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CONVOLUTION PROPERTIES OF A SLANTED RIGHT HALF-PLANE MAPPING

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Abstract. In the present paper, the authors identify some specific harmonic functions whose convolution with slanted right half-plane mapping is harmonic close-to-convex.

1. Introduction

A complex valued continuous function $f = u + iv$ is harmonic in a domain $D \subset C$ (complex plane) if both u and v are real harmonic in D. Clunie and Shiel-Small [2] showed that, in the unit disc $E = \{z : |z| < 1\}$, such function can be written in the form $f = h + \overline{g}$, where h and g are, respectively, known as analytic and co-analytic parts of the function f. Further, jacobian of the function f is denoted by J_f and is defined as,

$$J_f = |h'|^2 - |g'|^2.$$

The mapping f is sense preserving and locally one-to-one in E if and only if $J_f > 0$ in E. Such mappings are called locally univalent. It is well-known that the function $f = h + \bar{g}$ is locally univalent if and only if the function $w(z)= g'/h'$ (known as second dilatation of f) satisfies $|w(z)| < 1$.

We denote by S_H the class of harmonic, sense preserving and univalent functions in E, normalized by the conditions $f(0) = 0$ and $f_z(0) = 1$. So, a harmonic mapping in the class S_H has the representation $f = h + \overline{g}$, where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n. \tag{1}$$

Let S_H^0 be the subclass of S_H whose members f satisfy the additional condition $f_{\bar{z}}(0) = \bar{b_1} = 0$. Denote by K_H and C_H (respectively, K_H^0 and C_H^0), the subclasses

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of S_H (S_H^0) consisting of harmonic functions which map E onto convex and close-to-convex domains, respectively. A domain Ω is said to be convex in the direction $\phi, 0 \le \phi < \pi$, if every line parallel to the line through 0 and $e^{i\phi}$ has a connected intersection with Ω.

Convolution (Hadamard Product) of two harmonic functions $f(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \overline{z}^n$ and $F(z) = H(z) + \overline{G(z)} = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \overline{B}_n \overline{z}^n$ is denoted by $f * F$ and defined as follows:

$$\begin{aligned}(f * F)(z) &= (h * H)(z) + \overline{(g * G)(z)} \\ &= z + \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} \overline{b_n B_n} \overline{z}^n.\end{aligned}$$

Convolution of analytic univalent functions is an extensively studied subject, but not much is known about convolution of harmonic functions. Clunie and Shiel-Small [2] proved that if $\phi \in K$ and $F \in K_H$ then,

$$(\alpha\overline{\phi} + \phi) * F \in C_H \quad (|\alpha| \le 1).$$

They posed a question: if $F \in K_H$, then what is the collection of harmonic functions f, such that $F * f \in K_H$? Ruscheweyh and Salinas [5] presented a partial reply to their question by proving that if ϕ is analytic in the unit disk E then $F * \phi = \text{Re}(F) * \phi + \overline{\Im(F) * \phi} \in K_H$ for all $F \in K_H$ if and only if for each real number γ, the function $(\phi + i\gamma z\phi')$ is convex in the direction of imaginary axis.

M. Goodloe [3] proved that if $f \in K_H^0$ and ϕ is the vertical strip mapping defined by

$$\phi(z) = \frac{1}{2i \sin\alpha} \log\left[\frac{1 + ze^{i\alpha}}{1 + ze^{-i\alpha}}\right],$$

then $f * \phi \in K_H^0$.

M. Dorff et al. [4] defined the class $S^0(H_\alpha) \subset S_H^0$ which consists of harmonic functions f that map E onto slanted right half-plane $H_\alpha = \{z \in C : \text{Re}(e^{i\alpha} z) > -\frac{1}{2}, 0 \le \alpha < 2\pi\}$. Such mappings can be expressed as $f = h + \overline{g}$ where

$$h(z) + e^{-2i\alpha} g(z) = \frac{z}{1 - ze^{i\alpha}}. \tag{2}$$

M. Dorff et al. in the above paper, proved that if $f_1 \in S^0(H_{\alpha_1})$ and $f_2 \in S^0(H_{\alpha_2})$, then $f_1 * f_2 \in S^0(H_{-(\alpha_1+\alpha_2)})$.

