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THE TOTAL GRAPH OF A MODULE

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Abstract. A generalization of the total graph of a ring is presented. Various properties are proved and some relations to the total graph of a ring as well as to the zero-divisor graph are established.

1. Introduction

The idea of associating a graph to a ring first appears in [5]. For the vertices of the graph, Beck takes all elements of a commutative ring R. Two distinct vertices $x, y \in R$ are adjacent if $xy = 0$. This paper primarily deals with the questions of coloring and the computation of the chromatic number for some rings. Other authors have been motivated by the results of this article to research the interrelations between properties of graphs and rings, the question of the connectivity of the graph, its diameter, radius and other interesting invariants of graphs. Their interpretations in the theory of commutative rings, make Beck's paper the founding paper of a new and interesting field of algebra.

Of course, there are many ways to associate a graph to a given ring R. The most well-known is certainly the *zero-divisor graph* $\Gamma(R)$ introduced in [3]. In this paper, the set of vertices consists only of non-zero zero-divisors. The authors show that $\Gamma(R)$ is always connected, of diameter at most 3. Some other investigations into properties of this graph may be found in [1, 4, 7–10].

In [2], the notion of the *total graph of a commutative ring* $T(\Gamma(R))$ is introduced. We use the notation $\mathrm{T}\Gamma(R)$. The vertices of this graph are all elements of the ring R. Two vertices are adjacent if their sum is a zero-divisor. This graph, unlike the zero-divisor graph, need not be connected. Even in the case when the total graph is connected, its diameter may have arbitrary value n, for $n \geq 1$. The structure and the properties of the total graph are thoroughly examined in [2].

We define the total graph of a module in an analogous way. Let R be a commutative ring with identity, $R^* = R\backslash\{0\}$, $Z(R)$ the set of its zero-divisors, and

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$Z(R)^* = Z(R)\backslash\{0\}$. Let M be an R-module, $M^* = M\backslash\{0\}$, and $T(M) = \{m \in M \mid rm = 0 \text{ for some } r \in R^*\}$ the set of its torsion elements. We define the *total graph of a module* $\mathrm{T}\Gamma(M)$ as follows:

$$V(\mathrm{T}\Gamma(M)) = M, \quad E(\mathrm{T}\Gamma(M)) = \{\{m_1, m_2\} : m_1 + m_2 \in \mathrm{T}(M)\},$$

where $V(\Gamma)$ $(E(\Gamma))$ denote the set of vertices (edges) of the graph Γ. In the case $M = R$, i.e., when we look at R as an R-module, $T(M) = Z(R)$; therefore we arrive at the total graph of a ring $\mathrm{T}\Gamma(R)$ introduced in [2].

By a graph Γ, we mean a simple unoriented graph without loops. Two vertices x and y of the graph Γ are *connected* if there is a path in Γ connecting them. If every two vertices are connected, we say that the graph Γ is *connected*. For vertices x and y in Γ, one defines the *distance* $d(x, y)$ as the length of a shortest path between x and y, if these vertices are connected, and $d(x, y) = \infty$ if there is no such a path. The *diameter* of the graph Γ is $\operatorname{diam}(\Gamma) = \sup\{d(x, y) \mid x, y \in \Gamma\}$. A graph is *complete* if any two distinct vertices are adjacent. If the vertices of the graph Γ are partitioned into two disjoint sets A and B of cardinality $|A| = m$ and $|B| = n$, and two vertices are adjacent if and only if are not in the same set, then Γ is a *complete bipartite* graph. For complete and complete bipartite graphs, we use the standard notation K^n and $K^{m,n}$, where we allow m and n to be infinite cardinals.

2. $T(M)$ is a submodule of M

The structure of the total graph $\mathrm{T}\Gamma(M)$ may be completely described in those cases when torsion elements form a submodule. Let us start with the extreme cases $T(M) = M$ and $T(M) = \{0\}$.

Theorem 2.1. *The total graph* $\mathrm{T}\Gamma(M)$ *is complete iff* $T(M) = M$.

Proof. If $T(M) = M$, then for any two vertices $m_1, m_2 \in M$, one has $m_1 + m_2 \in T(M)$. Therefore, they are adjacent in $\mathrm{T}\Gamma(M)$. On the other hand, if $\mathrm{T}\Gamma(M)$ is complete, then every vertex is adjacent to 0. Thus $m = m + 0 \in T(M)$, from which the claim follows. ∎

Remark. The condition of the previous theorem is necessarily fulfilled if $Ann(M) \neq 0$. Let us illustrate this with the following examples.

Example 2.2. Let $R = \mathbb{Z}_n \times \mathbb{Z}_m$ and $M = \mathbb{Z}_n$ an R-module, where the module structure is given by $(a, b) \cdot m = am$. Then $Ann(M) \neq (0, 0)$ since $(0, b) \in Ann(M)$ for every $b \in \mathbb{Z}_m$, and thus $\mathrm{T}\Gamma(M)$ is complete.

Example 2.3. Every finite Abelian group M is a torsion $\mathbb{Z}$-module. In particular, if $R = \mathbb{Z}$ and $M = \mathbb{Z}_n$, an R-module with the usual multiplication, then $\mathrm{T}\Gamma(M) \cong K^n$. Therefore, all finite complete graphs may be realized as total graphs of modules.

