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RELATIVE ORDER OF ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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Abstract. In this paper we introduce the idea of relative order of entire functions of several complex variables. After proving some basic results, we observe that the relative order of a transcendental entire function with respect to an entire function is the same as that of its partial derivatives. Further we study the equality of relative order of two functions when they are asymptotically equivalent.

1. Introduction

Let f and g be two non-constant entire functions and

$$F(r) = \max\{|f(z)| : |z| = r\}, \quad G(r) = \max\{|g(z)| : |z| = r\}$$

be the maximum modulus functions of f and g respectively. Then $F(r)$ is a strictly increasing and continuous function of r and its inverse

$$F^{-1} : (|f(0)|, \infty) \to (0, \infty) \text{ exists and } \lim_{R\to\infty} F^{-1}(R) = \infty.$$

Bernal [3] introduced the definition of relative order of f with respect to g as

$$\rho_g(f) = \inf\{\mu > 0 : F(r) < G(r^\mu) \text{ for all } r > r_0(\mu) > 0\}.$$

During the past decades, several authors made close investigations on the properties of entire functions related to relative order. In the case of relative order, it therefore seems reasonable to define suitably the relative order of entire functions of several complex variables and to investigate its basic properties, which we attempts in this paper. In this regards we first need the following definition of order of entire functions.

Let $f(z_1, z_2)$ be a non-constant entire function of two complex variables z_1 and z_2, holomorphic in the closed polydisc

$$\{ (z_1, z_2) : |z_j| \le r_j,\ j = 1, 2 \text{ for all } r_1 \ge 0, r_2 \ge 0 \}.$$

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Let

$$F(r_1, r_2) = \max\{\, |f(z_1, z_2)| : |z_j| \le r_j,\ j = 1, 2\,\}.$$

Then by the Hartogs theorem and maximum principle [4, p. 2, p. 51] $F(r_1, r_2)$ is an increasing function of r_1, r_2. The order $\rho = \rho(f)$ of $f(z_1, z_2)$ is defined [4, p. 338] as the infimum of all positive numbers μ for which

$$F(r_1, r_2) < \exp[(r_1 r_2)^{\mu}] \tag{1.1}$$

holds for all sufficiently large values of r_1 and r_2. In other words

$$\rho(f) = \inf\{\, \mu > 0 : F(r_1, r_2) < \exp[(r_1 r_2)^{\mu}] \text{ for all } r_1 \ge R(\mu), r_2 \ge R(\mu)\,\}.$$

Equivalent formula for $\rho(f)$ is [4, p. 339] (see also [1]) is

$$\rho(f) = \limsup_{r_1,\, r_2 \to \infty} \frac{\log\log F(r_1, r_2)}{\log(r_1 r_2)}.$$

A more general approach to the problem of relative order of entire functions has been demonstrated by Kiselman [7].

Let h and k be two functions defined on $\Re$ such that $h, k : \Re \to [-\infty, \infty]$. The *order of h relative to k* is

$$\operatorname{order}(h : k) = \inf[a > 0 : \exists\ c_a \in \Re, \forall x \in \Re, f(x) \le a^{-1} g(ax) + c_a].$$

If H is an entire function then the growth function of H is defined by

$$h(t) = \sup[\log|H(z)|, |z| \le e^t], t \in \Re.$$

If H and K are two entire functions then the *order of H relative to K* is now defined by

$$\operatorname{order}(H : K) = \operatorname{order}(h : k).$$

As observed by Kiselman [7], the expression $a^{-1}g(ax) + c_a$ may be replaced by $g(ax) + c_a$ if $g(t) = e^t$ because then the infimum in the cases coincide. Taking $c_a = 0$ in the above definition, one may easily verify that

$$\operatorname{order}(H : K) = \rho_K(H)$$

i.e., the order $(H : K)$ coincides with the Bernal's definition of relative order.

Further if $K = \exp z$ then order $(H : K)$ coincides with the classical order of H.

In papers [5, 6, 7] detailed investigations on entire functions and relative order $(H : K)$ was made, but our analysis of relative order, generated from Bernal's relative order, made in the present paper have little relevance to the studies made in the above papers by Kiselman and others.

In 2007 Banerjee and Dutta [2] introduced the definition of relative order of an entire function $f(z_1, z_2)$ with respect to an entire function $g(z_1, z_2)$ as follows:

Definition 1.1. Let $g(z_1, z_2)$ be an entire function holomorphic in the closed polydisc $\{(z_1, z_2) : |z_j| \le r_j;\ j = 1, 2\}$ and let

$$G(r_1, r_2) = \max\{|g(z_1, z_2)| : |z_j| \le r_j,\ j = 1, 2\}.$$

The relative order of f with respect to g, denoted by $\rho_g(f)$ and is defined by

$$\rho_g(f) = \inf\{\, \mu > 0 : F(r_1, r_2) < G\,(r_1^{\mu},\ r_2^{\mu})\,;\ \text{for}\ \ r_1 \ge R(\mu),\ r_2 \ge R(\mu)\,\}.$$

The definition coincides with that of classical (1.1) if $g(z_1, z_2) = e^{z_1 z_2}$.