Aim of the present paper is to study the convolution of slanted right half plane mapping with some other special harmonic functions.

2. Preliminaries

To prove our main results, we need the following lemmas.

Lemma 2.1. *If f and g are analytic in E, with $|g'(0)| < |h'(0)|$ and $h + \epsilon g$ is close-to-convex for each ϵ with $|\epsilon| = 1$, then $f = h + \overline{g}$ is harmonic close-to-convex in E.*

Lemma 2.2. *A locally univalent harmonic function* $f = h + \overline{g}$ *in* E *is a univalent mapping of* E *onto a domain convex in the direction of real axis if and only if,* $h - g$ *is a conformal univalent mapping of* E *onto a domain convex in the direction of real axis.*

Lemma 2.1 and Lemma 2.2 are due to Clunie and Shiel-Small [2].

Lemma 2.3. *Let* f *be an analytic function in* E *with* $f(0) = 0$ *and* $f'(0) \neq 0$ *and let*

$$\phi(z) = \frac{z}{(1 + ze^{i\theta_1})(1 + ze^{i\theta_2})}, \tag{3}$$

where $\theta_1, \theta_2 \in R$. *If*

$$\text{Re}\left[\frac{zf'(z)}{\phi(z)}\right] > 0,$$

for all z *in* E, *then* f *is convex in the direction of the real axis.*

Lemma 2.4. *If* ξ *and* ψ *are respectively, convex and starlike functions, such that* $\xi(0) = \psi(0) = 0$, *then for each* F *analytic in* E *and satisfying* Re $F(z) > 0, z \in E$, *we have*

$$\text{Re}\left[\frac{\psi(z)F(z) * \xi(z)}{\psi(z) * \xi(z)}\right] > 0, \qquad z \in E.$$

Lemma 2.3 is due to Pommerenke [1], whereas Lemma 2.4 is due to Ruscheweyh and Shiel-Small [6].

3. Main results

In the following result, we identify a class of harmonic mappings whose convolution with slanted right half-plane mapping is harmonic close-to-convex.

Theorem 3.1 *Let* $f_1 = h_1 + \overline{g_1}$ *be a slanted right half-plane mapping given by* $h_1(z) + e^{-2i\alpha}g_1(z) = \frac{z}{1-ze^{i\alpha}}$, $0 \leq \alpha < 2\pi$. *If* $f_2 = h_2 + \overline{g_2} \in S_H^0$ *is such that, for every* $\phi \in \mathbb{R}$, $h_2 + e^{i\phi}g_2$ *is convex analytic in* E, *then* $f_1 * f_2$ *is harmonic close-to-convex. i.e* $f_1 * f_2 \in C_H^0$.

Proof. Let us assume that

$$F_1(z) = (h_1(z) + e^{-2i\alpha}g_1(z)) * (h_2(z) - e^{i\phi}g_2(z))$$

and

$$F_2(z) = (h_1(z) - e^{-2i\alpha}g_1(z)) * (h_2(z) + e^{i\phi}g_2(z)).$$

Then, a simple calculation yields

$$\frac{F_1(z) + F_2(z)}{2} = (h_1 * h_2)(z) - e^{(i\phi - 2\alpha)}(g_1 * g_2)(z) = H(z) + \lambda G(z) \quad \text{(say)},$$

where $H = h_1 * h_2$, $G = g_1 * g_2$ and $\lambda = -e^{(i\phi-2\alpha)}$. Clearly, H and G are analytic in E and it is easy to see that

