not in G'. This is the required contradiction. Thus, the situation $G'N = G$ is not possible. ■

3. Infinite Camina and generalized Camina groups

In this section we give infinite generalization of Camina and generalized Camina groups. We start with the following definitions for infinite groups.

Definition 3.1. An infinite group G such that non-trivial coset xG' is a single conjugacy class $cl(x)$ in G, is called an infinite Camina group.

Defintion 3.2. An infinite group G is called an infinite Frobenius group with Frobenius kernel N if N is a proper non-trivial normal subgroup of G and $C_G(x) \leq N$ for all non-identity elements x of N.

We now give construction of some infinite Camina groups.

Let F be a field. Consider the collection G of maps $f : F \longrightarrow F$ of type $f(r) = ar + b, a \neq 0, a, b \in F$. Then G becomes a group under composition if we note that $f^{-1}(r) = \frac{r-b}{a} \in G$. Excepting the case when $F = Z/(2)$, G is a non commutative group. Any commutator of G is of the type $fgf^{-1}g^{-1}(r) = r - d - bc + ad + b$, i.e., of type $r \mapsto r + \alpha, \alpha \in F$ where $f(r) = ar + b, g(r) = cr + d, a \neq 0, c \neq 0$, and $a, b, c, d \in F$.

Conversely, given $h \in G$ of the type $h(r) = r + \alpha$ for some $\alpha \in F$, we can choose a, b, c, d such that $\alpha = b - d + ad - bc$. Further, the collection of maps $r \mapsto r + \alpha, \alpha \in F$, itself is a group. Thus G', the derived subgroup of G is given by $G' = \{h \in G : h(r) = r + \alpha, \alpha \in F\}$. Now, if $f \in G$ and f is not in G', then f is given by $f(r) = ar + b, a \neq 1, a \neq 0, a \in F, b \in F$ (note that, in the case $F = Z/(2)$, $G = \{f_1, f_2 : f_1(r) = r, f_2(r) = r + 1\}$. We exclude the case $F = Z/(2)$).

Further, if $g \in G$, consider gfg^{-1} with $g(r) = cr + d$. Then $gfg^{-1}(r) = gf(\frac{r-d}{c}) = g(\frac{a(r-d)}{c} + b) = g(\frac{ar-ad+bc}{c}) = ar - ad + bc + d$, i.e., $gfg^{-1}(r) = ar - ad + bc + d$, which is of the type $f \circ h$ with $h \in G'$ and vice versa.

Therefore $cl(f) = fG', \forall f \in G, f$ not in G'. Hence G is a Camina group.

We note that G' is an abelian group. Thus G is solvable. As $Z(G)$ is trivial, G is not nilpotent.

Since $G' = \{h \in G : h(r) = r + \alpha, \alpha \in F\}$, any typical element of the conjugacy class of $r + \alpha, \alpha \neq 0 \in F$ is $r + a\alpha$ (if $f(r) = ar + b$ then $f \circ (r + \alpha) \circ f^{-1} = r + a\alpha$). So, if $r + \beta$ is a non identity element of G', then for $a \in F$ such that $a\alpha = \beta$, $r + \beta$ will be in the conjugacy class of $r + \alpha$. Thus G' consists of only two conjugacy classes. So, G is a 2-Camina group.

For any f such that $f(r) = ar + b$, $f^n(r) = a^n(r) + b\sum_{i=0}^{n-1} a^i$. Thus for elements of G of the type $-r + b$, $b \in F$, order is 2, and all these elements lie outside G' if and only if characteristic of F is not 2.

Note that centralizer of every element in G' is contained in G'. For, let $f \in G'$ such that $f(r) = r + \alpha$, and let $g(r) = ar + \beta \in C_G(f)$. Then we have $f \circ g(r) = ar + \beta + \alpha$, and $g \circ f(r) = ar + a\alpha + \beta$. So, $a = 1$. Hence g is in G'. This shows that G is a Frobenius group.

