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ON ALMOST COUNTABLY COMPACT SPACES

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Abstract. A space X is *almost countably compact* if for every countable open cover $\mathcal{U}$ of X, there exists a finite subset $\mathcal{V}$ of $\mathcal{U}$ such that $\bigcup\{\overline{V} : V \in \mathcal{V}\} = X$. In this paper, we investigate the relationship between almost countably compact spaces and countably compact spaces, and also study topological properties of almost countably compact spaces.

1. Introduction

By a space, we mean a topological space. Let us recall that a space X is *countably compact* if every countable open cover of X has a finite subcover. For T_1-spaces, countable compactness is equivalent to the condition saying that every infinite set has an accumulation point. As a generalization of countable compactness, Bonanzinga, Matveev and Pareek [2] defined a space X as *almost countably compact* if for every countable open cover $\mathcal{U}$ of X, there exists a finite subset $\mathcal{V}$ of $\mathcal{U}$ such that $\bigcup\{\overline{V} : V \in \mathcal{V}\} = X$. Clearly, every countably compact space is almost countably compact, but the converse does not hold (see Examples 2.3 and 2.4).

The purpose of this paper is to investigate the relationship between almost countably compact spaces and countably compact spaces, and also study topological properties of almost countably compact spaces.

Recall that the *extent* $e(X)$ of a space X is the smallest cardinal number κ such that the cardinality of every discrete closed subset of X is not greater than κ. The cardinality of a set A is denoted by $|A|$. Let ω be the first infinite cardinal, ω_1 the first uncountable cardinal and $\mathfrak{c}$ the cardinality of the set of all real numbers. As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. For a cardinal κ, $cf(\kappa)$ denotes the cofinality of κ. Every cardinal is often viewed as a space with the usual order topology. Other terms and symbols that we do not define follow [3].

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2. On almost countably compact spaces

In this section, we give some examples showing the relationship between almost countably compact spaces and countably compact spaces. First, we give a positive result. Recall from [1] that a subspace Y of a space X is relatively countably compact in X if every infinite subset of Y has a limit point in X. Equivalently, Y is relatively countably compact in X if and only if every countable open cover $\mathcal{U}$ of X there exists a finite subfamily $\mathcal{V}$ such that $Y \subseteq \cup\mathcal{V}$.

Proposition 2.1. *Let X be a space and D be a dense subset of X. If D is relatively countably compact in X, then X is almost countably compact.*

Proof. Let D be a dense subspace of X and D be relatively compact in X, and let $\mathcal{U}$ be any countable open cover of X. Then there exists a finite subset $\mathcal{V}$ of $\mathcal{U}$ such that $D \subseteq \bigcup\{\overline{V} : V \in \mathcal{V}\}$. Since D is relatively countably compact in X, hence $X = \{\overline{V} : V \in \mathcal{V}\}$, since D is dense in X, which completes the proof. ■

We get the following corollary by Proposition 2.1.

Corollary 2.2. *If X has a dense countably compact subspace, then X is almost countably compact.*

Example 2.3. There exists a Tychonoff almost countably compact space which is not countably compact.

Proof. Let $X = ((\omega_1+1)\times(\omega+1))\setminus\{\langle\omega_1,\omega\rangle\}$ be the Tychonoff plank. Then X is almost countably compact by Corollary 2.2, since $\omega_1\times(\omega+1)$ is a dense countably compact subset of X. But X is not countably compact, since $\{\langle\omega_1, n\rangle : n \in \omega\}$ is an infinite discrete closed subset of X, which completes the proof. ■

Example 2.4. The Isbell-Mrówka space X is almost countably compact, but it is not countably compact.

Proof. Let $X = \omega \cup \mathcal{R}$ be the Isbell-Mrówka space [4], where $\mathcal{R}$ is a maximal almost disjoint family of infinite subsets of ω such that $|\mathcal{R}| = \mathfrak{c}$. Then X is not countably compact, since $\mathcal{R}$ is a discrete closed in X, and X is almost countably compact, since ω is dense in X and every infinite subset of ω has a limit point in X, which completes the proof. ■

In the following, we give a well-known construction which produces almost countably compact spaces by using Tychonoff space. Let X be a Tychonoff space and τ a cardinal. The symbol $\beta(X)$ means the Čech-Stone compactification of the space X. Consider the Noble plank $N_\tau X$ of X (see [3])

$$N_\tau X = (\beta X \times (\tau+1)) \setminus ((\beta X \setminus X) \times \{\tau\}).$$

Since $\beta X\times\tau$ is a dense countably compact subset of $N_\tau X$, then we has the following result by Corollary 2.2.