$$f_1 * f_2 = H + \overline{G}.$$

First we show that $H(z)+\lambda G(z)$is close-to-convex in E for all λ with $|\lambda|=1$ (i.e. for each $\phi\in\mathbb{R}$ and $0\le\alpha<2\pi$). Now we write

$$\begin{aligned} zF_1'(z) &= (h_1(z)+e^{-2i\alpha}g_1(z))*z(h_2(z)-e^{i\phi}g_2(z))' \\ &= \left(\frac{z}{1-ze^{i\alpha}}\right) \; * \; z(h_2(z)-e^{i\phi}g_2(z))' \\ &= z[h_2(ze^{i\alpha})-e^{i\phi}g_2(ze^{i\alpha})]', \end{aligned} \tag{4}$$

and

$$zF_2'(z) = z(h_1(z)-e^{-2i\alpha}g_1(z))'*(h_2(z)+e^{i\phi}g_2(z)). \tag{5}$$

Adding (4) and (5), we get

$$\begin{aligned} &z(F_1(z)+F_2(z))' \\ &\quad = z\left[h_2(ze^{i\alpha})-e^{i\phi}g_2(ze^{i\alpha})\right]' \\ &\qquad + \left[z(h_1(z)-e^{-2i\alpha}g_1(z))'*(h_2(z)+e^{i\phi}g_2(z))\right] \\ &\quad = z\left[h_2(ze^{i\alpha})-e^{i\phi}g_2(ze^{i\alpha})\right]' \\ &\qquad + \left[\frac{z(h_1(z)-e^{-2i\alpha}g_1(z))'(h_1(z)+e^{-2i\alpha}g_1(z))'}{(h_1(z)+e^{-2i\alpha}g_1(z))'}*(h_2(z)+e^{i\phi}g_2(z))\right] \\ &\quad = z\left[h_2(ze^{i\alpha})-e^{i\phi}g_2(ze^{i\alpha})\right]' \\ &\qquad + \left[\frac{zP_1(z)}{(1-ze^{i\alpha})^2}*(h_2(z)+e^{i\phi}g_2(z))\right], \end{aligned}$$

where

$$P_1(z)=\frac{(h_1(z)-e^{-2i\alpha}g_1(z))'}{(h_1(z)+e^{-2i\alpha}g_1(z))'}=\frac{1-e^{-2i\alpha}\frac{g_1'(z)}{h_1'(z)}}{1+e^{-2i\alpha}\frac{g_1'(z)}{h_1'(z)}}=\frac{1-e^{-2i\alpha}w_1(z)}{1+e^{-2i\alpha}w_1(z)}.$$

Now $f_1(z)=h_1(z)+\overline{g_1(z)}$ being a slanted right half-plane mapping, it belongs to S_H^0 and so it is sense preserving. Therefore $|w_1(z)|=\left|\frac{g_1'(z)}{h_1'(z)}\right|<1$. Thus gives, $\operatorname{Re}P_1(z)>0$ in E. Let $\psi(z)=h_2(ze^{i\alpha})+e^{i\phi}g_2(ze^{i\alpha})$ be any analytic convex function such that,

$$\begin{aligned} &\operatorname{Re}\left[\frac{z(F_1(z)+F_2(z))'}{z\psi'(z)}\right] \\ &= \operatorname{Re}\left[\frac{(h_2(ze^{i\alpha})-e^{i\phi}g_2(ze^{i\alpha}))'}{(h_2(ze^{i\alpha})+e^{i\phi}g_2(ze^{i\alpha}))'}\right]+\operatorname{Re}\left[\frac{\frac{zP_1(z)}{(1-ze^{i\alpha})^2}*(h_2(z)+e^{i\phi}g_2(z))}{z(h_2(ze^{i\alpha})+e^{i\phi}g_2(ze^{i\alpha}))'}\right] \\ &= \operatorname{Re}\left[\frac{1-e^{i\phi}w_2(ze^{i\alpha})}{1+e^{i\phi}w_2(ze^{i\alpha})}\right]+\operatorname{Re}\left[\frac{\frac{zP_1(z)}{(1-ze^{i\alpha})^2}*(h_2+e^{i\phi}g_2)}{\frac{z}{(1-ze^{i\alpha})^2}*(h_2+e^{i\phi}g_2)}\right]. \end{aligned}$$