In this paper we introduce the idea of relative order of entire functions of several complex variables.

Definition 1.2. Let $f(z_1, z_2, \ldots, z_n)$ and $g(z_1, z_2, \ldots, z_n)$ be two entire functions of n complex variables $z_1, z_2, \ldots, z_n$ with maximum modulus functions $F(r_1, r_2, \ldots, r_n)$ and $G(r_1, r_2, \ldots, r_n)$ respectively then relative order of f with respect to g, denoted by $\rho_g(f)$ and is defined by

$$\rho_g(f) = \inf\{\, \mu > 0 : F(r_1, r_2, \ldots, r_n) < G(r_1^{\mu}, r_2^{\mu}, \ldots, r_n^{\mu});$$
$$\text{for}\ r_i \ge R(\mu);\ i = 1, 2, , \ldots, n\,\}.$$

Note 1.3. If we consider $n = 2$ then Definition 1.2 coincides with Definition 1.1.

The following definition will be needed.

Definition 1.4. The function $g(z_1, z_2, \ldots, z_n)$ is said to have the property (R) if for any $\sigma > 1$ and for all large $r_1,\ r_2, \ldots, r_n$,

$$[G(r_1, r_2, \ldots, r_n)]^2 < G(r_1^{\sigma}, r_2^{\sigma}, \ldots, r_n^{\sigma}).$$

The function $g(z_1, z_2, \ldots, z_n) = e^{z_1 z_2, \ldots, z_n}$ has the property (R) but $g(z_1, z_2, \ldots, z_n) = z_1 z_2, \ldots, z_n$ does not have the property (R).

Throughout the paper, we shall assume f, g, h etc. are non-constant entire functions of several complex variables and $F(r_1, r_2, \ldots, r_n)$, $G(r_1, r_2, \ldots, r_n)$, $H(r_1, r_2, \ldots, r_n)$ etc. denote respectively their maximum modulus in the polydisc $\{(z_1, z_2, \ldots, z_n) : |z_j| \le r_j,\ j = 1, 2, \ldots, n\}$. Also we shall consider non-constant polynomials.

2. Lemmas

The following lemmas will be required.

Lemma 2.1. *Let g have the property (R). Then for any positive integer p and for all $\sigma > 1$,*

$$[G(r_1, r_2, \ldots, r_n)]^p < G\,(r_1^{\sigma}, r_2^{\sigma}, \ldots, r_n^{\sigma})$$

holds for all large $r_1, r_2, \ldots, r_n$.

Proof. Let p be any positive integer. Then there exists an integer m such that $2^m > p$. Also we have $\sigma^{2^{-m}} > 1$. Now

$$\begin{aligned}
G(r_1^\sigma, r_2^\sigma, \ldots, r_n^\sigma) &= G((r_1^{\sigma^{1/2}})^{\sigma^{1/2}},\ (r_2^{\sigma^{1/2}})^{\sigma^{1/2}}, \ldots,\ (r_n^{\sigma^{1/2}})^{\sigma^{1/2}}) \\
&\geq [G(r_1^{\sigma^{1/2}}, r_2^{\sigma^{1/2}}, \ldots, r_n^{\sigma^{1/2}})]^2 \\
&= [G((r_1^{\sigma^{1/4}})^{\sigma^1/4}, (r_2^{\sigma^{1/4}})^{\sigma^{1/4}}, \ldots,\ (r_n^{\sigma^{1/4}})^{\sigma^{1/4}})]^2 \\
&\geq [G(r_1^{\sigma^{1/4}}, r_2^{\sigma^{1/4}}, \ldots,\ r_n^{\sigma^{1/4}})]^4 \\
&\geq \cdots \\
&\geq [G(r_1^{\sigma^{2^{-m}}}, r_2^{\sigma^{2^{-m}}}, \ldots,\ r_n^{\sigma^{2^{-m}}})]^{2^m} \\
&\geq [G(r_1, r_2, \ldots,\ r_n)]^{2^m} \text{ because } G(r_1, r_2, \ldots,\ r_n) \text{ is increasing,} \\
&\geq [G(r_1, r_2, \ldots,\ r_n)]^p.
\end{aligned}$$

This completes the proof. ∎

Lemma 2.2. *Let $f(z_1, z_2, \ldots, z_n)$ be nonconstant entire and $\alpha > 1$, $0 < \beta < \alpha$. Then*

$$F(\alpha r_1, \alpha r_2, \ldots, \alpha r_n) > \beta F(r_1, r_2, \ldots, r_n) \quad \textit{for all large} \ \ r_1, r_2, \ldots, r_n.$$

