МАТЕМАТИЧКИ ВЕСНИК
64, 1 (2012), 39–52
March 2012

оригинални научни рад
research paper

APPLICATION OF THE INFINITE MATRIX THEORY TO THE SOLVABILITY OF CERTAIN SEQUENCE SPACES EQUATIONS WITH OPERATORS

Bruno de Malafosse

Abstract. In this paper we deal with special *sequence spaces equations (SSE) with operators*, which are determined by an identity whose each term is a *sum or a sum of products of sets of the form* $\chi_a(T)$ *and* $\chi_{f(x)}(T)$ where f maps U^+ to itself, and χ is any of the symbols s, s^0, or $s^{(c)}$. We solve the equation $\chi_x(\Delta) = \chi_b$ where χ is any of the symbols s, s^0, or $s^{(c)}$ and determine the solutions of (SSE) with operators of the form $(\chi_a * \chi_x + \chi_b)(\Delta) = \chi_\eta$ and $[\chi_a * (\chi_x)^2 + \chi_b * \chi_x](\Delta) = \chi_\eta$ and $\chi_a + \chi_x(\Delta) = \chi_x$ where χ is any of the symbols s, or s^0.

1. Introduction

In the book entitled *Summability through Functional Analysis* [15], Wilansky introduced sets of the form $1/a * E$ where E is a BK space, where $a = (a_n)_{n\geq 1}$ is a sequence satisfying $a_n \neq 0$ for all n. Recall that $\xi = (\xi_n)_{n\geq 1}$ belongs to $1/a * E$ if $a\xi \in E$. In [12, 3] the sets s_r, s_r^0 and $s_r^{(c)}$ were defined by $((1/r^n)_n)^{-1} * E$ with $r > 0$, where E is ℓ_∞, c_0 and c respectively and the sets s_a, s_a^0 and $s_a^{(c)}$ by $(1/a)^{-1} * E$ with $a_n > 0$ for all n and E is ℓ_∞, c_0 and c respectively. The aim was to study an infinite linear system represented by the *matrix equation* $M\xi = \beta$ where ξ was the unknown and ξ, β were column matrices, and $M = (\mu_{nm})_{n,m\geq 1}$ was an infinite matrix mapping from $(1/a)^{-1} * E$ to itself, (cf. [12]). In [4, 13] the sum $\chi_a + \chi'_b$ and the product $\chi_a * \chi'_b$ were defined, where χ, χ' are any of the symbols s, s^0, or $s^{(c)}$, among other things characterizations of matrix transformations mapping in the sets $s_a + s_b^0(\Delta^q)$ and $s_a + s_b^{(c)}(\Delta^q)$ were given, where Δ is the *operator of the first difference*. In [7] characterizations of the sets $(s_a(\Delta^q), F)$ can be found, where F is any of the sets c_0, c and ℓ_∞. In [13] characterizations of matrix transformations mapping were given in the set $\widetilde{s_{\alpha,\beta}} = s_\alpha^0((\Delta - \lambda I)^h) + s_\beta^{(c)}((\Delta - \mu I)^l)$, in some cases the set $(\widetilde{s_{\alpha,\beta}}, s_\gamma)$ that can be reduced to a set of the form $S_{\alpha,\gamma}$. Also cite Hardy's results [9] extended by Móricz and Rhoades, (cf. [10, 11]), de Malafosse and

2010 AMS Subject Classification: 40C05, 46A15.
Keywords and phrases: Sequence space; operator of the first difference; BK space; infinite matrix; sequence spaces equations (SSE); (SSE) with operators.

Rakočević (cf. [8]) and formulated as follows. In [9] it is said that a series $\sum_{m=1}^{\infty} y_m$ is *summable* $(C,1)$ if $n^{-1}\sum_{m=1}^{n} s_m \to l$, where $s_m = \sum_{i=1}^{m} y_i$. It was shown by Hardy that if a series $\sum_{m=1}^{\infty} y_m$ is *summable* $(C,1)$ then $\sum_{m=1}^{\infty}(\sum_{i=m}^{\infty} y_i/i)$ is convergent. On the other hand cite *Hardy's Tauberian* theorem for *Cesàro means* where it was shown that if the sequence $(y_n)_n$ satisfies $\sup_n\{n|y_n - y_{n-1}|\} < \infty$, then

$$\frac{1}{n}s_n \to L \text{ implies } y_n \to L \text{ for some } L \in \mathbb{C}.$$

In this paper we are led to solve special *sequence spaces equations (SSE) with operators*, which are determined by an identity whose each term is a *sum or a sum of products of sets of the form* $\chi_a(T)$ *and* $\chi_{f(x)}(T)$, where f mapa U^+ to itself, and χ is any of the symbols s, or s^0, the sequence x is the unknown and T is a given triangle. Then we determine the set of all sequences $x \in U^+$ such that

$$u_n = O(a_n) \text{ and } v_n - v_{n-1} = O(x_n) \tag{1}$$

implies $u_n + v_n = O(x_n)$ $(n \to \infty)$ for all u, $v \in s$. Conversely, what are the sequences x for which $y_n = O(x_n)$ $(n \to \infty)$ implies there are sequences u and v such that $y = u + v$ and (1) holds. This problem leads to the solvability of the equation $s_a + s_x(\Delta) = s_x$. We also determine the set of all sequences $y \in s$ such that $(y_n - y_{n-1})/a_n \to l$ if and only if $y_n/b_n \to l'$. This statement can be written in the form $s_a^{(c)}(\Delta) = s_b^{(c)}$.

This paper is organized as follows. In Section 2 we recall some results on matrix transformations between sets of the form χ_ξ where χ is any of the symbols s, s^0, or $s^{(c)}$ and on the sum and the product of the previous sets. In Section 3 we recall characterizations of $\chi_a(\Delta) = \chi_b$ and determine the solutions of sequence spaces equations of the form $[\chi_a * \chi_x + \chi_b](\Delta) = \chi_\eta$ and $[\chi_a * (\chi_x)^2 + \chi_b * \chi_x](\Delta) = \chi_\eta$ and $\chi_a + \chi_x(\Delta) = \chi_x$ where χ is any of the symbols s, or s^0.