Here $w_2(ze^{i\alpha}) = \frac{g_2'(ze^{i\alpha})}{h_2'(ze^{i\alpha})}$ is the dilatation function of $f_2 \in S_H^0$ and so, $|w_2(ze^{i\alpha})| < 1$. Now, using Lemma 2.4 with $F(z) = P_1(z)$, we obtain

$$\mathrm{Re}\left[\frac{(F_1(z) + F_2(z))'}{\psi'(z)}\right] > 0,\ z \in E.$$

As $\psi(z) = h_2(ze^{i\alpha}) + e^{i\phi}g_2(ze^{i\alpha})$ is convex in E, we conclude that $F_1 + F_2$ is close-to-convex in E. Moreover

$$|G'(0)| = |[g_1(z) * g_2(z)]'|_{z=0} = 0$$

and

$$|H'(0)| = \left|[h_1(z) * h_2(z)]'\right|_{z=0} = 1.$$

Hence $|G'(0)| < |H'(0)|$.

Therefore in view of Lemma 2.1, we get $H + \overline{G} = f_1 * f_2$ is harmonic close-to-convex in E. ∎

Next, we give an example showing that there exist harmonic functions f_2 which satisfy the condition of above theorem .

Example 3.1. Consider the harmonic map $f_2(z) = z + \overline{\frac{z^2}{4}}$. For $\phi \in \mathbb{R}$ write $K(z) = h_2(z) + e^{i\phi}g_2(z) = z + e^{i\phi}\frac{z^2}{4}$. Obviously

$$\mathrm{Re}\left[1 + \frac{zK''(z)}{K(z)}\right] = \mathrm{Re}\left[1 + \frac{e^{i\phi}\frac{z}{2}}{1 + e^{i\phi}\frac{z}{2}}\right] > 0, \quad \text{for } z \in E \text{ and } \phi \in \mathbb{R}.$$

So, K is convex analytic in E. Therefore, in view of Theorem 3.1, $f_1 * f_2 \in C_H^0$ for all $f_1 \in S^0(H_\alpha)$.

Before stating our next result, we define a square mapping as follows:

Let $f_0 = h_0 + \overline{g}_0$ be a harmonic map given by

$$h_0(z) + g_0(z) = tan^{-1}z, \text{ with dilatation } \quad w_0(z) = -z^2. \tag{6}$$

By using shearing technique of Clunie and Shiel-Small [2], we easily obtain,

$$h_0(z) = \frac{1}{4}\log\left[\frac{1+z}{1-z}\right] + \frac{i}{4}\log\left[\frac{i+z}{i-z}\right] \tag{7}$$

and

$$g_0(z) = -\frac{1}{4}\log\left[\frac{1+z}{1-z}\right] + \frac{i}{4}\log\left[\frac{i+z}{i-z}\right].$$

Using `Mathematica` (Version 7.0) one can verify that f_0 maps the unit disc E onto a square region, as shown in Figure 1.

