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SCORE LISTS IN BIPARTITE MULTI HYPERTOURNAMENTS

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Abstract. Given non-negative integers m, n, h and k with $m \geq h \geq 1$ and $n \geq k \geq 1$, an $[h,k]$-bipartite multi hypertournament (or briefly $[h,k]$-BMHT) on $m+n$ vertices is a triple $(U, V, \mathbf{A})$, where U and V are two sets of vertices with $|U| = m$ and $|V| = n$ and $\mathbf{A}$ is a set of $(h+k)$-tuples of vertices, called arcs with exactly h vertices from U and exactly k vertices from V, such that for any $h+k$ subset $U_1 \cup V_1$ of $U \cup V$, $\mathbf{A}$ contains at least one and at most $(h+k)!$ $(h+k)$-tuples whose entries belong to $U_1 \cup V_1$. If $\mathbf{A}$ is a set of $(r+s)$-tuples of vertices, called arcs for r $(1 \leq r \leq h)$ vertices from U and s $(1 \leq s \leq k)$ vertices from V such that $\mathbf{A}$ contains at least one and at most $(r+s)!$ $(r+s)$-tuples, then the bipartite multi hypertournament is called an (h,k)-bipartite multi hypertournament (or briefly (h,k)-BMHT). We obtain necessary and sufficient conditions for a pair of sequences of non-negative integers in non-decreasing order to be losing score lists and score lists of $[h,k]$-BMHT and (h,k)-BMHT.

1. Introduction

A k-hypertournament is a complete k-hypergraph with each k-edge endowed with an orientation, that is, a linear arrangement of the vertices contained in the hyperedge. In a k-hypertournament, the score $s(v_i)$ (losing score $r(v_i)$) of a vertex v_i is the number of arcs containing v_i and in which v_i is not the last element (v_i is the last element). The score (losing score) sequence is formed by listing the scores (losing scores) in non-decreasing order. Zhou et al. [10] obtained a characterization of score and losing score sequences in k-hypertournaments, a result analogous to Landau's theorem [5] on tournament scores. More results on scores in k-hypertournaments can be found in [1, 2, 3, 4, 9, 11].

A bipartite hypergraph is a generalization of a bipartite graph. If $U = \{u_1, u_2, \ldots, u_m\}$ and $V = \{v_1, v_2, \ldots, v_n\}$ are vertex sets of a bipartite hypergraph G, the hyper edge of G is a subset of U and V, containing at least one vertex from U and at least one vertex from V. If an hyperedge has exactly h vertices from U and exactly k vertices from V, it is called an $[h,k]$-edge. If all edges of G are $[h,k]$-edges, G is said to be an $[h,k]$-bipartite hypergraph. In case G has all its hyper edges as $[i,j]$-edges for every i,j with $1 \leq i \leq h$ and $1 \leq j \leq k$, G is called an (h,k)-bipartite hypergraph.

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An $[h,k]$-bipartite hypertournament or breifly $[h,k]$-BHT ((h,k)-bipartite hypertournament or breifly (h,k)-BHT) is a complete $[h,k]$-bipartite hypergraph (complete (h,k)-bipartite hypergraph) with each $[h,k]$-edge endowed with an orientation, that is, a linear arrangement of the vertices contained in the hyper edge. The score and losing score of a vertex in $[h,k]$-BHT or (h,k)-BHT is defined in the same way as in a k-hypertournament.

Throughout this paper, m, n, h and k are non-negative integers with $m \geq h \geq 1$ and $n \geq k \geq 1$; $A = [a_i]_{i=1}^m$, $B = [b_j]_{j=1}^n$, $C = [c_i]_{i=1}^m$, $D = [d_j]_{j=1}^n$ are lists of non-negative integers in non-decreasing order, unless otherwise stated.

The following two results [8] provide characterizations of losing score lists and score lists in $[h,k]$-BHT.

Theorem 1. *(A,B) is a pair of losing score lists of an $[h,k]$-BHT if and only if for each p and q,*

$$\sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j \geq \binom{p}{h}\binom{q}{k}, \tag{1}$$

with equality when $p = m$ and $q = n$.

Theorem 2. *(C,D) is a pair of score lists of an (h,k)-BHT if and only if for each p and q,*

$$\sum_{i=1}^{p} c_i + \sum_{j=1}^{q} d_j \geq p\binom{m-1}{h-1}\binom{n}{k} + q\binom{m}{h}\binom{n-1}{k-1} + \binom{m-p}{h}\binom{n-q}{k} - \binom{m}{h}\binom{n}{k}, \tag{2}$$

with equality when $p = m$ and $q = n$.