Proof. Let the $\max\{|f(z_1, z_2, \ldots, z_n)| : |z_j| \leq r_j; j = 1, 2, \ldots, n\}$ be attained at $(s_1, s_2, \ldots, s_n)$ where $|s_1| = r_1$, $|s_2| = r_2$, $\ldots$, $|s_n| = r_n$. If the maximum is attained at more then one point, we choose any one of them. Consider the function

$$h(z_1) = f(z_1, s_2, \ldots, s_n).$$

Then $h(z_1)$ is an entire function of one variable z_1 and

$$\begin{aligned}
H(r_1) &= \max\{|h(z_1)| : |z_1| \leq r_1\} \\
&= \max\{|f(z_1, s_2, \ldots, s_n)| : |z_1| \leq r_1\} \\
&= |f(s_1, s_2, \ldots, s_n)| \\
&= F(r_1, r_2, \ldots, r_n). \qquad (2.1)
\end{aligned}$$

On the other hand if $g(z_1) = h(z_1) - h(0)$ then $g(0) = 0$ and so by Schwarz Lemma

$$|g(z_1)| \leq \frac{G(R)}{R}|z_1| \ \text{ for } \ |z_1| \leq R.$$

If $R = \alpha r_1$, then

$$G(r_1) \leq \frac{r_1}{\alpha r_1}\, G(\alpha r_1) = \frac{G(\alpha r_1)}{\alpha}$$

and so

$$H(r_1) - |h(0)| \leq G(r_1) \leq \frac{G(\alpha r_1)}{\alpha} \leq \frac{H(\alpha r_1) + |h(0)|}{\alpha}.$$

Let $q = \frac{\alpha-\beta}{1+\alpha}$. There exists $r_0 > 0$ such that $|h(0)| < qH(r_1)$. So for $r_1 > r_0$, we have

$$H(\alpha r_1) > [\alpha - (\alpha+1)q)]H(r_1) = \beta H(r_1). \tag{2.2}$$

Combining (2.1) and (2.2) we see that

$$F(\alpha r_1, \alpha r_2, \dots, \alpha r_n) > F(\alpha r_1, r_2, \dots, r_n) = H(\alpha r_1) > \beta H(r_1) = \beta F(r_1, r_2, \dots, r_n).$$

This proves the lemma. ■

LEMMA 2.3. *Let $f(z_1, z_2, \dots, z_n)$ be nonconstant entire function, $s > 1$, $0 < \mu < \lambda$ and n is a positive integer. Then*

(a) $\exists K = K(s, f) > 0$ such that $[F(r_1, r_2, \dots, r_n)]^s \le KF(r_1^s, r_2^s, \dots, r_n^s)$ for $r_1, r_2, \dots, r_n > 0$;

(b) $\displaystyle\lim_{r_1, r_2, \dots, r_n \to \infty} \frac{F(r_1^s,\ r_2^s, \dots, r_n^s)}{F(r_1,\ r_2, \dots, r_n)} = \infty = \lim_{r_1, r_2, \dots, r_n \to \infty} \frac{F(r_1^\lambda, r_2^\lambda, \dots, r_n^\lambda)}{F(r_1^\mu, r_2^\mu, \dots, r_n^\mu)}$.

The proof is omitted.

3. Preliminary theorem

THEOREM 3.1. *Let f, g, h be entire functions of several complex variables. Then*

(a) if f is a polynomial and g is transcendental entire, then $\rho_g(f) = 0$;

(b) if $F(r_1, r_2, \dots, r_n) \le H(r_1, r_2, \dots, r_n)$ for all large $r_1, r_2, \dots, r_n$, then $\rho_g(f) \le \rho_g(h)$.

Proof. (a) If f is a polynomial and g is transcendental entire, then there exists a positive integer p such that

$$F(r_1, r_2, \dots, r_n) \le M r_1^p r_2^p, \dots, r_n^p$$

and

$$G(r_1, r_2, \dots, r_n) > K r_1^m r_2^m, \dots, r_n^m$$

for all large $r_1, r_2, \dots, r_n$, where M and K are constant and $m > 0$ may be any real number. We have then for all large $r_1, r_2, \dots, r_n$ and $\mu > 0$,

$$\begin{aligned} G(r_1^\mu, r_2^\mu, \dots, r_n^\mu) &> K(r_1^\mu r_2^\mu, \dots, r_n^\mu)^m \\ &> M r_1^p r_2^p, \dots, r_n^p, \text{ by choosing } m \text{ suitably} \\ &\ge F(r_1, r_2, \dots, r_n). \end{aligned}$$

Thus for all large $r_1, r_2, \dots, r_n$ and $\mu > 0$,

$$F(r_1, r_2, \dots, r_n) < G(r_1^\mu, r_2^\mu, \dots, r_n^\mu).$$

Since $\mu > 0$ is arbitrary, we must have

$$\rho_g(f) \le 0, \text{ i.e., } \rho_g(f) = 0.$$

(b) Let $\epsilon > 0$ be arbitrary then from the definition of relative order, we have

$$H(r_1, r_2, \ldots, r_n) < G(r_1^{\rho_g(h)+\epsilon}, r_2^{\rho_g(h)+\epsilon}, \ldots, r_n^{\rho_g(h)+\epsilon}).$$