1.1. The sets s_a, s_a^0 and $s_a^{(c)}$ for $a \in U^+$

For a given infinite matrix $M = (\mu_{nm})_{n,m\geq 1}$ we define the operators A_n for any integer $n \geq 1$, by

$$M_n(\xi) = \sum_{m=1}^{\infty} \mu_{nm}\xi_m \tag{2}$$

where $\xi = (\xi_m)_{m\geq 1}$, and the series are assumed convergent for all n. So we are led to the study of the operator M defined by $M\xi = (M_n(\xi))_{n\geq 1}$ mapping between sequence spaces.

A Banach space E of complex sequences with the norm $\|\|_E$ is a *BK space* if each projection $P_n : \xi \to P_n\xi = \xi_n$ is continuous. A BK space E is said to have AK if every sequence $\xi = (\xi_n)_{n\geq 1} \in E$ has a unique representation $\xi = \sum_{n=1}^{\infty} \xi_n e_n$ where e_n is the sequence with 1 in the n-th position and 0 otherwise.

We will denote by s the sets of all sequences. By c_0, c, ℓ_∞ we denote the subsets of s that converge to zero, that are convergent and that are bounded respectively. We shall use the set $U^+ = \{(u_n)_{n\geq 1} \in s \ : \ u_n > 0 \text{ for all } n\}$. Using Wilansky's notations [15], we define for any sequence $a = (a_n)_{n\geq 1} \in U^+$ and for any set of sequences E, the set

$$(1/a)^{-1} * E = \{ (\xi_n)_{n\geq 1} \in s : (\xi_n/a_n)_n \in E \}.$$

To simplify, we use the diagonal infinite matrix D_a defined by $[D_a]_{nn} = a_n$ for all n and write $D_a * E = (1/a)^{-1} * E$ and define $s_a = D_a * \ell_\infty$, $s_a^0 = D_a * c_0$ and $s_a^{(c)} = D_a * c$, see [1, 3, 4–6, 10, 13, 14]. Each of the previous spaces $D_a * E$ is a BK space normed by $\|\xi\|_{s_a} = \sup_{n\geq 1}(|\xi_n|/a_n)$ and s_a^0 has AK, see [6].

Now let $a = (a_n)_{n\geq 1}$, $b = (b_n)_{n\geq 1} \in U^+$. By $S_{a,b}$ we denote the set of infinite matrices $M = (\mu_{nm})_{n,m\geq 1}$ such that

$$\|M\|_{S_{a,b}} = \sup_{n\geq 1}\left(\frac{1}{b_n}\sum_{m=1}^{\infty} |\mu_{nm}|a_m\right) < \infty.$$

The set $S_{a,b}$ is a Banach space with the norm $\|M\|_{S_{a,b}}$. Let E and F be any subsets of s. When M maps E into F we write $M \in (E, F)$, see [2]. So for every $\xi \in E$, we have $M\xi \in F$, ($M\xi \in F$ will mean that for each $n \geq 1$ the series defined by $M_n(\xi) = \sum_{m=1}^{\infty} \mu_{nm}\xi_m$ is convergent and $(M_n(\xi))_{n\geq 1} \in F$). It can easily be seen that $(s_a, s_b) = S_{a,b}$.

When $s_a = s_b$ we obtain the Banach algebra with identity $S_{a,b} = S_a$, (see for instance [1, 5, 6]) normed by $\|M\|_{S_a} = \|M\|_{S_{a,a}}$. We also have $M \in (s_a, s_a)$ if and only if $M \in S_a$.

If $a = (r^n)_{n\geq 1}$, we denote by s_r, s_r^0 and $s_r^{(c)}$ the sets s_a, s_a^0 and $s_a^{(c)}$ respectively. When $r = 1$, we obtain $s_1 = \ell_\infty$, $s_1^0 = c_0$ and $s_1^{(c)} = c$, and putting $e = (1, 1, \dots)$ we have $S_1 = S_e$. Recall that $(\ell_\infty, \ell_\infty) = (c_0, \ell_\infty) = (c, \ell_\infty) = S_1$. We have $M \in (c_0, c_0)$ if and only if $M \in S_1$ and $\lim_{n\to\infty} \mu_{nm} = 0$ for all $m \geq 1$; and $M \in (c, c)$ if and only if $M \in S_1$ and $\lim_{n\to\infty} M_n(e) = l$ and $\lim_{n\to\infty} \mu_{nm} = l_m$ for all $m \geq 1$ and for some scalars l and l_m. Finally for any given subset F of s, we define the *domain of* M by

$$F_M = F(M) = \{ \xi \in s : M\xi \in F \}.$$

1.2. Sum of sets of the form s_ξ, or s_ξ^0

In this subsection among other things we recall some properties of the sum $E + F$ of sets of the form s_ξ, or s_ξ^0.

Let E, $F \subset s$ be two linear vector spaces, we write $E + F$ for the set of all sequences $w = u + v$ where $u \in E$ and $v \in F$. From [4, Proposition 1, p. 244] and [5, Theorem 4, p. 293] we deduce the next results.

Proposition 1. *Let a, $b \in U^+$ and let χ be either of the symbols s, or s^0. Then we have*

(i) $\chi_a \subset \chi_b$ if and only if there is $K > 0$ such that

$$a_n \leq K b_n \text{ for all } n.$$

(ii) α) $\chi_a = \chi_b$ if and only if there are K_1, $K_2 > 0$ such that

$$K_1 \leq \frac{a_n}{b_n} \leq K_2 \text{ for all } n.$$

β) $s_a^{(c)} = s_b^{(c)}$ *if and only if there is* $l \neq 0$ *such that* $\dfrac{a_n}{b_n} \to l \ (n \to \infty)$.

(iii) $\chi_a + \chi_b = \chi_{a+b}$.