The next two results [6] provide characterizations of losing score lists and score lists in (h,k)-BHT.

Theorem 3. *(A,B) is a pair of losing score lists of an (h,k)-BHT if and only if for each p and q,*

$$\sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j \geq \sum_{r=1}^{h}\sum_{s=1}^{k} \binom{p}{r}\binom{q}{s},$$

with equality when $p = m$ and $q = n$.

Theorem 4. *(C,D) is a pair of score lists of an (h,k)-BHT if and only if for each p and q,*

$$\sum_{i=1}^{p} c_i + \sum_{j=1}^{q} d_j \geq \sum_{r=1}^{h}\sum_{s=1}^{k} \left[p\binom{m-1}{r-1}\binom{n}{s} + q\binom{m}{r}\binom{n-1}{s-1} + \binom{m-p}{r}\binom{n-q}{s} - \binom{m}{r}\binom{n}{s}\right],$$

with equality when $p = m$ and $q = n$.

More results on bipartite hypertournaments can be found in [7].

2. Scores of $[h, k]$-bipartite multi hypertournaments

An $[h, k]$-bipartite multi hypertournament (or briefly $[h, k]$-BMHT) H on $m+n$ vertices is a triple $(U, V, \mathbf{A})$, where U and V are two sets of vertices with $|U| = m$ and $|V| = n$ and $\mathbf{A}$ is a set of $(h+k)$-tuples of vertices, called arcs with exactly h vertices from U and exactly k vertices from V, such that any $h+k$ subset $U_1 \cup V_1$ of $U \cup V$, $\mathbf{A}$ contains at least one and at most $(h+k)!$ $(h+k)$-tuples whose entries belong to $U_1 \cup V_1$. An arc in H with exactly h vertices from U and exactly k vertices from V is called an $[h, k]$-arc.

In an $[h, k]$-BMHT H, for a given vertex $u_i \in U$, the score $d_H^+(u_i)$ or simply $d^+(u_i)$ (losing score $d_H^-(u_i)$ or simply $d^-(u_i)$) is the number of $[h, k]$-arcs containing u_i and in which u_i is not the last element (in which u_i is the last element). Similarly we define by $d^+(v_j)$ and $d^-(v_j)$ as the score and losing score of a vertex $v_j \in V$. If the losing scores (scores) of vertices $u_i \in U$ and $v_j \in V$ are arranged in non-decreasing order as a pair of lists A and B (C and D), then (A, B) $((C, D))$ is called the pair of losing score lists (score lists) of an $[h, k]$-BMHT.

We note that in $[h, k]$-BMHT if the arc set $\mathbf{A}$ contains exactly one of $(h+k)!$ $(h+k)$-tuples for each $h+k$ subset $U_1 \cup V_1$ of $U \cup V$, then $[h, k]$-BMHT becomes $[h, k]$-BHT. Further, if there are exactly $(h+k)!$ $(h+k)$-tuples in the arc set $\mathbf{A}$ of $[h, k]$-BMHT for each $h+k$ subset $U_1 \cup V_1$ of $U \cup V$, then we say $[h, k]$-BMHT is complete.

Clearly there are exactly $(h+k)!\binom{m}{h}\binom{n}{k}$ arcs in a complete $[h, k]$-BMHT and so

$$\sum_{i=1}^{m} d^-(u_i) + \sum_{j=1}^{n} d^-(v_j) = (h+k)!\binom{m}{h}\binom{n}{k}.$$

From the above discussion, we have the following observation.

Lemma 5. *If (A, B) is a pair of losing score lists of a complete $[h, k]$-BMHT, then*

$$\sum_{i=1}^{m} a_i + \sum_{j=1}^{n} b_j = (h+k)!\binom{m}{h}\binom{n}{k}.$$

Evidently, in a complete $[h, k]$-BMHT, there are exactly $(h+k-1)!\binom{m-1}{h-1}\binom{n}{k}$ arcs containing a vertex $u_i \in U, (i = 1, 2, \ldots, m)$ as the last entry and there are exactly $(h+k-1)!\binom{m}{h}\binom{n-1}{k-1}$ arcs containing a vertex $v_j \in V$, $(i = 1, 2, \ldots, n)$ as the last entry. Therefore the losing scores a_i and b_j, and scores c_i and d_j in a complete [h,k]-BMHT are given as follows.