So for all large $r_1, r_2, \ldots, r_n$,

$$F(r_1, r_2, \ldots, r_n) \leq H(r_1, r_2, \ldots, r_n) < G(r_1^{\rho_g(h)+\epsilon}, r_2^{\rho_g(h)+\epsilon}, \ldots, r_n^{\rho_g(h)+\epsilon}).$$

So, $\rho_g(f) \leq \rho_g(h) + \epsilon$. Since $\epsilon > 0$ is arbitrary,

$$\rho_g(f) \leq \rho_g(h).$$

This completes the proof. ■

4. Sum and product theorems

THEOREM 4.1. *Let f_1 and f_2 be entire functions of several complex variables having relative orders $\rho_g(f_1)$ and $\rho_g(f_2)$ respectively. Then*

(i) $\rho_g(f_1 \pm f_2) \leq \max\{\rho_g(f_1),\ \rho_g(f_2)\}$

and

(ii) $\rho_g(f_1.f_2) \leq \max\{\rho_g(f_1),\ \rho_g(f_2)\}$,

provided g has the property (R). The equality holds in (i) if $\rho_g(f_1) \neq \rho_g(f_2)$.

Proof. First suppose that relative order of f_1 and f_2 are finite, if one of them of both are infinite then the theorem is trivial. Let $f = f_1 + f_2$, $\rho = \rho_g(f)$, $\rho_i = \rho_g(f_i)$, $i = 1, 2$ and $\rho_1 \leq \rho_2$. Therefore for any $\epsilon > 0$ and for all large $r_1, r_2, \ldots, r_n$

$$F_1(r_1, r_2, \ldots, r_n) < G(r_1^{\rho_1+\epsilon}, r_2^{\rho_1+\epsilon}, \ldots, r_n^{\rho_1+\epsilon}) \leq G(r_1^{\rho_2+\epsilon}, r_2^{\rho_2+\epsilon}, \ldots, r_n^{\rho_2+\epsilon})$$

and

$$F_2(r_1, r_2, \ldots, r_n) < G(r_1^{\rho_2+\epsilon}, r_2^{\rho_2+\epsilon}, \ldots, r_n^{\rho_2+\epsilon})$$

hold. So for all large $r_1, r_2, \ldots, r_n$,

$$\begin{aligned} F(r_1, r_2, \ldots, r_n) &\leq F_1(r_1, r_2, \ldots, r_n) + F_2(r_1, r_2, \ldots, r_n) \\ &< 2G\left(r_1^{\rho_2+\epsilon}, r_2^{\rho_2+\epsilon}, \ldots, r_n^{\rho_2+\epsilon}\right) \\ &< G\left(3r_1^{\rho_2+\epsilon}, 3r_2^{\rho_2+\epsilon}, \ldots, 3r_n^{\rho_2+\epsilon}\right), \text{ by Lemma 2.2} \\ &< G\left(r_1^{\rho_2+3\epsilon}, r_2^{\rho_2+3\epsilon}, \ldots, r_n^{\rho_2+3\epsilon}\right). \\ \therefore \quad \rho &\leq \rho_2 + 3\epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary,

$$\rho \leq \rho_2. \tag{4.1}$$

Next let $\rho_1 < \rho_2$ and suppose $\rho_1 < \mu < \lambda < \rho_2$. Then for all large $r_1, r_2, \ldots, r_n$

$$F_1(r_1, r_2, \ldots, r_n) < G(r_1^{\mu},\ r_2^{\mu}, \ldots, r_n^{\mu}) \tag{4.2}$$

and there exists a nondecreasing sequence $\{r_{ip}\}$, $r_{ip} \to \infty$; $i = 1, 2, \ldots, n$ as $p \to \infty$ such that

$$F_2(r_{1p}, r_{2p}, \ldots, r_{np}) > G(r_{1p}^{\lambda}, r_{2p}^{\lambda}, \ldots, r_{np}^{\lambda}) \text{ for } p = 1, 2, \ldots \tag{4.3}$$

Using Lemma 2.3(b), we see that

$$G(r_1^{\lambda},\ r_2^{\lambda}, \ldots, r_n^{\lambda}) > 2G(r_1^{\mu}, r_2^{\mu}, \ldots, r_n^{\mu}) \text{ for all large } r_1, r_2, \ldots, r_n. \tag{4.4}$$

So from (4.2), (4.3) and (4.4),

$$F_2(r_{1p}, r_{2p}, \ldots, r_{np}) > 2F_1(r_{1p}, r_{2p}, \ldots, r_{np}) \text{ for } \ p = 1, 2, \ldots$$