(iv) $\chi_a + \chi_b = \chi_a$ *if and only if* $b/a \in \ell_\infty$.

We immediately deduce the next corollary that will be useful in the following.

Lemma 2. *The next statements are equivalent.*

i) $a \in s_b$,

ii) $s_a \subset s_b$,

iii) $s_a^0 \subset s_b^0$,

iv) $a_n \leq K b_n$ *for all* n *and for some* $K > 0$.

In the following our aim is to determine the set of all sequences $x = (x_n)_{n\geq 1} \in U^+$ such that

$$\frac{y_n}{b_n} = O(1) \ (n \to \infty)$$

if and only if there are u, $v \in s$ such that $y = u + v$ and

$$u_n = O(a_n) \text{ and } v_n = O(x_n) \ (n \to \infty).$$

We have the next result.

Theorem 3. *Let* $a = (a_n)_{n\geq 1}$, $b = (b_n)_{n\geq 1} \in U^+$ *and let* χ *be any of the symbols* s, *or* s^0. *Consider the equation*

$$\chi_a + \chi_x = \chi_b, \tag{3}$$

where $x = (x_n)_{n\geq 1} \in U^+$ *is the unknown. Then*

(i) if $a/b \in c_0$ *then equation* (3) *holds if and only if there are* K_1, $K_2 > 0$ *depending on* x, *such that*

$$K_1 b_n \leq x_n \leq K_2 b_n \text{ for all } n,$$

that is $s_x = s_b$;

(ii) if a/b, $b/a \in \ell_\infty$ *then equation* (3) *holds if and only if there is* $K > 0$ *depending on* x *such that*

$$0 < x_n \leq K b_n \text{ for all } n,$$

that is $x \in s_b$;

(iii) if $a/b \notin \ell_\infty$ *then equation* (3) *has no solution in* U^+.

Proof. The proof in the case when $\chi = s$ was given in [1]. When $\chi = s^0$ the proof follows exactly the same lines as in the previous case since we have the equivalence of (ii) and (iii) in Lemma 2 and by Proposition 1 we have $s_\xi = s_\eta$ if and only if $s_\xi^0 = s_\eta^0$ for ξ, $\eta \in U^+$. ∎

We deduce the next corollary.

Corollary 4. *Let r, $u > 0$ and let χ be any of the symbols s, or s^0. Consider the equation*

$$\chi_r + \chi_x = \chi_u \tag{4}$$

where $x = (x_n)_{n\geq 1}$ is the unknown. Then we have

i) If $r < u$ equation (4) is equivalent to

$$K_1 u^n \leq x_n \leq K_2 u^n \text{ for all } n$$

for some K_1, $K_2 > 0$;

ii) if $r = u$ equation (4) is equivalent to

$$x_n \leq K u^n \text{ for all } n$$

for some $K > 0$;

iii) if $r > u$ equation (4) has no solution.

1.3. Product of sequence spaces

In this subsection we will deal with some properties of the *product* $E * F$ of particular subsets E and F of s. For any sequences $\xi \in E$ and $\eta \in F$ we put $\xi\eta = (\xi_n \eta_n)_{n\geq 1}$. Most of the next results were shown in [4]. For any sets of sequences E and F, we put

$$E * F = \bigcup_{\xi \in E} (1/\xi)^{-1} * F = \{\xi\eta \in s \ : \ \xi \in E \text{ and } \eta \in F\}.$$

We immediately have the following results, where we put

$$\mathcal{S} = \{s_a : a \in U^+\} \text{ and } \mathcal{S}^0 = \{s_a^0 : a \in U^+\}.$$

Proposition 5. *The set $\mathcal{S}$, (resp. $\mathcal{S}^0$) with multiplication $*$ is a commutative group and ℓ_∞, (resp. c_0) is the unit element for $\mathcal{S}$, (resp. $\mathcal{S}^0$).*

Proof. We only deal with the set $\mathcal{S}$ the case of the set $\mathcal{S}^0$ can be treated similarly. By [4, Proposition 1, p. 244] we have $\chi_a * \chi_b = \chi_{ab}$. We deduce that the map $\psi : U^+ \mapsto \mathcal{S}$ defined by $\psi(a) = s_a$ is a surjective homomorphism and since U^+ with the multiplication of sequences is a group it is the same for $\mathcal{S}$. Then the unit element of $\mathcal{S}$ is $\psi(e) = s_1 = \ell_\infty$. ■

Remark 6. Note that the inverse of χ_a is $\chi_{1/a}$ where χ be any of the symbols s, or s^0.

As a direct consequence of Proposition 5 we deduce the next corollary.

Corollary 7. *Let a, b, $b' \in U^+$ and let χ be any of the symbols s, or s^0. We successively have*

(i) $\chi_a * \chi_b = \chi_{ab}$.

(ii) $\chi_a * \chi_b = \chi_a * \chi_{b'}$ if and only if $s_b = s_{b'}$.

(iii) The sequence $x = (x_n)_{n\geq 1} \in U^+$ satisfies the equation

$$\chi_a * \chi_x = s_b \tag{5}$$

if and only if

$$K_1 \frac{b_n}{a_n} \le x_n \le K_2 \frac{b_n}{a_n} \text{ for all } n \tag{6}$$

for some K_1, $K_2 > 0$ depending only on x.