$$a_i = (h+k-1)!\binom{m-1}{h-1}\binom{n}{k},$$
$$b_j = (h+k-1)!\binom{m}{h}\binom{n-1}{k-1}$$

and

$$c_i = (h+k-1)!(h+k-1)\binom{m-1}{h-1}\binom{n}{k},$$
$$d_j = (h+k-1)!(h+k-1)\binom{m}{h}\binom{n-1}{k-1}.$$

Lemma 5 can also be proved as follows.

$$\begin{aligned}\sum_{i=1}^{m} a_i + \sum_{j=1}^{n} b_j &= \sum_{i=1}^{m}\left[(h+k-1)!\binom{m-1}{h-1}\binom{n}{k}\right] + \sum_{j=1}^{n}\left[(h+k-1)!\binom{m}{h}\binom{n-1}{k-1}\right] \\ &= (h+k-1)!\left[m\binom{m-1}{h-1}\binom{n}{k} + n\binom{m}{h}\binom{n-1}{k-1}\right] \\ &= (h+k-1)!\left[m\frac{(m-1)!}{(h-1)!(m-h)!}\binom{n}{k} + n\frac{(n-1)!}{(k-1)!(n-k)!}\binom{m}{h}\right] \\ &= (h+k-1)!\left[\frac{m!}{h!(m-h)!}h\binom{n}{k} + \frac{n!k}{k!(n-k)!}\binom{m}{h}\right] \\ &= (h+k-1)!\left[h\binom{m}{h}\binom{n}{k} + k\binom{m}{h}\binom{n}{k}\right] \\ &= (h+k)(h+k-1)!\binom{m}{h}\binom{n}{k} \\ &= (h+k)!\binom{m}{h}\binom{n}{k}.\end{aligned}$$

THEOREM 6. *If (C, D) is a pair of score lists of a complete $[h, k]$-BMHT, then*

$$\sum_{i=1}^{m} c_i + \sum_{j=1}^{n} d_j = (h+k)!(h+k-1)\binom{m}{h}\binom{n}{k}.$$

Proof. Since there are exactly

$$(h+k)!\binom{m-1}{h-1}\binom{n}{k}$$

and exactly

$$(h+k)!\binom{m}{h}\binom{n}{k}$$

arcs respectively containing a vertex $u_i \in U$ and a vertex $v_j \in V$, we have

$$\begin{aligned}\sum_{i=1}^{m} c_i + \sum_{j=1}^{n} d_j &= \sum_{i=1}^{m}\left[(h+k)!\binom{m-1}{h-1}\binom{n}{k}\right] + \sum_{j=1}^{n}\left[(h+k)!\binom{m}{h}\binom{n-1}{k-1}\right] \\ &\quad - (h+k)!\binom{m}{h}\binom{n}{k} \\ &= (h+k)!\left[m\binom{m-1}{h-1}\binom{n}{k} + n\binom{m}{h}\binom{n-1}{k-1} - \binom{m}{h}\binom{n}{k}\right]\end{aligned}$$

$$= (h+k)!\left[h\binom{m}{h}\binom{n}{k} + k\binom{m}{h}\binom{n}{k} - \binom{m}{h}\binom{n}{k}\right]$$
$$= (h+k)!(h+k-1)\binom{m}{h}\binom{n}{k}. \quad \blacksquare$$

The following results are immediate consequences of the above observations.

THEOREM 7. *(A, B) is a pair of losing score lists of a complete $[h, k]$-BMHT if and only if for each p $(h \le p \le m)$ and each q $(k \le q \le n)$*

$$\sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j = (h+k-1)!\left[p\binom{m-1}{h-1}\binom{n}{k} + q\binom{m}{h}\binom{n-1}{k-1}\right].$$

THEOREM 8. *(C, D) is a pair of score lists of a complete $[h, k]$-BMHT if and only if for each p $(h \le p \le m)$ and each q $(k \le q \le n)$*

$$\sum_{i=1}^{p} c_i + \sum_{j=1}^{q} d_j = (h+k-1)!(h+k-1)\left[p\binom{m-1}{h-1}\binom{n}{k} + q\binom{m}{h}\binom{n-1}{k-1}\right].$$

The next two results give the necessary and sufficient conditions for a pair of lists $(A, B)((C, D))$ of non-negative integers in non-decreasing order to be the pair of losing score lists (score lists) of some $[h, k]$-BMHT.