Proposition 2.5. *If X is a Tychonoff space and $cf(\tau) > \omega$, then $N_\tau X$ is almost countably compact.*

By the above construction, we easily get the following proposition.

Proposition 2.6. *Every Tychonoff space X can be embedded in a Tychonoff almost countably compact space Y as a closed subspace.*

Proof. Let X be a Tychonoff space. If we put $Y = N_{\omega_1}X$, then $\overline{X} = X \times \{\omega_1\}$ is a closed subset of Y, which is homeomorphic to X. Since $\beta X \times \omega_1$ is a dense countably compact subset of Y, then Y is almost countably compact by Corollary 2.2, which completes the proof. ∎

It is well-known that the extent of a countably compact space is finite, but the following example shows that the extent of a Tychonoff almost countably compact spaces can be arbitrarily big.

Proposition 2.7. *For every regular uncountable cardinal κ, there exists a Tychonoff almost countably compact space X such that $e(X) \geq \kappa$.*

Proof. Let D be with a discrete space of cardinality κ. If we put $X = N_{\omega_1}D$, then X is Tychonoff almost countably compact by Corollary 2.2, since $\beta D \times \omega_1$ is a dense countably compact subset of X. Since $D \times \{\omega_1\}$ is a discrete closed subset of X with $|D \times \{\omega_1\}| = \kappa$, then $e(X) \geq \kappa$, which completes the proof. ∎

It is well known that a space X is compact if and only if X is a countably compact space with the Lindelöf property. About the class of almost compact spaces, we have the following result.

Proposition 2.8. *Every regular almost countably compact and Lindelöf space is compact.*

Proof. Let X be a regular almost countably compact and Lindelöf space and $\mathcal{U}$ be any open cover of X. For each $x \in X$, there exists an $U_x \in \mathcal{U}$ such that $x \in U_x$, and there exists an open neighbourhood V_x of x such that $x \in V_x \subseteq \overline{V_x} \subseteq U_x$. Let $\mathcal{V} = \{V_x : x \in X\}$. Then $\mathcal{V}$ is an open cover of X. Hence $\mathcal{V}$ has a countable subcover, since X is Lindeöf, saying $\{V_{x_n} : n \in \omega\}$. Thus $\{V_{x_n} : n \in \omega\}$ has a finite subset $\{V_{x_{n_j}} : j = 1, 2, \ldots, m\}$ such that $X = \bigcup\{\overline{V_{x_{n_j}}} : j = 1, 2, \ldots, m\}$, since X is almost countably compact. Clearly, $\{U_{x_{n_j}} : j = 1, 2, \ldots, m\}$ is a finite subcover of $\mathcal{U}$, which completes the proof. ∎

The following example shows that the condition of regularity in Proposition 2.8 is necessary.

Example 2.9. There exists a Hausdorff almost countably compact and Lindelöf space which is not compact.

Proof. Let

$$A = \{a_n : n \in \omega\} \text{ and } B = \{b_m : m \in \omega\}$$
$$Y = \{\langle a_n, b_m\rangle : n \in \omega, m \in \omega\},$$

and let

$$X = Y \cup A \cup \{a\} \text{ where } a \notin Y \cup A.$$

We topologize X as follows: every point of Y is isolated; a basic neighborhood of a point $a_n \in A$ for each $n \in \omega$ takes the from

$$U_{a_n}(m) = \{a_n\} \cup \{\langle a_n, b_i\rangle : i > m\} \text{ for } m \in \omega$$

and a basic neighborhood of a takes the from

$$U_a(F) = \{a\} \cup \bigcup \{\langle a_n, b_m\rangle : a_n \in A \setminus F, m \in \omega\} \text{ for a finite subset } F \text{ of } A .$$

Clearly, X is a Hausdorff space by the construction of the topology of X. However, X is not regular, since the point a cannot be separated from the closed subset A by disjoint open subsets of X. Since $|X| = \omega$, then X is Lindelöf, and since A is a discrete closed subset of X with $|A| = \omega$, then X is not compact.