Therefore

$$\begin{aligned} F(r_{1p}, r_{2p}, \ldots, r_{np}) &\geq F_2(r_{1p}, r_{2p}, \ldots, r_{np}) - F_1(r_{1p}, r_{2p}, \ldots, r_{np}) \\ &> \frac{1}{2} F_2(r_{1p}, r_{2p}, \ldots, r_{np}) \\ &> \frac{1}{2} G\left(r_{1p}^{\lambda},\ r_{2p}^{\lambda}, \ldots, r_{np}^{\lambda}\right), \text{ from (4.3)} \\ &> G\left((1/3) r_{1p}^{\lambda},\ (1/3) r_{2p}^{\lambda}, \ldots, (1/3) r_{np}^{\lambda}\right) \\ &\qquad \text{for all large } p \text{ and by Lemma 2.2} \\ &> G\left(r_{1p}^{\lambda-\epsilon}, r_{2p}^{\lambda-\epsilon}, \ldots, r_{np}^{\lambda-\epsilon}\right), \end{aligned}$$

where $\epsilon > 0$ is arbitrary. This gives $\rho \geq \lambda - \epsilon$ and since $\lambda \in (\rho_1, \rho_2)$ and $\epsilon > 0$ is arbitrary, we have

$$\rho \geq \rho_2. \tag{4.5}$$

Combining (4.1) and (4.5),

$$\rho_g(f_1 + f_2) = \rho_g(f_2) = \max\{\rho_g(f_1),\ \rho_g(f_2)\}.$$

For the second part, we let $f = f_1 \cdot f_2$, $\rho = \rho_g(f)$ and $\rho_g(f_1) \leq \rho_g(f_2)$. Then

$$\begin{aligned} F(r_1, r_2, \ldots, r_n) &\leq F_1(r_1, r_2, \ldots, r_n) \cdot F_2(r_1, r_2, \ldots, r_n) \\ &< G\left(r_1^{\rho_1+\epsilon}, r_2^{\rho_1+\epsilon}, \ldots, r_n^{\rho_1+\epsilon}\right) \cdot G\left(r_1^{\rho_2+\epsilon}, r_2^{\rho_2+\epsilon}, \ldots, r_n^{\rho_2+\epsilon}\right) \\ &\qquad \text{for arbitrary } \epsilon > 0 \\ &\leq \left[G\left(r_1^{\rho_2+\epsilon}, r_2^{\rho_2+\epsilon}, \ldots, r_n^{\rho_2+\epsilon}\right)\right]^2 \\ &< G\left(r_1^{\sigma(\rho_2+\epsilon)}, r_2^{\sigma(\rho_2+\epsilon)}, \ldots, r_n^{\sigma(\rho_2+\epsilon)}\right), \text{ for any } \sigma > 1 \end{aligned}$$

since g has the property (R). So

$$\rho \leq \sigma(\rho_2 + \epsilon).$$

Now letting $\epsilon \to 0$ and $\sigma \to 1_+$, we have

$$\rho \leq \rho_2.$$

$$\therefore \quad \rho_g(f_1 \cdot f_2) \leq \rho_g(f_2) = \max\{\rho_g(f_1),\ \rho_g(f_2)\}.$$

This completes the proof. ■

5. Relative order of the partial derivatives

Regarding the relative order of f and its partial derivatives $\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \dots, \frac{\partial f}{\partial z_n}$ with respect to g and $\frac{\partial g}{\partial z_1}, \frac{\partial g}{\partial z_2}, \dots, \frac{\partial g}{\partial z_n}$, we prove the following theorem.

THEOREM 5.1. *If f and g are transcendental entire functions of several complex variables and g has the property (R) then*

$$\rho_g\left(\frac{\partial f}{\partial z_1}\right) = \rho_g(f) = \rho_{\frac{\partial g}{\partial z_1}}(f).$$

Proof. We write

$$\overline{F}(r_1, r_2, \dots, r_n) = \max_{|z_j|=r_j,\, j=1,2,\dots,n} \left|\frac{\partial f(z_1, z_2, \dots, z_n)}{\partial z_1}\right|$$

and

$$\overline{G}(r_1, r_2, \dots, r_n) = \max_{|z_j|=r_j,\, j=1,2,\dots,n} \left|\frac{\partial g(z_1, z_2, \dots, z_n)}{\partial z_1}\right|.$$

Let $(z_1', z_2', \dots, z_n')$ be such that

$$|f(z_1', z_2', \dots, z_n')| = \max_{|z_j|=r_j,\, j=1,2,\dots,n} |f(z_1, z_2, \dots, z_n)|.$$