THEOREM 9. *(A, B) is a pair of losing score lists of an $[h, k]$-BMHT if and only if for each p $(h \le p \le m)$ and each q $(k \le q \le n)$*

$$\binom{p}{h}\binom{q}{k} \le \sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j \le (h+k-1)!\left[p\binom{m-1}{h-1}\binom{n}{k} + q\binom{m}{h}\binom{n-1}{k-1}\right], \quad (3)$$

and

$$0 \le a_i \le (h+k-1)!\binom{m-1}{h-1}\binom{n}{k},$$
$$0 \le b_j \le (h+k-1)!\binom{m}{h}\binom{n-1}{k-1}.$$

THEOREM 10. *(C, D) is a pair of score lists of an $[h, k]$-BMHT if and only if for each p $(h \le p \le m)$ and each q $(k \le q \le n)$*

$$p\binom{m-1}{h-1}\binom{n}{k} + q\binom{m}{h}\binom{n-1}{k-1} + \binom{m-p}{h}\binom{n-q}{k} - \binom{m}{h}\binom{n}{k}$$
$$\le \sum_{i=1}^{p} c_i + \sum_{j=1}^{q} d_j$$
$$\le (h+k-1)!(h+k-1)\left[p\binom{m-1}{h-1}\binom{n}{k} + q\binom{m}{h}\binom{n-1}{k-1}\right], \quad (4)$$

and

$$0 \le c_i \le (h+k-1)!(h+k-1)\binom{m-1}{h-1}\binom{n}{k},$$
$$0 \le d_j \le (h+k-1)!(h+k-1)\binom{m}{h}\binom{n-1}{k-1}.$$

Before coming to the proofs of Theorems 9 and 10, we make the following important observation.

We define an order relation $\to$ on all pairs of lists of non-negative integers in non-decreasing order satisfying (3) and therefore including all pairs of lists satisfying (1) as follows.

Let $A = [a_i]_1^m$ and $B = [b_j]_1^n$, and let t be the smallest index for which $a_t = a_m = \max\{a_i : 1 \le i \le m\}$. Let $A' = [a'_i]_1^m$, where

$$a'_i = \begin{cases} a_i, & \text{for all } i,\ i \ne t,\ 1 \le i \le m, \\ a_i - 1, & \text{for } i = t. \end{cases}$$

Then we say that the pair (A, B) strictly covers the pair (A', B). Here the list B has been kept fixed. Note that the same argument can be used by fixing the list A and choosing $B' = [b'_j]_1^n$, as was chosen in A'.

Clearly, if $t > 1$, then

$$a_{t-1} < a_t = a_{t+1} = \cdots = a_m$$

and if $t = 1$, then

$$a_1 = a_2 = \cdots = a_m.$$

Further, if (A, B) covers (A', B), then

$$\sum_{i=1}^{m} a'_i = (\sum_{i=1}^{m} a_i) - 1.$$

This implies by Theorem 1, that if the pair (X, Y) of non-negative integers in non-decreasing order satisfies (3), then (X, Y) is the losing pair of score lists of some $[h, k]$-BHT if and only if (X, Y) covers no pair of lists satisfying (3). In case (A, B) satisfies (3) and (A, B) is not a pair of losing score lists of any $[h, k]$-BHT, then (A, B) covers exactly one pair (A', B) satisfying (3).

For any two pairs of non-negative integer lists (X, Y) and (X', Y) in non-decreasing order satisfying (3), define $(X', Y) \to (X, Y)$ if either $(X', Y) = (X, Y)$ or there is a sequence

$$(X', Y) = (X_l, Y), X_{l-1}, Y), \ldots, (X_1, Y), (X_0, Y) = (X, Y),$$

$(1 \le i \le l)$ of pairs of non-negative integer lists in non-decreasing order, each satisfying conditions (3) such that (X_i, Y) covers (X_{i-1}, Y).

In case the two lists are (X, Y) and (X, Y') then $(X, Y') \to (X, Y)$ if either $(X, Y') = (X, Y)$ or there is a sequence

$$(X, Y') = (X, Y_w), (X, Y_{w-1}), \ldots, (X, Y_1), (X, Y_0) = (X, Y),$$

$(1 \leq i \leq l)$ of pairs of non-negative integer lists in non-decreasing order, each satisfying conditions (3) such that (X, Y_j) covers (X, Y_{j-1}).