M. Dorff et al. [4] studied convolution of a slanted right half-plane mapping with another slanted right half-plane mapping. As stated in Section 1, they proved that if $f_1 \in S^0(H_{\alpha_1})$ and $f_2 \in S^0(H_{\alpha_2})$, then $f_1 * f_2 \in S^0(H_{-(\alpha_1+\alpha_2)})$. In the

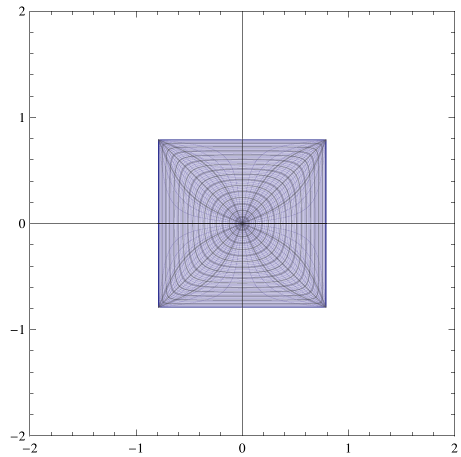

Fig. 1. Image of E under f_0

following result, we study the convolution of a slanted right half-plane mapping with the square map defined above.

THEOREM 3.2. *Let $f_1 = h_1 + \overline{g_1}$ be a slanted right half-plane mapping given by $h_1(z) + e^{-2i\alpha}g_1(z) = \frac{z}{1-ze^{i\alpha}}$, $0 \le \alpha < 2\pi$ and let $f_0 = h_0 + \overline{g_0}$ given by (6) be a square map. Then $f_1 * f_0$ is convex in the direction (-α), provided $f_1 * f_0$ is sense preserving.*

Proof. Let us write

$$F_1(z) = (h_1(z) + e^{-2i\alpha}g_1(z)) * (h_0(z) - g_0(z))$$

and

$$F_2(z) = (h_1(z) - e^{-2i\alpha}g_1(z)) * (h_0(z) + g_0(z)).$$

Then

$$\frac{F_1(z) + F_2(z)}{2} = (h_1 * h_0)(z) - e^{-2i\alpha}(g_1 * g_0)(z) = H(z) - \lambda G(z) \tag{8}$$

(say), where $H = h_1 * h_0$, $G = g_1 * g_0$ and $\lambda = e^{-2i\alpha}$. It can be easily seen that

$$f_1 * f_0 = H + \overline{G}.$$

We shall show that, $e^{i\alpha}\frac{F_1(z)+F_2(z)}{2}$ is convex in horizontal direction.

$$\begin{aligned} zF_1'(z) &= (h_1(z) + e^{-2i\alpha}g_1(z)) * z(h_0(z) - g_0(z))' \\ &= \left(\frac{z}{1 - ze^{i\alpha}}\right) * z(h_0(z) - g_0(z))' \\ &= z[h_0(ze^{i\alpha}) - g_0(ze^{i\alpha})]' \\ &= z[h_0(ze^{i\alpha}) - g_0(ze^{i\alpha})]' \frac{[h_0(ze^{i\alpha}) + g_0(ze^{i\alpha})]'}{[h_0(ze^{i\alpha}) + g_0(ze^{i\alpha})]'} \end{aligned}$$

$$= z[h_0(ze^{i\alpha}) + g_0(ze^{i\alpha})]' \frac{[h_0(ze^{i\alpha}) - g_0(ze^{i\alpha})]'}{[h_0(ze^{i\alpha}) + g_0(ze^{i\alpha})]'}$$
$$= \frac{z}{1+(ze^{i\alpha})^2} P_0(ze^{i\alpha}),$$

where

$$P_0(ze^{i\alpha}) = \frac{(h_0(ze^{i\alpha}) - g_0(ze^{i\alpha}))'}{(h_0(ze^{i\alpha}) + g_0(ze^{i\alpha}))'} = \frac{1 - \frac{g_0'(ze^{i\alpha})}{h_0'(ze^{i\alpha})}}{1 + \frac{g_0'(ze^{i\alpha})}{h_0'(ze^{i\alpha})}} = \frac{1 - w_0(ze^{i\alpha})}{1 + w_0(ze^{i\alpha})}.$$