Without loss of generality we may assume that $f(0, z_2', \dots, z_n') = 0$. Otherwise we set

$$h(z_1, z_2, \dots, z_n) = z_1 f(z_1, z_2, \dots, z_n).$$

Then $h(0, z_2', \dots, z_n') = 0$ and $\rho_g(f) = \rho_g(h)$. We may write, for fixed z_i on $|z| = r_i;\ i = 2, 3, \dots, n$

$$f(z_1, z_2, \dots, z_n) = \int_0^{z_1} \frac{\partial f(t, z_2, \dots, z_n)}{\partial t}\, dt,$$

where the line of integration is the segment from $z = 0$ to $z = re^{i\theta_0}$, $r > 0$. Now

$$\begin{aligned} F(r_1, r_2, \dots, r_n) &= |f(z_1', z_2', \dots, z_n')| \\ &= \left|\int_0^{z_1'} \frac{\partial f(t, z_2', \dots, z_n')}{\partial t}\, dt\right| \\ &\le r_1 \max_{|z_1|=r_1} \left|\frac{\partial f(z_1, z_2', \dots, z_n')}{\partial z_1}\right| \\ &= r_1\, \overline{F}(r_1, r_2, \dots, r_n). \end{aligned} \tag{5.1}$$

Let $(z_1'', z_2'', \dots, z_n'')$ be such that

$$\left|\frac{\partial f(z_1'', z_2'', \dots, z_n'')}{\partial z_1}\right| = \max_{|z_j|=r_j,\, j=1,2,\dots,n} \left|\frac{\partial f(z_1, z_2, \dots, z_n)}{\partial z_1}\right|.$$

Let C denote the circle $|t - z_1''| = r_1$. So,

$$
\begin{aligned}
\overline{F}(r_1, r_2, \ldots, r_n) &= \max_{|z_j|=r_j,\, j=1,2,\ldots,n} \left| \frac{\partial f(z_1, z_2, \ldots, z_n)}{\partial z_1} \right| \\
&= \left| \frac{\partial f(z_1'', z_2'', \ldots, z_n'')}{\partial z_1} \right| \\
&= \left| \frac{1}{2\pi i} \oint_C \frac{f(t, z_2'', \ldots, z_n'')}{(t - z_1'')^2}\, dt \right| \\
&\le \frac{1}{2\pi} \frac{F(2r_1, r_2, \ldots, r_n)}{r_1^2} 2\pi r_1 \\
&= \frac{F(2r_1, r_2, \ldots, r_n)}{r_1}. \qquad (5.2)
\end{aligned}
$$

From (5.1) and (5.2) we obtain

$$
\frac{F(r_1, r_2, \ldots, r_n)}{r_1} \le \overline{F}(r_1, r_2, \ldots, r_n) \le \frac{F(2r_1, r_2, \ldots, r_n)}{r_1} \le F(2r_1, r_2, \ldots, r_n) \qquad (5.3)
$$

for $r_1, r_2, \ldots, r_n \ge 1$.

Now by the definition of $\rho_g(\frac{\partial f}{\partial z_1})$, for given $\epsilon > 0$

$$
\overline{F}(r_1, r_2, \ldots, r_n) < G\left(r_1^{\rho_g\left(\frac{\partial f}{\partial z_1}\right)+\epsilon}, r_2^{\rho_g\left(\frac{\partial f}{\partial z_1}\right)+\epsilon}, \ldots,\ r_n^{\rho_g\left(\frac{\partial f}{\partial z_1}\right)+\epsilon} \right)
$$

for $r_1, r_2, \ldots, r_n \ge r_0(\epsilon)$. Hence from (5.3)

$$
\begin{aligned}
F(r_1, r_2, \ldots, r_n) &\le r_1\ G\left(r_1^{\rho_g\left(\frac{\partial f}{\partial z_1}\right)+\epsilon}, r_2^{\rho_g\left(\frac{\partial f}{\partial z_1}\right)+\epsilon}, \ldots,\ r_n^{\rho_g\left(\frac{\partial f}{\partial z_1}\right)+\epsilon} \right) \\
&\le \left[G\left(r_1^{\rho_g\left(\frac{\partial f}{\partial z_1}\right)+\epsilon}, r_2^{\rho_g\left(\frac{\partial f}{\partial z_1}\right)+\epsilon}, \ldots,\ r_n^{\rho_g\left(\frac{\partial f}{\partial z_1}\right)+\epsilon} \right) \right]^2 \\
&\le G\left(r_1^{\sigma\left[\rho_g\left(\frac{\partial f}{\partial z_1}\right)+\epsilon\right]}, r_2^{\sigma\left[\rho_g\left(\frac{\partial f}{\partial z_1}\right)+\epsilon\right]}, \ldots, .\ r_n^{\sigma\left[\rho_g\left(\frac{\partial f}{\partial z_1}\right)+\epsilon\right]} \right)
\end{aligned}
$$

for every $\sigma > 1$, by Lemma 2.1.