2. On some sequence spaces equations with operators

In this section we consider among other things the equations $s_a^{(c)}(\Delta) = s_b^{(c)}$, $s_{ax+b}(\Delta) = s_\eta$, $s_{ax^2+bx}(\Delta) = s_\eta$ and $s_a + s_x(\Delta) = s_x$ for given sequences $a, b \in U^+$. The resolution of the equation $s_{ax+b}(\Delta) = s_\eta$ is equivalent to determine the set of all sequences $x \in U^+$ such that

$$y_n - y_{n-1} = O(a_n x_n + b_n)$$

if and only if $y_n = O(\eta_n)$ $(n \to \infty)$ for all $y \in s$. Solving the equation $s_a + s_x(\Delta) = s_x$ leads to know the set of all sequences $x \in U^+$ such that for each sequence y we have

$$y_n = O(x_n) \tag{7}$$

if and only if there are sequences u, v such that $y = u + v$ and

$$u_n = O(a_n) \text{ and } v_n - v_{n-1} = O(x_n) \ (n \to \infty).$$

2.1. On the identities $\chi_a(\Delta) = \chi_b$ where $\chi \in \{s^0, s^{(c)}, s\}$

To solve the next equations we need additional definitions and properties. The infinite matrix $T = (t_{nm})_{n,m\ge1}$ is said to be a triangle if $t_{nm} = 0$ for $m > n$ and $t_{nn} \ne 0$ for all n. Now let U be the set of all sequences $(u_n)_{n\ge1} \in s$ with $u_n \ne 0$ for all n. The infinite matrix $C(a)$ with $a = (a_n)_{n\ge1} \in U$ is defined by

$$[C(a)]_{nm} = \begin{cases} 1/a_n, & \text{if } m \le n, \\ 0, & \text{otherwise.} \end{cases}$$

It can be shown that the matrix $\Delta(a)$ defined by

$$[\Delta(a)]_{nm} = \begin{cases} a_n, & \text{if } m = n, \\ -a_{n-1}, & \text{if } m = n-1 \text{ and } n \ge 2, \\ 0, & \text{otherwise,} \end{cases}$$

is the inverse of $C(a)$, that is $C(a)(\Delta(a)\xi) = \Delta(a)(C(a)\xi)$ for all $\xi \in s$. If $a = e$ we get the well known operator of the first difference represented by $\Delta(e) = \Delta$. We then have $\Delta\xi_n = \xi_n - \xi_{n-1}$ for all $n \ge 1$, with the convention $\xi_0 = 0$. It is usually written

$$\Sigma = C(e) = \begin{pmatrix} 1 & & & \\ 1 & 1 & & 0 \\ 1 & 1 & 1 & \\ . & . & . & . \end{pmatrix}.$$

Note that $\Delta = \Sigma^{-1}$ and Δ, $\Sigma \in S_R$ for any $R > 1$. Consider the sets where $[C(a)a]_n = (\sum_{m=1}^n a_m)/a_n$,

$$\widehat{C_1} = \{a \in U^+ : \quad C(a)a \in \ell_\infty\},$$

$$\widehat{C} = \{a \in U^+ : \quad [C(a)a]_n \to l \text{ for some } l \in \mathbb{C}\},$$
$$\widehat{\Gamma} = \{a \in U^+ : \lim_{n\to\infty}(\frac{a_{n-1}}{a_n}) < 1\},$$
$$\Gamma = \{a \in U^+ : \limsup_{n\to\infty}(\frac{a_{n-1}}{a_n}) < 1\}.$$

and

$$G_1 = \{x \in U^+ : x_n \geq k\gamma^n \text{ for all } n \text{ and for some } k > 0 \text{ and } \gamma > 1\}.$$

By [3, Proposition 2.1, p. 1786] and [6] we obtain the next lemma.

Lemma 8. *We have*
(i) $\widehat{\Gamma} = \widehat{C}$.
(ii) $\Gamma \subset \widehat{C_1} \subset G_1$.

Since $\widehat{\Gamma} \subset \Gamma$ we deduce $\widehat{\Gamma} = \widehat{C} \subset \Gamma \subset \widehat{C_1} \subset G_1$.

Here among other things we study the equivalence

$$\frac{y_n - y_{n-1}}{a_n} \to l \text{ if and only if } \frac{y_n}{b_n} \to l' \ (n \to \infty) \text{ for all } y \in s \text{ and for some } l,\ l' \in \mathbb{C}.$$

This statement can written in the form $s_a^{(c)}(\Delta) = s_b^{(c)}$. We will use the next elementary lemma.

Lemma 9. *Let T_1, T_2 be triangles and E, F be sequence spaces. Then for any triangles T we have $T \in (E(T_1), F(T_2))$ if and only if $T_2TT_1^{-1} \in (E, F)$.*

The proof is based on the fact that T_1, T_2 and T being triangles we have $E(T_1) = T_1^{-1}E$ and for every $\xi \in E$ we have

$$T_2[T(T_1^{-1}\xi)] = (T_2TT_1^{-1})\xi.$$

Let us state the next results.

Theorem 10. *Let a, $b \in U^+$. We have*
(i) The following statements are equivalent
a) $s_a(\Delta) = s_b$,
b) $s_a^0(\Delta) = s_b^0$,
c) $s_a = s_b$ *and* $b \in \widehat{C_1}$.
(ii) Assume $(b_{n-1}/b_n)_n \in c$. *Then*

$$s_a^{(c)}(\Delta) = s_b^{(c)} \tag{8}$$

if and only if

$$\frac{a_n}{b_n} \to l \neq 0 \text{ for some } l \in \mathbb{C} \text{ and } b \in \widehat{\Gamma}.$$

Proof. The statement (i) was shown in [5, Proposition 9, p. 300]. It remains to show (ii). The first identity (8) means that Δ is bijective from $s_a^{(c)}$ to $s_b^{(c)}$.