Thus, if the underlying field F is infinite, then G is an infinite 2-Camina group which is Frobenius with abelian kernel G'. It is solvable but not nilpotent and non-torsion-free group.

By taking different infinite fields F, we get numerous examples of infinite 2-Camina groups which are solvable but not nilpotent.

The above is not the only way in which infinite Camina groups could be constructed. As was suggested to us by Prof. Evgeny Khukhro, University of Manchester, U.K. and Novosibirsk Institute of Mathematics, Russia, following is the sketch of possible construction of nilpotent infinite Camina groups.

Let G be a finite p group which is also a Camina group. By Dark and Scoppola [4], its class of nilpotency would be less than or equal to three. If $p > 3$, then it is possible to construct a Lie ring L by using inversions of the Baker-Campbell-Hausdorff (Lazard's Correspondence) formula. An infinite Lie ring with the same property can be obtained by extending the ring L. Applying Lazard's correspondence in the opposite direction one obtains an infinite Camina group for which the nilpotency class would be preserved.

Thus, one can have infinite Camina groups which are nilpotent or non nilpotent. The question however remains that , if an infinite Camina group is nilpotent then is its class of nilpotency necessarily less or equal to three? As proposed by us this question has appeared as an unsolved problem in group theory ([5]).

Unsolved problem. Let G be a nilpotent group in which every coset xG' for $x \in G \setminus G'$, is equal to the conjugacy class $cl(x)$. Is there a bound for the nilpotency class of G ? If G is finite, then its class is at most 3 (R.Dark and C. Scoppola [4]).

Marcel Herzog of Tel-Aviv University and his group has also come out with other examples of infinite Camina groups [6]. They have constructed infinite Camina groups which are non-solvable.

In his paper [8] Mark Lewis defined generalized Camina group (denoted by GCG) as a finite group G in which the conjugacy class of every element $g \in G \setminus Z(G)G'$ is gG'. These groups are isoclinic to Camina groups. In particular he showed that if G is a nilpotent generalized Camina group then its nilpotence class is at most 3. Obviously, every Camina group is a generalized Camina group.

As an example of generalized Camina group consider $G = Z_3 \rtimes Z_4$ described via generators a, b with relations $a^6 = 1, b^2 = a^3, ba = a^{-1}b$. Here $G' = \{1, a^2, a^4\}$ and $Z(G) = \{1, a^3\}$. Simple computations show that G is a generalized Camina group.

We extend the concept of generalized Camina group to infinite case and give one example in support. We define

Definition 3.3. A group G (finite or infinite) is called a generalized Camina group if the conjugacy class of every element $g \in G \setminus Z(G)G'$ is $G'g$.

We give an example of infinite generalized Camina group.

$G = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid ad \neq 0, a, b, d \in R \right\}$ where R is the field of reals. Then $G' = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in R \right\}$ is a commutator group. Here G/G' is abelian. $Z(G) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in R \right\}$ Let $g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G \setminus Z(G)G'$ with $a \neq d$. Then coset of G' w.r.t. g is $G'g = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid ad \neq 0, a, b, d, x \in R \right\} = \left\{ \begin{pmatrix} a & b + dx \\ 0 & d \end{pmatrix} \mid ad \neq 0 \right\}$. We can obtain conjugacy class of g as $cl(g) = \left\{ \begin{pmatrix} a & \frac{(aq+bs)-dq}{p} \\ 0 & d \end{pmatrix} \mid ad \neq 0 \right\}$.

Comparing the two sets $G'g$ and $cl(g)$, for every a, b, d and x we can always find p, q and s such that $b + dx = \frac{(aq+bs)-dq}{p}$ and vice versa. Thus $G'g = cl(g), \forall g \in G \setminus Z(G)G'$. As a result G is an infinite generalized Camina group.

This shows that G is an infinite generalized Camina group which is not a Camina group as $Z(G)$ is not contained in G'. Moreover, this construction is valid for any field. One only needs to take an infinite field to get the infinite group.