LEMMA 11. *If (A, B) is a pair of lists satisfying (3), then there exists an $[h, k]$-BHT H with a pair of losing score lists (P, Q) where $P = [p_i]_1^m$, $p_1 \leq p_2 \leq \cdots \leq p_m$ and $Q = [q_j]_1^n$, $q_1 \leq q_2 \leq \cdots \leq q_n$ such that $p_i \leq a_i$ and $q_j \leq b_j$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.*

Proof. Let the pair of lists (A, B) satisfy (3). We use induction

$$l(A, B) = \sum_{i=1}^{m} a_i + \sum_{j=1}^{n} b_j - \binom{m}{h}\binom{n}{k}.$$

If $l(A, B) = 0$, then by Theorem 1, (A, B) is itself a pair of losing score lists of some $[h, k]$-BHT H^*. If $l(A, B) = 1$, then by above observations, (A, B) covers exactly one pair of lists (X, B), with $X = [x_i]_1^m$, satisfying (3) and such that $x_i \leq a_i$ for $1 \leq i \leq m$ and

$$\begin{aligned} l(X, B) &= \sum_{i=1}^{m} x_i + \sum_{j=1}^{n} b_j - \binom{m}{h}\binom{n}{k} \\ &= \sum_{i=1}^{m} a_i + \sum_{j=1}^{n} b_j - \binom{m}{h}\binom{n}{k} - 1 \\ &= l(A, B) - 1. \end{aligned}$$

By induction hypothesis applied to (X, B), there is a pair of losing score lists (X, B) of some $[h, k]$-BHT H^{**} such that $x_i \leq a_i$ for $1 \leq i \leq m$.

If $l(A, B) \geq 2$, starting from the pair (X, B), it follows by using similar arguments that there is a pair of losing score lists (X, Y), where $Y = [y_j]_1^n$ with $y_j \leq b_j$ for all $1 \leq j \leq n$, of some $[h, k]$-BHT H^{***}.

Repeating the above process and by applying transitivity of $\to$ it follows that there is a pair of losing score lists (P, Q) of some $[h, k]$-BHT H, where $p_i \leq x_i \leq a_i$ and $q_j \leq y_j \leq b_j$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. ∎

LEMMA 12. *If (C, D) is a pair of lists satisfying (4), then there exists an $[h, k]$-BHT H with a pair of score lists (P, Q) where $P = [p_i]_1^m$ and $Q = [q_j]_1^n$ such that $p_i \leq c_i$ and $q_j \leq d_j$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.*

Proof of Theorem 9. Necessity. Let (A, B) be a pair of losing score lists of some $[h, k]$-BMHT H of order $m + n$, and let U and V be the two sets of vertices of H with $|U| = m$ and $|V| = n$. Then any p vertices from U and any q vertices from V induce an $[h, k]$-BMHT of order $p + q$ which, in turn, contains an $[h, k]$-BHT H^* of order $p + q$. Therefore the sum of the losing scores in H^* of these

$p+q$ vertices is at least $\binom{p}{h}\binom{m}{k}$. Also in H, a vertex $u \in U$ can be at the last entry in at most $(h+k-1)!\binom{m-1}{h-1}\binom{n}{k}$ and a vertex $v \in V$ can be at the last entry in at most $(h+k-1)!\binom{m}{h}\binom{n-1}{k-1}$ arcs. So any losing score in A cannot exceed $(h+k-1)!\binom{m-1}{h-1}\binom{n}{k}$ and any losing score in B cannot exceed $(h+k-1)!\binom{m}{h}\binom{n-1}{k-1}$.

Sufficiency. If (A,B) is a pair of lists satisfying (3), then by Lemma 11, there is an $[h,k]$-BHT H^{**} of order $m+n$ with a pair of losing score lists (P,Q) where $(P,Q) \to (A,B)$. In H^{**},denote the vertex with losing score p_i by $u_i, 1 \le i \le m$, and the vertex with losing score q_j by $v_j, 1 \le j \le n$. Since the number of arcs in which u_i and v_j respectively are not at the last entry are

$$(h+k-1)!\binom{m-1}{h-1}\binom{n}{k} - p_i \ge (h+k-1)!\binom{m-1}{h-1}\binom{n}{k} - a_i$$

and

$$(h+k-1)!\binom{m}{h}\binom{n-1}{k-1} \ge (h+k-1)!\binom{m}{h}\binom{n-1}{k-1} - b_j$$

therefore $(h+k-1)!\binom{m-1}{h-1}\binom{n}{k} - a_i$ and $(h+k-1)!\binom{m}{h}\binom{n-1}{k-1} - b_j$ arcs respectively can be added in which u_i and v_j are at the last entry. This produces an $[h,k]$-BMHT H^{***} with a pair of losing score lists (A,B). ■

Proof of Theorem 10 can be now established by using Lemma 12 and by using the same argument as in Theorem 9.