Since g has the property (R). So,

$$
\rho_g(f) \le \left[\rho_g\left(\frac{\partial f}{\partial z_1} \right) + \epsilon \right] \sigma.
$$

Letting $\sigma \to 1_+$, since $\epsilon > 0$ is arbitrary, we have

$$
\rho_g(f) \le \rho_g\left(\frac{\partial f}{\partial z_1} \right). \qquad (5.4)
$$

Similarly from $\overline{F}(r_1, r_2, \ldots, r_n) \le F(2r_1, r_2, \ldots, r_n)$ of (5.3) gives

$$
\rho_g\left(\frac{\partial f}{\partial z_1} \right) \le \rho_g(f). \qquad (5.5)
$$

So from (5.4) and (5.5)

$$\rho_g\left(\frac{\partial f}{\partial z_1}\right)=\rho_g(f),$$

which proves first part of the theorem.

For the second part we see that under the hypothesis, we obtain

$$\frac{G(r_1,r_2,\ldots,r_n)}{r_1}\leq \overline{G}(r_1,r_2,\ldots,r_n)\leq G(2r_1,r_2,\ldots,r_n). \tag{5.6}$$

Now by the definition of $\rho_{\frac{\partial g}{\partial z_1}}(f)$, for given $\epsilon>0$

$$\begin{aligned}
F(r_1,r_2,\ldots,r_n) &< \overline{G}\left(r_1^{\rho_{\frac{\partial g}{\partial z_1}}(f)+\epsilon},\ r_2^{\rho_{\frac{\partial g}{\partial z_1}}(f)+\epsilon},\ldots,\ r_n^{\rho_{\frac{\partial g}{\partial z_1}}(f)+\epsilon}\right)\\
&\leq G\left(2r_1^{\rho_{\frac{\partial g}{\partial z_1}}(f)+\epsilon},\ r_2^{\rho_{\frac{\partial g}{\partial z_1}}(f)+\epsilon},\ldots,\ r_n^{\rho_{\frac{\partial g}{\partial z_1}}(f)+\epsilon}\right),\ \text{using (5.6)}\\
&< G\left(r_1^{\rho_{\frac{\partial g}{\partial z_1}}(f)+2\epsilon},\ r_2^{\rho_{\frac{\partial g}{\partial z_1}}(f)+2\epsilon},\ldots,\ r_n^{\rho_{\frac{\partial g}{\partial z_1}}(f)+2\epsilon}\right).
\end{aligned}$$

So

$$\rho_g(f)\leq \rho_{\frac{\partial g}{\partial z_1}}(f)+2\epsilon.$$

Since $\epsilon>0$ be arbitrary, this gives

$$\rho_g(f)\leq \rho_{\frac{\partial g}{\partial z_1}}(f).$$

Again from (5.6)

$$\begin{aligned}
F(r_1,r_2,\ldots,r_n) &< G\left(r_1^{\rho_g(f)+\epsilon},r_2^{\rho_g(f)+\epsilon},\ldots,\ r_n^{\rho_g(f)+\epsilon}\right)\\
&< r_1\cdot\overline{G}\left(r_1^{\rho_g(f)+\epsilon},r_2^{\rho_g(f)+\epsilon},\ldots,r_n^{\rho_g(f)+\epsilon}\right)\\
&< \left[\overline{G}\left(r_1^{\rho_g(f)+\epsilon},r_2^{\rho_g(f)+\epsilon},\ldots,r_n^{\rho_g(f)+\epsilon}\right)\right]^2\\
&\leq \overline{G}\left(r_1^{\sigma(\rho_g(f)+\epsilon)},r_2^{\sigma(\rho_g(f)+\epsilon)},\ldots,r_n^{\sigma(\rho_g(f)+\epsilon)}\right),\ \text{for any } \sigma>1.
\end{aligned}$$

So

$$\rho_{\frac{\partial g}{\partial z_1}}(f)\leq\sigma[\rho_g(f)+\epsilon].$$

Now letting $\sigma\to 1_+$, since $\epsilon>0$ is arbitrary

$$\rho_{\frac{\partial g}{\partial z_1}}(f)\leq\rho_g(f)$$

and so

$$\rho_{\frac{\partial f}{\partial z_1}}(f)=\rho_g(f).$$

Consequently,

$$\rho_g\left(\frac{\partial f}{\partial z_1}\right)=\rho_g(f)=\rho_{\frac{\partial g}{\partial z_1}}(f).$$

This proves the theorem. ■

Note 5.2. Similar result holds for other partial derivatives.

6. Asymptotic behavior

Definition 6.1. Two entire functions g_1 and g_2 are said to be asymptotically equivalent if there exists l, $0 < l < \infty$ such that

$$\frac{G_1(r_1,\ r_2,\ldots,r_n)}{G_2(r_1,\ r_2,\ldots,r_n)} \to l \quad \text{as } r_1,r_2,\ldots,r_n \to \infty,$$

and in this case we write $g_1 \sim g_2$.

If $g_1 \sim g_2$ then clearly $g_2 \sim g_1$.