We shall show that X is almost countably compact. Let $\mathcal{U}$ be a countable open cover of X. Then there exists an $U_a \in \mathcal{U}$ such that $a \in U_a$. Hence there exists a finite subset F of A such that $U_a(F) \subseteq U_a$ by the construction of the topology of X. Thus

$$\{a\} \cup (A \setminus F) \cup \{\langle a_n, b_m\rangle : a_n \in A \setminus F, m \in \omega\} \subset \overline{U_a};$$

On the other hand, for each $a_n \in F$, $\{a_n\} \cup \{\langle a_n, b_m\rangle : m \in \omega\}$ is a compact subset of X, and there exists a finite subset $\mathcal{V}_{a_n} \subseteq \mathcal{U}$ such that

$$\{a_n\} \cup \{\langle a_n, b_m\rangle : m \in \omega\} \subset \cup\{\overline{V} : V \in \mathcal{V}_{a_n}\}.$$

If we put

$$\mathcal{V} = \{U_a\} \cup \bigcup \{\mathcal{V}_{a_n} : a_n \in F\},$$

then $\mathcal{V}$ is a finite subset of $\mathcal{U}$ such that

$$X = \bigcup \{\overline{V} : V \in \mathcal{V}\},$$

which completes the proof. ∎

In Example 2.3, the closed subset $\{\langle \omega_1, n\rangle : n \in \omega\}$ of a Tychonoff almost countably compact space X is not almost countably compact, which shows that a closed subset of an almost countably compact space need not be almost countably compact. In the following, we give a positive result, which can be easily proved.

Proposition 2.10. *If X is an almost countably compact space, then every open and closed subset of X is almost countably compact.*

Proposition 2.11. *The sum $\bigoplus_{s \in S} X_s$, where $X_s \neq \emptyset$ for $s \in S$, is almost countably compact if and only if all spaces X_s are almost countably compact and the set S is finite.*

Since a continuous image of a countably compact space is countably compact, similarly, we have the following result.

Proposition 2.12. *A continuous image of an almost countably compact space is almost countably compact.*

Proof. Suppose that X is an almost countably compact space and $f : X \to Y$ a continuous map. Let $\mathcal{U} = \{U_n : n \in \omega\}$ be a countable open cover of Y. Then $\mathcal{V} = \{f^{-1}(U) : U \in \mathcal{U}\}$ is a countable open cover of X. Since X is almost countably compact, there exists a finite subset $\{n_i : i = 1, 2, \ldots, m\}$ such that

$$\bigcup\{\overline{f^{-1}(U_{n_i})} : i = 1, 2, \ldots, m\} = X.$$

Hence

$$Y = f(X) = f(\bigcup\{\overline{f^{-1}(U_{n_i})} : i = 1, 2, \ldots, m\}) = \bigcup\{f(\overline{f^{-1}(U_{n_i})}) : i = 1, 2, \ldots, m\}$$
$$= \bigcup\{\overline{f(f^{-1}(U_{n_i}))} : i = 1, 2, \ldots, m\} = \bigcup\{\overline{U_{n_i}} : i = 1, 2, \ldots, m\}.$$

This shows that Y is almost countably compact. ■

Next, we turn to consider preimages. To show that the preimage of an almost countably compact space under a closed 2-to-1 continuous map need not be almost countably compact. We use the Alexandorff duplicate $A(X)$ of a space X. The underlying set of $A(X)$ is $X \times \{0, 1\}$; each point of $X \times \{1\}$ is isolated and a basic neighborhood of a point $\langle x, 0\rangle \in X \times \{0\}$ is of the from $(U \times \{0\}) \cup ((U \times \{1\}) \setminus \{\langle x, 1\rangle\})$, where U is a neighborhood of x in X. It is well-known that X is countably compact if and only if $A(X)$ is countably compact. But the statement is not true for almost countably compact spaces.

Example 2.13. There exists a closed 2-to-1 continuous map $f : X \to Y$ such that Y is an almost countably compact space, but X is not almost countably compact.