Since Δ is a triangle and its inverse is equal to Σ, by Lemma 9 equality (8) is equivalent to $\Sigma \in (s_a^{(c)}, s_b^{(c)})$ and to $\Delta \in (s_b^{(c)}, s_a^{(c)})$. Then also by Lemma 9 we have $D_{1/b}\Sigma D_a \in (c,c)$ and $D_{1/a}\Delta D_b \in (c,c)$. From the characterization of (c,c) we deduce

$$[C(b)a]_n = \frac{\sum_{m=1}^{n} a_m}{b_n} \to L \text{ for some } L, \tag{9}$$

and

$$\frac{b_n + b_{n-1}}{a_n} \leq K \text{ for all } n, \tag{10}$$

Conditions (9) and (10) imply there is K' such that

$$\frac{a_n}{b_n} \leq K' \text{ and } \frac{b_n}{a_n} \leq K \text{ for all } n \tag{11}$$

that is $s_a = s_b$. Then we have $a \in \widehat{C_1}$ since (11) implies

$$[C(a)a]_n = [C(b)a]_n \frac{b_n}{a_n} \leq \frac{1}{K'}[C(b)a]_n \text{ for all } n.$$

Then b_{n-1}/b_n cannot tend to 1. Indeed we have

$$\frac{[C(b)a]_n}{[C(b)a]_{n-1}} = \frac{\sum_{m=1}^{n-1} a_m + a_n}{\sum_{m=1}^{n-1} a_m} \frac{b_{n-1}}{b_n} = (1 + \frac{a_n}{\sum_{m=1}^{n-1} a_m})\frac{b_{n-1}}{b_n}.$$

Then $L \neq 0$ since

$$[C(b)a]_n \geq \frac{a_n}{K' a_n} = \frac{1}{K'} > 0 \text{ for all } n$$

and $\lim_{n\to\infty} \frac{[C(b)a]_n}{[C(b)a]_{n-1}} = \frac{L}{L} = 1$. So if b_{n-1}/b_n tend to 1 we should have

$$1 + \frac{a_n}{\sum_{m=1}^{n-1} a_m} \to 1 \ (n \to \infty)$$

and

$$[C(a)a]_n = \frac{\sum_{m=1}^{n-1} a_m}{a_n} + 1 \to \infty \ (n \to \infty)$$

which is contradictory. So we have $b_{n-1}/b_n \to L' \neq 1$. Then

$$\frac{a_n}{b_n} = \frac{1}{b_n}(\sum_{m=1}^{n} a_m - \sum_{m=1}^{n-1} a_m) = [C(b)a]_n - [C(b)a]_{n-1}\frac{b_{n-1}}{b_n}$$

tends to $L - LL' = L(1 - L') \neq 0$ and a_n/b_n has a nonzero limit l. We deduce

$$[C(a)a]_n = [C(b)a]_n \frac{b_n}{a_n} \to \frac{L}{l} \neq 0$$

and $a \in \widehat{C} = \widehat{\Gamma}$. So $\frac{a_{n-1}}{a_n} \to \chi < 1 (n \to \infty)$ and

$$\frac{b_{n-1}}{b_n} = \frac{b_{n-1}}{a_{n-1}} \frac{1}{\frac{b_n}{a_n}} \frac{a_{n-1}}{a_n} \to \frac{1}{l}\frac{1}{\frac{1}{l}}\chi < 1$$

which implies $b \in \widehat{\Gamma}$. This concludes the proof.

Conversely assume $a_n/b_n \to l \neq 0$ for some $l \in \mathbb{C}$ and $\lim_{n\to\infty}(b_{n-1}/b_n) < 1$. Then $s_a^{(c)} = s_b^{(c)}$ and $b \in \widehat{\Gamma}$ implies $s_a^{(c)}(\Delta) = s_a^{(c)} = s_b^{(c)}$. ■

We can state the next result which is a direct consequence of Theorem 10 (i) b).

Corollary 11. *(i)* $s_a^{(c)}(\Delta) = s_a^{(c)}$ *if and only if* $a \in \widehat{\Gamma}$.

(ii) $c(\Delta) \neq s_a^{(c)}$ *for any* $a \in U^+$.

(iii) Let $r,\ u > 0$. *Then* $s_r^{(c)}(\Delta) = s_u^{(c)}$ *if and only if* $r = u > 1$.

Let us cite the next lemma where $[\Sigma^q]_{nm} = \binom{q+n-m-1}{n-m}$ with $m \leq n$.

Corollary 12. [5] *Let* $q \geq 1$ *be an integer. Then the following statements are equivalent*

(i) $a \in \widehat{C_1}$,

(ii) $s_a(\Delta) = s_a$,

(iii) $s_a^0(\Delta) = s_a^0$,

(iv) $s_a(\Delta^q) = s_a$,

(v) $s_a^0(\Delta^q) = s_a^0$,

(vi) $\dfrac{1}{a_n}\sum\limits_{m=1}^{n}\binom{q+n-m-1}{n-m}a_k = O(1)\ (n \to \infty)$.

2.2. On the (SSE) with operators $(\chi_a * \chi_x + \chi_b)(\Delta) = \chi_\eta$ **and** $[\chi_a * (\chi_x)^2 + \chi_b * \chi_x](\Delta) = \chi_\eta$ **with** $\chi \in \{s^0, s\}$

As consequences of the preceding we can state the next results.

Proposition 13. *Let* $a,\ b,\ \eta \in U^+$. *Then*

i) a) If $b/\eta \in c_0$ *the (SSE) with operator*

$$(s_a * s_x + s_b)(\Delta) = s_\eta \tag{12}$$

is equivalent to $s_x = s_{\eta/a}$ *and* $\eta \in \widehat{C_1}$;

b) If $s_b = s_\eta$ *then (SSE)* (12) *is equivalent to* $x \in s_{\eta/a}$ *and* $\eta \in \widehat{C_1}$;

c) If $b/\eta \notin \ell_\infty$ *then (SSE)* (12) *has no solution.*

ii) Assume

$$a \in s_\eta^0 \tag{13}$$

and

$$b \in s_a. \tag{14}$$

Then the (SSE)

$$[s_a * (s_x)^2 + s_b * s_x](\Delta) = s_\eta \tag{15}$$

is equivalent to $\eta \in \widehat{C_1}$ *and* $s_x = s_{\sqrt{\eta/a}}$.