3. Scores of (h,k)-bipartite multi hypertournaments

An (h,k)-bipartite multi hypertournament (or briefly (h,k)-BMHT) H on $m+n$ vertices is a triple $(U,V,\mathbf{A})$, where U and V are two sets of vertices with $|U| = m$ and $|V| = n$ and $\mathbf{A}$ is a set of $(r+s)$-tuples of vertices, called arcs with r $(1 \le r \le h)$ vertices from U and s $(1 \le s \le k)$ vertices from V, such that any $r+s$ subset $U_1 \cup V_1$ of $U \cup V, A$ contains at least one and at most $(h+k)!$ $(h+k)$-tuples whose entries belong to $U_1 U V_1$. The score and losing score of a vertex in (h,k)-BMHT is defined in the same way as in $[h,k]$-BMHT.

An (h,k)-BMHT is said to be complete if for every set of vertices U_1(for all $h = 1,2,\ldots,|U_1|$) and every set of vertices V_1(for all $k = 1,2,\ldots,|V_1|,$) there are exactly $(h+k)!$ $(h+k)$-tuples in the arc set A. Clearly in a complete (h,k)-BMHT H there are exactly $\sum_{r=1}^{h}\sum_{s=1}^{k}\left[(r+s)!\binom{m}{r}\binom{n}{s}\right]$ arcs and therefore

$$\sum_{i=1}^{m} d_H^-(u_i) + \sum_{j=1}^{n} d_H^-(v_j) = \sum_{r=1}^{h}\sum_{s=1}^{k}\left[(r+s)!\binom{m}{r}\binom{n}{s}\right].$$

Thus,

$$\sum_{i=1}^{m} a_i + \sum_{j=1}^{n} b_j = \sum_{r=1}^{h}\sum_{s=1}^{k}\left[(r+s)!\binom{m}{r}\binom{n}{s}\right].$$

Further, we observe that the maximum arcs in (h,k)-BMHT is

$$\sum_{r=1}^{h}\sum_{s=1}^{k}\left[(r+s)!\binom{m}{r}\binom{n}{s}\right].$$

From the above facts, we have the following result.

LEMMA 13. *If (C, D) is a pair of score lists of a complete (h, k)-BMHT H, then*

$$\sum_{i=1}^{m} c_i + \sum_{j=1}^{n} d_j = \sum_{r=1}^{h}\sum_{s=1}^{k}\left[(r+s)!(r+s-1)!\binom{m}{r}\binom{n}{s}\right].$$