Proof. Let Y be the space X in the proof of Example 2.3. Then Y is almost countably compact and has an infinite discrete closed subset $F = \{\langle\omega_1, n\rangle : n \in \omega\}$. Let X be the Alexandroff duplicate $A(Y)$ of Y. Then X is not almost countably compact, since $F \times \{1\}$ is an infinite discrete, open and closed set in X. Let $f : X \to Y$ be the natural map. Then f is a closed 2-to-1 continuous map, which completes the proof. ■

Remark 1. The proof of Example 2.13 also shows that the Alexandorff duplicate $A(X)$ need not be almost countably compact for an almost countably compact space X.

We give a positive result. Recall from [5] that a mapping f from a space X to a space Y is called *almost open* if $f^{-1}(\overline{U}) \subseteq \overline{f^{-1}(U)}$ for each open subset U of Y.

Proposition 2.14. *Let Y be an almost countably compact space and $f : X \to Y$ be an almost open and perfect mapping. Then X is almost countably compact.*

Proof Let $\mathcal{U}$ be a countable open cover of X and let

$$\mathcal{V} = \{V : \text{ there exists a finite subfamily } \mathcal{F} \text{ of } \mathcal{U} \text{ such that } V = \bigcup\mathcal{F}\}.$$

Then $\mathcal{V}$ is countable, since $\mathcal{U}$ is countable. Hence we can enumerate $\mathcal{V}$ as $\{V_n : n \in \omega\}$. For each $n \in \omega$, let

$$W_n = Y \setminus f(X \setminus V_n),$$

then W_n is an open subset of Y, since f is closed. Let $\mathcal{W} = \{W_n : n \in \omega\}$, then $\mathcal{W}$ is a countable open cover of Y. In fact, for every $y \in Y$, there exists a $V_n \in \mathcal{V}$ such that $f^{-1}(y) \subseteq V_n$. Since $f^{-1}(y)$ is compact, then $W_n = Y \setminus f(X \setminus V_n)$ is an open neighborhood of y. Since Y is almost countably compact, then there exists a finite subfamily $\{W_{n_i} : i = 1, 2, \ldots, m\}$ of $\mathcal{W}$ such that

$$Y = \bigcup_{i \leq m} \overline{W_{n_i}}.$$

Since f is almost open, then

$$\begin{aligned} X = f^{-1}(Y) = f^{-1}(\bigcup_{i \leq m} \overline{W_{n_i}}) &\subseteq \bigcup_{i \leq m} f^{-1}(\overline{W_{n_i}}) \\ \subseteq \bigcup_{i \leq m} \overline{f^{-1}(W_{n_i})} &\subseteq \bigcup_{i \leq m} \overline{V_{n_i}}, \end{aligned}$$

and since every element of $\mathcal{V}$ is the union of a finite subfamily of $\mathcal{U}$. This shows that X is almost countably compact, which completes the proof. ∎

We get the following corollary by Proposition 2.14.

Corollary 2.15. *The product of an almost countably compact space and a compact space is almost countably compact.*

It is well-known that the product of two countably compact spaces is countably compact. In the following, we show that the product of two countably compact spaces is almost countably compact by using the example. Here, we give the proof roughly for the sake of completeness (see [3, Example 3.10.19]).

Example 2.16. There exist two countably compact spaces X and Y such that $X \times Y$ is not almost countably compact.

Proof. Let N be with the discrete topology. We can define

$$X = \bigcup_{\alpha < \omega_1} E_\alpha \text{ and } Y = \beta N \setminus X,$$

where E_α are the subsets of βN which are defined inductively so as to satisfy the following conditions (1), (2) and (3):

(1) $E_0 = N$;

(2) $|E_\alpha| \leq \mathfrak{c}$ for each $\alpha < \omega_1$;

(3) every infinite subset of E_α has an accumulation point in $E_{\alpha+1}$.

Those sets E_α and are well-defined since every infinite closed set in βN has the cardinality $2^{\mathfrak{c}}$ (see [3]). Then X and Y are countably compact. But $X \times Y$ is not almost countably compact, because the diagonal $\{\langle n, n\rangle : n \in \omega\}$ is a discrete open and closed subset of $X \times Y$ with the cardinality ω and almost countably compactness is preserved by open and closed subsets. ∎

Remark 2. Since every countably compact space is almost countably compact, thus Example 2.16 shows that the product of two almost countably compact spaces need be not almost countably compact.

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