Proof. i) We have $s_a * s_x + s_b = s_{ax} + s_b = s_{ax+b}$. So $(s_a * s_x + s_b)(\Delta) = s_{ax+b}(\Delta)$. By Theorem 10 (ii) we have that (12) is equivalent to

$$\begin{cases} s_{ax+b} = s_\eta \\ \eta \in \widehat{C_1}, \end{cases} \tag{16}$$

and $s_{ax+b} = s_\eta$ is equivalent to $s_b + s_{ax} = s_\eta$. For the study of the (SSE) it is enough to apply Theorem 3. If $b/\eta \in c_0$ then $s_{ax} = s_\eta$ and $s_x = s_{\eta/a}$. The remainder of the proof can be shown similarly.

ii) First show the necessity. Since we have $s_a * (s_x)^2 + s_b * s_x = s_{ax^2+bx}$, by Theorem 10 (iii) identity (15) is equivalent to

$$\begin{cases} s_{ax^2+bx} = s_\eta \\ \eta \in \widehat{C_1}. \end{cases} \tag{17}$$

Then $s_{x^2+\frac{b}{a}x} = s_{\frac{\eta}{a}}$. Let us show $x_n \to \infty$ $(n \to \infty)$. Since $\eta \in \widehat{C_1}$ we have $\eta_n \to \infty$ and by (17) there is $K > 0$ such that $a_n x_n^2 + b_n x_n \geq K\eta_n$ and

$$y_n = x_n^2 + \frac{b_n}{a_n} x_n \geq K \frac{\eta_n}{a_n} \text{ for all } n$$

Then condition (13) implies $\eta_n/a_n \to \infty$ $(n \to \infty)$ and $y_n \to \infty$ $(n \to \infty)$. Now by the identity $y_n = x_n^2 + (b_n/a_n)x_n$ we have

$$x_n = \frac{1}{2}\Big(-\frac{b_n}{a_n} + \sqrt{\Big(\frac{b_n}{a_n}\Big)^2 + 4y_n}\Big) \text{ for all } n,$$

and by (14) we deduce $x_n \to \infty$ $(n \to \infty)$. We then have

$$\frac{a_n x_n^2 + b_n x_n}{a_n x_n^2} = 1 + \frac{b_n}{a_n}\frac{1}{x_n} = 1 + O(1)o(1) = 1 + o(1) \ (n \to \infty),$$

and $\dfrac{a_n x_n^2 + b_n x_n}{a_n x_n^2} \to 1$ $(n \to \infty)$, which shows $s_{ax^2+bx} = s_{ax^2}$. By Corollary 7 iii) we conclude $s_x = s_{\sqrt{\eta/a}}$.

Sufficiency. Assume $s_x = s_{\sqrt{\eta/a}}$ and $\eta \in \widehat{C_1}$. Then $s_{ax^2+bx} = s_\eta$. But (14) implies $s_b \subset s_a$ and

$$s_{b\sqrt{\frac{\eta}{a}}} \subset s_{\sqrt{a\eta}}$$

and by (13) we have $\sqrt{a_n\eta_n}/\eta_n = \sqrt{a_n/\eta_n} = o(1)$ $(n \to \infty)$. We conclude $s_{ax^2+bx} = s_\eta$ and since $\eta \in \widehat{C_1}$ we have $s_{ax^2+bx}(\Delta) = s_\eta$. This concludes the proof of i). ■

We deduce the next corollaries.

Corollary 14. *Let u, $p > 0$ and $R > 1$. Consider the (SSE)*

$$(s_{(u^n x_n)_n} + s_{(n^p)_n})(\Delta) = s_R \text{ with } x \in U^+. \tag{18}$$

Then

(i) if $R > u$ then the solutions x of (18) *satisfy $x_n \to \infty$ $(n \to \infty)$ and for any $\alpha > 0$ we have $\lim_{n\to\infty} \frac{x_n}{n^\alpha} = \infty$;*

(ii) if $R = u$ then the solutions of (18) *satisfy $x_n = O(1)$ $(n \to \infty)$;*

(iii) if $R < u$ then for any given $\beta > 0$ the solutions of (18) *satisfy*

$$\lim_{n\to\infty} n^\beta x_n = 0.$$

Proof. (i) We have $a_n = u^n$, $\eta_n = R^n$ and $b_n = n^p$. Since $n^p R^{-n} \to 0$ $(n \to \infty)$ we have $b/\eta \in c_0$ and (18) is equivalent to $s_x = s_{R/u}$. Then putting $R/u = r$ there is K_1 such that $x_n n^{-\alpha} \geq K_1 r^n n^{-\alpha}$ and since $r > 1$ we have $r^n n^{-\alpha} \to \infty$ and $x_n n^{-\alpha} \to \infty$ $(n \to \infty)$.

(ii) We have $R = u$ and as we have seen above we have $s_x = s_1$ which implies $x_n = O(1)$ $(n \to \infty)$.

(iii) Here we have $s_x = s_{R/u} = s_r$ with $r < 1$ so there is K_2 such that $x_n n^\beta \leq K_2 r^n n^\beta$ and since $r^n n^\beta$ tends to naught we conclude it is the same for $n^\beta x_n$. ∎

Corollary 15. *Let $x \in U^+$ satisfy the (SSE) with operator*

$$(s_{(n^p x_n^2)_n} + s_{(x_n \ln n)_n})(\Delta) = s_R \tag{19}$$

with $p > 0$ and $R > 1$. Then for every $\alpha > 0$ we have $\lim_{n\to\infty} \frac{x_n}{n^\alpha} = \infty$.

Proof. Here we have $a_n = n^p$, $b_n = \ln n$, $\eta_n = R^n$ and conditions (13) and (14) hold since trivially we have $n^p/R^n = o(1)$ and $\ln n/n^p = O(1)$ $(n \to \infty)$, since $R > 1$ we also have $\eta \in \widehat{C_1}$. Then the solutions of (19) satisfy $x_n \geq K_1 R^{n/2} n^{-\frac{p}{2}}$ and $x_n/n^\alpha \geq K_1 R^{n/2}/n^{\frac{p}{2}+\alpha}$ then $R^{n/2}/n^{\frac{p}{2}+\alpha} \to \infty$ and $x_n/n^\alpha \to \infty$ $(n \to \infty)$. This concludes the proof. ∎

Using similar arguments we immediately obtain the following result.