Since $|w_0(ze^{i\alpha})| < 1$, therefore

$$\text{Re}[P_0(ze^{i\alpha})] > 0, \text{ in } E. \tag{9}$$

$$zF_2'(z) = z(h_1(z) - e^{-2i\alpha} g_1(z))' * (h_0(z) + g_0(z))$$
$$= z(h_1(z) - e^{-2i\alpha} g_1(z))' \frac{(h_1(z) + e^{-2i\alpha} g_1(z))'}{(h_1(z) + e^{-2i\alpha} g_1(z))'} * (h_0(z) + g_0(z))$$
$$= z(h_1(z) + e^{-2i\alpha} g_1(z))' \frac{(h_1(z) - e^{-2i\alpha} g_1(z))'}{(h_1(z) + e^{-2i\alpha} g_1(z))'} * (h_0(z) + g_0(z))$$
$$= \frac{z}{(1 - ze^{i\alpha})^2} P_1(z) * tan^{-1} z,$$

where

$$P_1(z) = \frac{(h_1(z) - e^{-2i\alpha} g_1(z))'}{(h_1(z) + e^{-2i\alpha} g_1(z))'} = \frac{1 - e^{-2i\alpha} \frac{g_1'(z)}{h_1'(z)}}{1 + e^{-2i\alpha} \frac{g_1'(z)}{h_1'(z)}} = \frac{1 - e^{-2i\alpha} w_1(z)}{1 + e^{-2i\alpha} w_1(z)}.$$

Obviously, $\text{Re}\, P_1(z) > 0$ in E as $|w_1(z)| = \frac{|g_1'(z)|}{|h_1'(z)|} < 1$. Now

$$z(F_1(z) + F_2(z))' = \frac{z}{1+(ze^{i\alpha})^2} P_0(ze^{i\alpha}) + \left[\frac{z}{(1 - ze^{i\alpha})^2} P_1(z) * tan^{-1} z\right]$$

Taking $\theta_1 = \frac{\pi}{2}$, and $\theta_2 = -\frac{\pi}{2}$ in (3), we get $\phi(z) = \frac{z}{1+z^2}$.

$$\text{Re}\left[e^{i\alpha} \frac{z(F_1(z) + F_2(z))'}{\phi(ze^{i\alpha})}\right]$$
$$= \text{Re}[P_0(ze^{i\alpha})] + \text{Re}\left[\frac{\frac{z}{(1-ze^{i\alpha})^2} P_1(z) * tan^{-1} z}{\left(\frac{z}{1+(ze^{i\alpha})^2}\right)}\right]$$
$$= \text{Re}[P_0(ze^{i\alpha})] + \text{Re}\left[\frac{\frac{z}{(1-ze^{i\alpha})^2} P_1(z) * tan^{-1} z}{\frac{z}{(1-ze^{i\alpha})^2} * tan^{-1} z}\right]. \tag{10}$$

Now Using Lemma 2.4 with $\xi(z) = \tan^{-1} z$, $\psi(z) = \frac{z}{(1-ze^{i\alpha})^2}$ and $F(z) = P_1(z)$ we get,

$$\text{Re}\left[\frac{\frac{z}{(1-ze^{i\alpha})^2} P_1(z) * \tan^{-1} z}{\frac{z}{(1-ze^{i\alpha})^2} * \tan^{-1} z}\right] > 0 \text{ in } E. \tag{11}$$

Using (9) and (11) in (10) we have

$$\operatorname{Re}\left[e^{i\alpha}\frac{z(F_1(z)+F_2(z))'}{\phi(ze^{i\alpha})}\right]>0,\ z\in E.$$

Thus $e^{i\alpha}(F_1(z)+F_2(z))$ is convex in horizontal direction (CHD) by Lemma 2.3. In view of (8)

$$\begin{aligned} e^{i\alpha}\frac{F_1(z)+F_2(z)}{2} &= e^{i\alpha}(H-e^{-2i\alpha}G) \\ &= e^{i\alpha}H-e^{-i\alpha}G \quad \text{is also CHD.} \end{aligned} \tag{12}$$

Thus if $f_1 * f_0 = H+\overline{G}$ is sense preserving, so is the map $e^{i\alpha}(H+\overline{G})$. Therefore by Lemma 2.2 and in view of (12), we have $e^{i\alpha}(H+\overline{G}) = e^{i\alpha}H+\overline{e^{-i\alpha}G}$ is CHD . Hence, $f_1 * f_0$ is convex in direction $(-\alpha)$. ■

We close this section by proving that, we can omit the condition of 'sense-preserving' of $f_1 * f_0$ by considering the right half-plane mapping instead of slanted right half-plane mapping.