Theorem 6.2. *If $g_1 \sim g_2$ and if f is an entire function of several complex variables then $\rho_{g_1}(f) = \rho_{g_2}(f)$.*

Proof. Let $\epsilon > 0$, then from Lemma 2.2 and for all large $r_1,r_2,\ldots,r_n$

$$G_1(r_1,r_2,\ldots,\ r_n) < (l+\epsilon)G_2(r_1,r_2,\ldots,\ r_n) < G_2(\alpha r_1,\alpha r_2,\ldots,\ \alpha r_n), \tag{6.1}$$

where $\alpha > 1$ is such that $l + \epsilon < \alpha$. Now,

$$\begin{aligned} F(r_1,r_2,\ldots,r_n) &< G_1\left(r_1^{\rho_{g_1}(f)+\epsilon}, r_2^{\rho_{g_1}(f)+\epsilon},\ldots,\ r_n^{\rho_{g_1}(f)+\epsilon}\right) \\ &< G_2\left(r_1^{\rho_{g_1}(f)+2\epsilon}, r_2^{\rho_{g_1}(f)+2\epsilon},\ldots,\ r_n^{\rho_{g_1}(f)+2\epsilon}\right) \quad \text{using (6.1)}. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have for all large $r_1,r_2,\ldots,r_n$

$$\rho_{g_2}(f) \le \rho_{g_1}(f).$$

The reverse inequality is clear because $g_2 \sim g_1$ and so $\rho_{g_1}(f) = \rho_{g_2}(f)$. ∎

Note 6.3. Converse of the Theorem 6.2 is not always true and the condition $g_1 \sim g_2$ is not necessary, which are shown by the following examples.

Example 6.4. Consider the functions

$$\begin{aligned} f(z_1,z_2,\ldots,z_n) &= z_1z_2,\ldots,z_n, \\ g_1(z_1,z_2,\ldots,z_n) &= z_1z_2,\ldots,z_n \ \text{ and} \\ g_2(z_1,z_2,\ldots,z_n) &= (z_1z_2,\ldots,z_n)^2. \end{aligned}$$

Then we have

$$g_1 \nsim g_2 \text{ and } \ \rho_{g_1}(f) = 1,\ \rho_{g_2}(f) = 1/2.$$

Example 6.5. Consider the functions

$$\begin{aligned} f(z_1,z_2,\ldots,z_n) &= e^{z_1z_2,\ldots,z_n}, \\ g_1(z_1,z_2,\ldots,z_n) &= e^{z_1z_2,\ldots,z_n} \ \text{ and} \\ g_2(z_1,z_2,\ldots,z_n) &= e^{2z_1z_2,\ldots,z_n}. \end{aligned}$$

Then $g_1 \nsim g_2$ but $\rho_{g_1}(f) = \rho_{g_2}(f)$.

Theorem 6.6. *Let f_1, f_2 , g be entire functions of several complex variables and $f_1 \sim f_2$. Then $\rho_g(f_1) = \rho_g(f_2)$.*

The proof is similar as the one of Theorem 6.2.

Note 6.7. Converse of the Theorem 6.6 is not always true and the condition $f_1 \sim f_2$ is not necessary, which are shown by the following examples.

Example 6.8. Consider the functions

$$f_1(z_1, z_2, \dots, z_n) = z_1 z_2, \dots, z_n,$$
$$f_2(z_1, z_2, \dots, z_n) = (z_1 z_2, \dots, z_n)^2 \text{ and}$$
$$g(z_1, z_2, \dots, z_n) = z_1 z_2, \dots, z_n.$$

Then $f_1 \nsim f_2$ and $\rho_g(f_1) \neq \rho_g(f_2)$.

Example 6.9. Consider the functions

$$f_1(z_1, z_2, \dots, z_n) = e^{z_1 z_2, \dots, z_n},$$
$$f_2(z_1, z_2, \dots, z_n) = e^{2 z_1 z_2, \dots, z_n} \text{ and}$$
$$g(z_1, z_2, \dots, z_n) = e^{z_1 z_2, \dots, z_n}.$$

Then $f_1 \nsim f_2$ but $\rho_g(f_1) = \rho_g(f_2)$.

REFERENCES

[**1**] A.K. Agarwal, *On the properties of an entire function of two cpmplex variables*, Canadian J. Math. **20** (1968), 51–57.

[**2**] D. Banerjee, R.K. Dutta, *Relative order of entire functions of two complex variables*, Internat. J. Math. Sci. & Engg. Appl. **1** (2007), 141–154

[**3**] L. Bernal, *Orden relative de crecimiento de funciones enteras*, Collect. Math. **39** (1988), 209–229.

[**4**] B.A. Fuks, *Theory of analytic functions of several complex variables*, Moscow, 1963.

[**5**] S. Halvarsson, *Growth properties of entire functions depending on a parameter*, Annales Polonici Math. **14** (1996), 71–96.

[**6**] C.O. Kiselman, *Order and type as measures of growth for convex or entire functions*, Proc. Lond. Math. Soc. **66** (1993), 152–186.

[**7**] C.O. Kiselman, *Plurisubharmonic functions and potential theory in several complex variables*, a contribution to the book project, Development of Mathematics 1950-2000, edited by Jean-Paul Pier.

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