Proposition 16. *Let a, b, $\eta \in U^+$. Then*

i) α) If $b/\eta \in c_0$ then the (SSE)

$$s^0_{ax+b}(\Delta) = s^0_\eta \tag{20}$$

is equivalent to $s_x = s_{\eta/a}$ and $\eta \in \widehat{C_1}$.

β) If $s_b = s_\eta$ then (SSE) (20) *is equivalent to $x \in s_\eta$ and $\eta \in \widehat{C_1}$;*

γ) If $b/\eta \notin \ell_\infty$ then (SSE) (20) *has no solution.*

ii) Assume $a \in s^0_\eta$ and $b \in s_a$. Then the (SSE)

$$s^0_{ax^2+bx}(\Delta) = s^0_\eta \tag{21}$$

is equivalent to $\eta \in \widehat{C_1}$ and $s_x = s_{\sqrt{\eta/a}}$.

We immediately deduce the following.

Corollary 17. *The (SSE) with operator*

$$\chi_{x^2+x}(\Delta) = s_\eta \text{ with } \chi = s^0, \text{ or } s \tag{22}$$

is equivalent to $\eta \in \widehat{C_1}$ *and* $s_x = s_{\sqrt{\eta}}$.

Proof. We only consider the (SSE) (22) where $\chi = s$, the other case can be shown similarly. We have $s_{x^2+x}(\Delta) = s_\eta$ equivalent to $s_{x^2+x} = s_\eta$ and $\eta \in \widehat{C_1}$. So $a = e \in s_\eta^0$ since $1/\eta \in c_0$ and $b = e \in s_a = \ell_\infty$, then by Proposition 12 we conclude $s_x = s_{\sqrt{\eta}}$. Conversely. Assume $s_x = s_{\sqrt{\eta}}$ and $\eta \in \widehat{C_1}$. Then $\eta_n \to \infty$, so we have $(\eta_n + \sqrt{\eta_n})/\eta_n \to 1$ $(n \to \infty)$ and $s_{x^2+x} = s_{\eta+\sqrt{\eta}} = s_\eta$. We conclude $s_{x^2+x}(\Delta) = s_\eta(\Delta) = s_\eta$. ■

2.3. On the (SSE) $\chi_{ax^2+x}(\Delta) = \chi_x$ and $\chi_a + \chi_x(\Delta) = \chi_x$ with $\chi \in \{s^0, s\}$

Now we are interested in the study of sequence spaces equations with a second member depending on x such as the (SSE) $\chi_{ax^2+x}(\Delta) = s_x$ and $\chi_a + \chi_x(\Delta) = \chi_x$. We will see that the last equation is equivalent to the equation $s_a^0 + s_x^0(\Delta) = s_x^0$.

Proposition 18. *The (SSE)*

$$\chi_{ax^2+x}(\Delta) = \chi_x \tag{23}$$

where χ *is any of the symbols* s^0, *or* s *is equivalent to* $x \in \widehat{C_1}$ *and to*

$$x_n \leq \frac{K}{a_n} \text{ for all } n \text{ and for some } K > 0$$

Proof. We only show the proposition for $\chi = s$. The proof being similar for the other case. We have that (23) is equivalent to

$$\begin{cases} s_{ax^2+x} = s_x, \\ x \in \widehat{C_1}. \end{cases}$$

Since we have $s_{ax^2+x} = s_{ax^2} + s_x$ the identity $s_{ax^2+x} = s_x$ is equivalent to $s_{ax^2} \subset s_x$ and to $s_x \subset s_{1/a}$ by Proposition 1 i). This concludes the proof of the proposition. ■

Using similar arguments we deduce the following result.

Remark 19. We immediately deduce that $s_{x^2+x}(\Delta) = s_x$ has no solution since we have $x \in \widehat{C_1}$ implies $x_n \to \infty$ $(n \to \infty)$ and we cannot have $s_x \subset s_{1/a} = \ell_\infty$. It is the same for the equation $s_{x^2+x}^0(\Delta) = s_x^0$.

In the following we will use the set $s_a^* = \{x \in U^+ : a/x \in \ell_\infty\}$. We can state the next result.

Proposition 20. *Assume*

$$\varliminf_{n\to\infty} \left(\frac{r^n}{a_n}\right) > 0 \text{ for all } r > 1. \tag{24}$$

Then

$$\{x \in U^+ : \chi_a + \chi_x(\Delta) = \chi_x\} = \widehat{C_1} \tag{25}$$

where χ *is either* s, *or* s^0.

Proof. First show identity (25) with $\chi = s$. Let A_a be the set

$$A_a = \{x \in U^+ : s_a + s_x(\Delta) = s_x\}.$$

Show that $A_a = \widehat{C_1} \cap s_a^*$. First let $x \in A_a$. Then $s_x(\Delta) \subset s_x$ and $I \in (s_x(\Delta), s_x)$, by Lemma 9 we have $\Sigma \in (s_x, s_x)$ that is

$$\frac{1}{x_n}(x_1 + \cdots + x_n) = O(1)\ (n \to \infty). \tag{26}$$

We conclude $A_a \subset \widehat{C_1}$. Then show $A_a \subset s_a^*$. We have $x \in A_a$ also implies

$$s_a \subset s_a + s_x(\Delta) = s_x$$

we deduce $a \in s_a \subset s_x$ and $x \in s_a^*$. We conclude $A_a \subset \widehat{C_1} \cap s_a^*$. Now show the inclusion $\widehat{C_1} \cap s_a^* \subset A_a$. Let $x \in \widehat{C_1} \cap s_a^*$. First $x \in \widehat{C}_1$ implies $s_x(\Delta) = s_x$, then $x \in s_a^*$ implies $s_a \subset s_x$ and $s_a + s_x = s_x$. We conclude $s_a + s_x(\Delta) = s_x$ and $x \in A_a$. This shows $\widehat{C_1} \cap s_a^* \subset A_a$. Now show $\widehat{C}_1 \subset s_a^*$. Since by Lemma 8 (ii) we have $\widehat{C_1} \subset G_1$, the condition $x \in \widehat{C}_1$ implies there are $k > 0$ and $\gamma > 1$ such that $x_n \geq k\gamma^n$. Since we have $\underline{\lim}_{n\to\infty}(r^n/a_n) > 0$ then $\inf_n(r^n/a_n) > 0$ for all $r > 1$ and there is $r_0 \in]1, \gamma[$ such that

$$\frac{x_n}{a_n} \geq k\left(\frac{\gamma^n}{a_n}\right) \geq k \inf_n \left(\frac{r_0^n}{a_n}\right) > 0 \text{ for all } n$$

and $x \in s_a^*$. So we have shown $\widehat{C_1} \subset s_a^*$ and $A_a = \widehat{C_1}$. This completes the first part of the proof.