PROOF. Since there are

$$\sum_{r=1}^{h}\sum_{s=1}^{k}\left[(r+s)!\binom{m-1}{r-1}\binom{n}{s}\right]$$

arcs containing a vertex $u_i \in U$ and

$$\sum_{r=1}^{h}\sum_{s=1}^{k}\left[(r+s)!\binom{m}{r}\binom{n-1}{s-1}\right]$$

arcs containing a vertex $v_j \in V$, so

$$\begin{aligned}
\sum_{i=1}^{m} c_i + \sum_{j=1}^{n} d_j &= \sum_{i=1}^{m}\left(\sum_{r=1}^{h}\sum_{s=1}^{k}\left[(r+s)!\binom{m-1}{r-1}\binom{n}{s}\right]\right) \\
&\quad + \sum_{j=1}^{n}\left(\sum_{r=1}^{h}\sum_{s=1}^{k}\left[(r+s)!\binom{m}{r}\binom{n-1}{s-1}\right]\right) - \sum_{r=1}^{h}\sum_{s=1}^{k}\left[(r+s)!\binom{m}{r}\binom{n}{s}\right] \\
&= m\sum_{r=1}^{h}\sum_{s=1}^{k}\left[(r+s)!\binom{m-1}{r-1}\binom{n}{s}\right] + n\sum_{r=1}^{h}\sum_{s=1}^{k}\left[(r+s)!\binom{m}{r}\binom{n-1}{s-1}\right] \\
&\quad - \sum_{r=1}^{h}\sum_{s=1}^{k}\left[(r+s)!\binom{m}{r}\binom{n}{s}\right] \\
&= \sum_{r=1}^{h}\sum_{s=1}^{k}\left[(r+s)!m\binom{m-1}{r-1}\binom{n}{s} + (r+s)!n\binom{m}{r}\binom{n-1}{s-1}\right. \\
&\quad \left. - (r+s)!\binom{m}{r}\binom{n}{s}\right] \\
&= \sum_{r=1}^{h}\sum_{s=1}^{k}\left[(r+s)!r\binom{m}{r}\binom{n}{s} + (r+s)!s\binom{m}{r}\binom{n}{s} - (r+s)!\binom{m}{r}\binom{n}{s}\right] \\
&= \sum_{r=1}^{h}\sum_{s=1}^{k}\left[(r+s)!(r+s-1)!\binom{m}{r}\binom{n}{s}\right]. \quad \blacksquare
\end{aligned}$$

Now we have the following results which are immediate consequences of the above facts.

THEOREM 14. *(A, B) is a pair of losing score lists of a complete (h, k)-BMHT if and only if for each $p(h \le p \le m)$ and each $q(k \le q \le n)$*

$$\sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j = \sum_{r=1}^{h}\sum_{s=1}^{k}\left[(r+s-1)!\left(p\binom{m-1}{r-1}\binom{n}{s} + q\binom{m}{r}\binom{n-1}{s-1}\right)\right].$$

THEOREM 15. *(C,D) is a pair of score lists of a complete (h,k)-BMHT if and only if for each $p(h \le p \le m)$ and each $q(k \le q \le n)$*

$$\sum_{i=1}^{p} c_i + \sum_{j=1}^{q} d_j = \sum_{r=1}^{h}\sum_{s=1}^{k}\left[(r+s)!(r+s-1)!\left(p\binom{m-1}{r-1}\binom{n}{s} + q\binom{m}{r}\binom{n-1}{s-1}\right)\right].$$

From the above discussions and observations, we conjecture the following two necessary and sufficient conditions for a pair of lists (A,B) and (C,D) respectively to be the pair of losing score lists and score lists of some (h,k)-BMHT.

CONJECTURE 16. (A,B) is a pair of losing score lists of a complete (h,k)-BMHT if and only if for each p $(h \le p \le m)$ and each q $(k \le q \le n)$

$$\sum_{r=1}^{h}\sum_{s=1}^{k}\binom{p}{r}\binom{q}{s} \le \sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j \le \sum_{r=1}^{h}\sum_{s=1}^{k}\left[(r+s-1)!\left(p\binom{m-1}{r-1}\binom{n}{s} + q\binom{m}{r}\binom{n-1}{s-1}\right)\right]$$

and

$$0 \le a_i \le \sum_{r=1}^{h}\sum_{s=1}^{k}\left[(r+s-1)!\binom{m-1}{r-1}\binom{n}{s}\right],$$

$$0 \le b_j \le \sum_{r=1}^{h}\sum_{s=1}^{k}\left[(r+s-1)!\binom{m}{r}\binom{n-1}{s-1}\right].$$

CONJECTURE 17. (C,D) is a pair of score lists of a complete (h,k)-BMHT if and only if for each $p(h \le p \le m)$ and each q $(k \le q \le n)$

$$\sum_{r=1}^{h}\sum_{s=1}^{k}\left[p\binom{m-1}{r-1}\binom{n}{s} + q\binom{m}{r}\binom{n-1}{s-1} + \binom{m-p}{r}\binom{n-q}{s} - \binom{m}{r}\binom{n}{s}\right] \le \sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j \le \sum_{r=1}^{h}\sum_{s=1}^{k}\left[(r+s-1)!\left(p\binom{m-1}{r-1}\binom{n}{s} + q\binom{m}{r}\binom{n-1}{s-1}\right)\right]$$

and

$$0 \le c_i \le \sum_{r=1}^{h}\sum_{s=1}^{k}\left[(r+s-1)!\binom{m-1}{r-1}\binom{n}{s}\right],$$

$$0 \le d_j \le \sum_{r=1}^{h}\sum_{s=1}^{k}\left[(r+s-1)!\binom{m}{r}\binom{n-1}{s-1}\right].$$

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