Remark. In the above example $G = H \times Z(G)$ where $H = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a, b \in R, a \neq 0 \right\}$ is an infinite Frobenius group.

4. Or-Cos and Or-Con Groups

Xing-Zhong, Guo-hua Qian and Wu-jie Shi [16] have shown that "If G is a finite group in which elements of same order outside the center are conjugate then either G is abelian or $G \cong S_3$.". Such groups are called OC-groups. A more general question would be to determine the structure of a finite group G with a normal subgroup N such that elements of same order in $G \setminus N$ are conjugate. In this section we try to address this and similar problems . We start with the following definitions followed by examples.

Definition 4.1. A group G is said to be an Or-Con group with respect to a normal subgroup N if for every $x \in G \setminus N$ the conjugacy class $cl(x)$ equals order class of x restricted to $G \setminus N$.

Definition 4.2. A group G is said to be an Or-Cos group with respect to a normal subgroup N if for every $x \in G \setminus N$ the coset xN equals order class of x restricted to $G \setminus N$.

In the above definitions it may happen that for some $a \in G \setminus N$ there exists $b \in N$ such that $|a| = |b|$. This observation leads us to the following definitions.

Definition 4.3. An Or-Con group with respect to normal subgroup N such that N consists of complete order classes is called a complete Or-Con group.

Definition 4.4. An Or-Cos group with respect to normal subgroup N such that N consists of complete order classes is called a complete Or-Cos group.

Definition 4.5. If a is an element of a group G, then a is said to be a real element of G if a and a^{-1} are in the same conjugacy class in G.

Examples.

1. Consider the group $G = Z_3 \rtimes Z_4$ described via generators a and b satisfying the relations $a^6 = 1$, $b^2 = a^3$, and $ba = a^{-1}b$. Then $N = \{1, a, a^2, a^3, a^4, a^5\}$ is its maximal normal subgroup. G is complete Or-Cos with respect to N, as all the elements of order 4 are in $G \setminus N$ and form a single coset of N. Since the elements $b, ab, a^2b, a^3b, a^4b, a^5b$ are in the same order class but not in a single conjugacy class, G cannot be an Or-Con group. Similarly, as $Z(G) = \{1, a^3\}$, and $G' = \{1, a^2, a^4\}$, it cannot be a Camina group. This is an example of a non abelian Or-Cos group which is neither an Or-Con group nor a Camina group.

2. Dihedral groups D_p of order $2p$, p an odd prime, described via generators a and b with relations, $a^p = 1$, $b^2 = 1$, and $ba = a^{-1}b$ are non abelian complete Or-Cos, complete Or-Con, and Camina groups with respect to $G' = Z_p$.

3. Z_4 with $N = \{1, a^2\}$ and $Z_4 \times Z_2$ with $N = \{1, a^2, b, a^2b\}$ are complete Or-Cos groups. $Z_3^2 = Z_2 \times Z_2 \times Z_2$ described via generators a, b and c with relations $a^2 = b^2 = c^2 = 1$, $ba = ab$, $ca = ac$, $cb = bc$ has 7 normal subgroups namely, $\{1, a, b, ab\}$, $\{1, a, c, ac\}$, $\{1, a, bc, abc\}$, $\{1, b, c, bc\}$, $\{1, b, ac, abc\}$, $\{1, ab, c, abc\}$,

$\{1, ab, ac, bc\}$ each of order 4 with respect to which it is Or-Cos. This is not a complete Or-Cos group. Q_8 described via generators a and b with relations $a^4 = b^2 = 1$ and $ba = a^{-1}b$ has 3 normal subgroups namely, $\{1, a, a^2, a^3\}$, $\{1, b, a^2, a^2b\}$, and $\{1, ab, a^2, a^2b\}$ for being Or-Cos. This is not a complete Or-Cos group. None of these four groups are Or-Con. Further, although each of them is a Camina group, none of them is Or-Cos with respect to G'.

Thus, a Camina group G which is also Or-Cos may not be Or-Cos with respect to G'.