Now show identity (25) holds with $\chi = s^0$. Let A_a^0 be the set

$$A_a^0 = \{x \in U^+ : s_a^0 + s_x^0(\Delta) = s_x^0\}.$$

Show that $A_a^0 = \widehat{C_1} \cap s_a^*$. First let $x \in A_a^0$. Again by Lemma 9 we have $s_x^0(\Delta) \subset s_x^0$ and $\Sigma \in (s_x^0, s_x^0)$. So we have

$$\frac{1}{x_n}(x_1 + \cdots + x_n) = O(1) \text{ and } \frac{1}{x_n} = o(1)\ (n \to \infty). \tag{27}$$

But since we have $x \in \widehat{C_1}$ implies $1/x_n \to 0$, conditions given by (27) are equivalent to $x \in \widehat{C_1}$. So we have shown $A_a^0 \subset \widehat{C}_1$. Then show $A_a^0 \subset s_a^*$. We have $x \in A_a^0$ implies $s_a^0 \subset s_a^0 + s_x^0(\Delta) = s_x^0$ and $s_a^0 \subset s_x^0$. By Lemma 2 we deduce $s_a \subset s_x$ and $a \in s_a \subset s_x$, this means that $x \in s_a^*$. We conclude $A_a^0 \subset \widehat{C_1} \cap s_a^*$. The proof of the inclusion $\widehat{C_1} \cap s_a^* \subset A_a$ follows exactly the same lines that in the proof of $\widehat{C_1} \cap s_a^* \subset A_a^0$. So $A_a^0 = \widehat{C_1} \cap s_a^*$. Finally reasoning as above condition (24) permits us to conclude (25) holds with $\chi = s^0$. ■

The next corollary can be easily deduced.

Corollary 21. *We have*

(i) $s_a + s_x(\Delta) \subset s_x$ *if and only if* $x \in \widehat{C_1} \cap s_a^*$;

(ii) if $x \in \widehat{C_1}$ *then* $s_x \subset s_a + s_x(\Delta)$.

Example 22. Let $\alpha > 0$. Then the set of all sequences $x \in U^{+}$ such that

$$u_n = O(n^{\alpha}) \text{ and } v_n - v_{n-1} = O(x_n)$$

implies

$$u_n + v_n = O(x_n) \ (n \to \infty) \text{ for all } u, \ v \in s,$$

is equal to $\widehat{C_1}$. Indeed for any $r > 1$ we have $\underline{\lim}_{n\to\infty}(r^n/n^{\alpha}) > 0$ and $s_a + s_x(\Delta) \subset s_x$.

REFERENCES

[**1**] A. Farés, B. de Malafosse, *Sequence spaces equations and application to matrix transformations*, Intern. Forum **3** (2008), 911–927.

[**2**] I.J. Maddox, *Infinite Matrices of Operators*, Springer-Verlag, Berlin, Heidelberg and New York, 1980.

[**3**] B. de Malafosse, *On some BK space*, Intern. J. Math. Math. Sci. **28** (2003), 1783–1801.

[**4**] B. de Malafosse, *Sum and product of certain BK spaces and matrix transformations between these spaces*, Acta Math. Hung. **104** (2004), 241–263.

[**5**] B. de Malafosse, *Sum of sequence spaces and matrix transformations*, Acta Math. Hung. **113** (2006), 289–313.

[**6**] B. de Malafosse, *The Banach algebra $B(X)$, where X is a BK space and applications*, Mat. Vesnik **57** (2005), 41–60.

[**7**] B. de Malafosse, E. Malkowsky, *Sets of difference sequences of order m*, Acta Sci. Math. (Szeged) **70** (2004), 659–682.

[**8**] B. de Malafosse, V. Rakočević, *A generalization of a Hardy theorem*, Linear Algebra Appl. **421** (2007), 306–314

[**9**] G.H. Hardy, *Divergent Series*, Oxford University Press, Oxford, 1949.

[**10**] F. Móricz, B.E. Rhoades, *An equivalent reformulation of summability by weighted mean methods*, Linear Algebra Appl. **268** (1998), 171-181.

[**11**] F. Móricz, B.E. Rhoades, *An equivalent reformulation of summability by weighted mean methods, revisited*, Linear Algebra Appl. **349** (2002), 187-192.

[**12**] B. de Malafosse, *Contribution à l'étude des systèmes infinis*, Thèse de Doctorat de 3$^{\text{è}}$ cycle, Université Paul Sabatier, Toulouse III, 1980.

[**13**] B. de Malafosse, *Properties of some sets of sequences and application to the spaces of bounded difference sequences of order μ*, Hokkaido Math. J. **31** (2002), 283–299.

[**14**] B. de Malafosse, V. Rakočević, *Applications of measure of noncompactness in operators on the spaces s_a, s_a^0, $s_a^{(c)}$ and l_a^p*, J. Math. Anal. Appl. **323** (2006), 131–145.

[**15**] A. Wilansky, *Summability through Functional Analysis*, North-Holland Mathematics Studies **85**, 1984.

(received 13.09.2010; available online 20.02.2011)

LMAH Université du Havre, I.U.T Le Havre BP 4006 76610, Le Havre, France.

E-mail: `bdemalaf@wanadoo.fr`