Theorem 3.3. *Let $f_1 = h_1+\overline{g_1}$ be the right half-plane mapping given by $h_1(z)+g_1(z) = \frac{z}{1-z}$ with dilatation $w_1(z)=-z$ and f_0 be harmonic square map given by (4). Then $f_1 * f_0$ is convex in horizontal direction.*

Proof. In view of Theorem 3.2, it is sufficient to prove that $f_1 * f_0$ is sense preserving. As $f_1 = h_1+\overline{g_1}$ where

$$h_1(z)+g_1(z) = \frac{z}{1-z} \quad \text{with} \quad w_1(z)=-z,$$

a simple calculations gives

$$h_1(z) = \frac{z-\frac{1}{2}z^2}{(1-z)^2} \quad \text{and} \quad g_1(z) = \frac{-\frac{1}{2}z^2}{(1-z)^2}.$$

Now

$$\begin{aligned} f_1 * f_0 &= h_1(z) * h_0(z) + \overline{g_1(z) * g_0(z)} \\ &= H(z)+\overline{G(z)} \quad \text{(say)}, \end{aligned}$$

where

$$H(z) = h_1(z) * h_0(z) = \frac{1}{2}\left[h_0(z)+zh_0'(z)\right]$$

and

$$G(z) = g_1(z) * g_0(z) = \frac{1}{2}\left[g_0(z)-zg_0'(z)\right].$$

Let $\widetilde{W}$ be the dilatation of $f_1 * f_0$. Then

$$\widetilde{W}(z) = \frac{[g_1(z) * g_0(z)]'}{[h_1(z) * h_0(z)]'} = -\frac{zg_0''(z)}{2h_0'(z)+zh_0''(z)}.$$

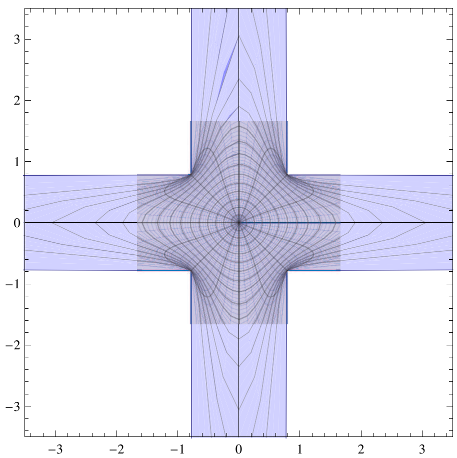

Fig. 2. Image of E under $f_1 * f_0$

Now, $g_0'(z) = w_0(z)h_0'(z)$ gives $g_0''(z) = w_0(z)h_0''(z) + w_0'(z)h_0'(z)$. So

$$\widetilde{W}(z) = \frac{-zw_0'(z)h_0'(z) - zw_0(z)h_0''(z)}{2h_0'(z) + zh_0''(z)}. \tag{13}$$

Substituting the values of the dilatation w_0 from (6) and the analytic part h_0 from (7) of f_0, respectively, in (13) we get

$$\widetilde{W}(z) = \frac{2z^2 + 2z^6}{2 + 2z^4} = z^2.$$

So $\left|\widetilde{W}(z)\right| < 1$ for all $z \in E$. This completes the proof. Image of E under $f_1 * f_0$ is plotted in Figure 2, using `Mathematica`. ■

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