4. The infinite groups with $N = 1$ which are HNN extensions are Or-Con groups which are neither Or-Cos groups nor Camina groups.

It is clear from the definitions that every OC- group is an Or-Con group with $N = Z(G)$.

We now present some basic properties and inter-relations between the different group structures defined above. $O_x|_{G\setminus N}$ will denote order class of x restricted to $G \setminus N$.

Proposition 4.1. *Z_2 is the only abelian Or-Con group.*

Proof. For an abelian group to be Or-Con, it is necessary that at least half the number of elements of G be each of different order. This is not possible, except when $|G| \leq 2$. Because if $|G| = 3$, then G has no non-trivial normal subgroup, and all non-identity elements are of same order. In case $|G| > 3$ and G is Or-Con, then there exists a normal subgroup N such that elements of same order outside N are conjugate . Thus, not only all elements outside N are singleton conjugacy classes, but order classes outside N are all singleton sets. Also, $|N| \leq |G|/2$. Hence, at least $|G|/2$ number of elements should be each of different order, and other than 1. This is not possible as all these numbers are required to be divisors of order of G. Thus Z_2 is the only abelian Or-Con group. ∎

Proposition 4.2. *If G is Or-Cos with respect to N then $G' \subseteq N$.*

Proof. We have $O_x|_{G\setminus N} = xN$ for every $x \in G \setminus N$. Thus every coset of N will contain at least one conjugacy class. Hence $G' \subseteq N$. ∎

Proposition 4.3. *If G is Or-Cos and Or-Con with respect to same normal subgroup N, then G is a Camina group with $N = G'$.*

Proof. Let G be Or-Cos as well as Or-Con with respect to N. Then $O_x|_{G\setminus N} = xN$ and $O_x|_{G\setminus N} = cl(x)$. Thus $xN = cl(x)$ for all $x \in G \setminus N$. Therefore G is a Camina group and $N = G'$. ∎

Proposition 4.4. *Let G be a Camina group.*

(1) *If G is Or-Con with respect to G', then it is Or-Cos with respect to G'.*

(2) *If G is Or-Cos with respect to G', then it is Or-Con with respect to G'.*

Proof. (1) For every $x \in G \setminus G'$, $xG' = cl(x)$ and $O_x|_{G\setminus G'} = cl(x)$. Therefore $O_x|_{G\setminus G'} = xG'$.

(2) For every $x \in G\setminus G'$, $xG' = cl(x)$ and $O_x|_{G\setminus G'} = xG'$. Therefore $O_x|_{G\setminus G'} = cl(x)$. ∎

THEOREM 4.1. *If G is a complete Or-Cos group with respect to N then N is unique.*

Proof. Let N_1 and N_2 be two normal subgroups such that G is Or-Cos with respect to N_1 and N_2. We first show that we cannot have $N_1 \subset N_2$. If possible, let $N_1 \subset N_2$. Then for $x \in G \setminus N_2$ we shall have $O_x = xN_2$, and $O_x = xN_1$. This implies $|N_1| = |N_2|$ which is not possible. Similarly, we cannot have $N_2 \subset N_1$.

Let, if possible $x \in N_1$ but x is not in N_2, and $y \in N_2$ but y is not in N_1. So, $O_x = xN_2$, and as $x \in N_1$ and G is complete Or-Cos with respect to N_1, we have $O_x \subset N_1$. This implies $|N_2| = |O_x| < |N_1|$, i.e. $|N_2| < |N_1|$. Similarly, using y we get $|N_1| < |N_2|$. This is a contradiction. Therefore $N_1 = N_2$. ∎

REMARK. We notice that if G is not a complete Or-Cos group with respect to N then N may not be unique, as can be seen from examples Q_8 and $Z_2^3 = Z_2 \times Z_2 \times Z_2$.

Following results have been suggested by Prof. Mark Lewis, Kent State University, Ohio.

THEOREM 4.2. *Let G be an Or-Cos group with respect to the normal subgroup N, then G/N is an elementary abelian 2-group.*

Proof. Let $a \in G \setminus N$. Then a and a^{-1} belong to the same coset of N , as order of a and a^{-1} is same. Hence $aN = a^{-1}N$, i.e. aN has order 2 in G/N. Thus G/N has exponent 2. Therefore G/N is an elementary abelian 2-group. ∎

THEOREM 4.2. *If G is an Or-Con group with respect to the normal subgroup N, then $G/G'N$ is an elementary abelian 2-group and G is Or-Cos with respect to $G'N$.*

Proof. Let $a \in G \setminus N$. Then a and a^{-1} are in the same conjugacy class of G. Hence, aN and $a^{-1}N$ are in the same conjugacy class of G/N. So, all conjugacy classes of G/N are real. Therefore, all irreducible characters of $G/G'N$ are real. Hence, $G/G'N$ is an elementary abelian 2-group. If a and b are elements of $G\setminus G'N$ and are of same order, then a and b are conjugate in G. All conjugates of a will lie in $aG'N$ as $G/G'N$ is abelian. Therefore $b \in aG'N$. ∎

COROLLARY 4.1. *Every Or-Con group is also an Or-Cos group.*

Proof is immediate from the above Theorem.

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REFERENCES

[**1**] B. Baumslag, B. Chandler, *Theory and problems of Group Theory*, Schaum's outline series, McGraw-Hill. Inc. 1968.

[**2**] A.R. Camina, *Some conditions which almost characterize Frobenius groups*, Israel J. Math **31** (1978), 153–160.

[**3**] L. Cangelmi, A.S. Muktibodh, *Camina kernels with few conjugacy classes*, Algebra Discrete Math. **9** (2010), 39–49.

[**4**] R. Dark, C.M. Scoppola, *On Camina groups of prime power order*, J. Algebra **181** (1996), 787–802.

[**5**] S.H. Ghate, A.S. Muktibodh, *The Kourovka Note book of Unsolved problems in Group Theory* **17** (2010), 17.47, p.115, Novosibirsk.

[**6**] M. Herzog, Patrizia Longobardi, Mercede Maj, *On infinite Camina groups*, Ischia group theory, 2010.

[**7**] N. Ito, *On finite groups with given conjugate types*, I, Nagoya Math. J. **6** (1953), 17–28.

[**8**] M.L. Lewis, *Generalizing Camina groups and their character tables*, J. Group Theory **12** (2009), 209–218.

[**9**] I.D. MacDonald, *Theory of Groups*, Oxford University Press, 1968.

[**10**] I.D. MacDonald, *Some p-groups of Frobenius and extra-special type*, Israel J. Math. **47** (1984), 350–364.

[**11**] A.S. Muktibodh, *Generalized Con-Cos groups*, Advances in Algebra, Proceedings of the ICM Satellite Conference in Algebra and related topics (2003), 434–441.

[**12**] A.S. Muktibodh, *Characterization of Frobenius groups of special type*, Math. J. Okayama Univ. **48** (2006), 73–76.

[**13**] D.J.S. Robinson, *A Course in the Theory of Groups* (GTM 80, second edition), 1996, Springer, New York.

[**14**] A. Vera Lopez, J. Vera Lopez, *Classification of finite groups according to number of conjugacy classes*, Israel J. Math. **51** (1985).

[**15**] L. Verardi, *Gruppi semiextraspeciali di esponente p*, Ann. Mat. Pura Appl. **148** (1987), 131–171.

[**16**] X. You, G. Qian, W. Shi, *Finite groups in which elements of same order outside the center are conjugate*, Science In China, Series A : Mathematics **50** (2007), 1493–1500.

[**17**] J.P. Zhang, *On Syskin problem of finite groups*, Sci. Sinica **2** (1988), 189–193.

[**18**] J.P. Zhang *On finite groups all of whose elements of the same order are conjugate in their automorphism groups*, J. Algebra **153** (1992), 22–36.

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