Let us look at the other extreme case, when the graph is totally disconnected. If $T(M) = \{0\}$, then the vertices m_1 and m_2 are adjacent iff $m_2 = -m_1$. If, in addition to that, $M \neq 0$, then $T\Gamma(M)$ is a disconnected graph and its only edges are those that connect vertices m_i and $-m_i$.

THEOREM 2.4. *Let R be a commutative ring and M an R-module. Then $T\Gamma(M)$ is totally disconnected iff R has characteristic 2 and M is torsion-free.*

Proof. If $T\Gamma(M)$ is totally disconnected, then 0 is not adjacent to any vertex, i.e., $0 + m = m \notin T(M)$ for every $m \in M^*$. So $T(M) = \{0\}$, and therefore M is torsion-free. Further, since $m + (-m) = 0$, from the total disconnectedness of the graph it follows that $m = -m$, i.e., $2m = 0$ for every $m \in M$. Since $T(M) = \{0\}$, it follows that $2 = 0$, i.e., $\mathrm{char}\,(R) = 2$. The reverse implication is obvious. ∎

LEMMA 2.5. *Let $T(M)$ be a submodule of an R-module M and $x \in M \setminus T(M)$. Then $2x \in T(M)$ iff $2 \in Z(R)$.*

Proof. First suppose that $2 \in Z(R)$, i.e., there exists an $r \in R^*$ such that $2r = 0$. Then $2x \in T(M)$ for all x since $r(2x) = (2r)x = 0$.

Let us now assume that $2x \in T(M)$. Since $x \notin T(M)$, we have that $x \neq 0$ and for all $r \in R$: $rx = 0 \Rightarrow r = 0$. Since $2x \in T(M)$, there exists $a \in R^*$ such that $a(2x) = 0$. So $(2a)x = 0$, and since $x \notin T(M)$, one must have $2a = 0$, i.e., $2 \in Z(R)$. ∎

THEOREM 2.6. *Let M be an R-module such that $T(M)$ is a proper submodule of M. Then $T\Gamma(M)$ is disconnected.*

Proof. If $T(M) = \{0\}$, then the vertex 0 evidently is not adjacent to any other vertex. If $T(M) \neq \{0\}$, then the subgraphs with vertices from $T(M)$ and $M \setminus T(M)$ are disjoint. For, if $x \in T(M)$ and $y \in M \setminus T(M)$ were adjacent, then $x + y \in T(M)$; so this, since $T(M)$ is a submodule, would lead to the contradiction $y \in T(M)$. ∎

The description of the structure of the graph $T\Gamma(M)$ when $T(M) \neq M$ is a submodule is almost identical to the description of the graph $T\Gamma(R)$ when $Z(R)$ is an ideal in R [2, Theorem 2.2]. Therefore, we present a proof without giving all the details.

Let us look at the quotient module $M/T(M)$. Let $|T(M)| = \alpha$ and $|M/T(M)| = \beta$. Let $x, y \in M \setminus T(M)$ be such that $x + T(M) \neq y + T(M)$. The elements $x + m_1$, $x + m_2$ from the same coset $x + T(M)$ are adjacent iff $2x \in T(M)$; so $2 \in Z(R)$, according to Lemma 2.5. Then $x + m_1$ and $y + m_2$ are not adjacent—otherwise we would have $x - y = x + y - 2y \in T(M)$, and therefore $x + T(M) = y + T(M)$. Since every coset has cardinality α, we arrive at the conclusion that $T\Gamma(M)$ is the disjoint union of β complete graphs K^α when $2 \in Z(R)$.

If $2 \notin Z(R)$, then the elements $x + m_1$, $x + m_2$ from $x + T(M)$ are obviously not adjacent. The elements $x + m_1$, $y + m_2$ from different cosets are adjacent iff $x + y \in T(M)$, or $y + T(M) = -x + T(M)$. In this way, we conclude that the subgraph spanned by the vertices from $M \setminus T(M)$ is a disjoint union of $\frac{\beta - 1}{2}$ ($= \beta$ if β is infinite) disjoint bipartite graphs $K^{\alpha,\alpha}$. Therefore, the following theorem holds.

Theorem 2.7. *Let R be a commutative ring and M an R-module such that $T(M)$ is a proper submodule of M. Suppose $|T(M)| = \alpha$ and $|M/T(M)| = \beta$.*

1. *If $2 \in Z(R)$, then $\mathrm{T}\Gamma(M)$ is a union of β disjoint complete graphs K^{α}.*
2. *If $2 \notin Z(R)$, then $\mathrm{T}\Gamma(M)$ is a union of $\frac{\beta-1}{2}$ disjoint bipartite graphs $K^{\alpha,\alpha}$ and one complete K^{α}.*

Example 2.8. Let R be a ring and $M = R \oplus R$ a module over R.

1. If $R = \mathbb{Z}_4$, then $\mathrm{T}\Gamma(M)$ is a union of 4 disjoint K^4.
2. If $R = \mathbb{Z}_9$, then $\mathrm{T}\Gamma(M)$ is a disjoint union of one complete graph K^9 and 4 bipartite $K^{9,9}$.

Theorems 2.1 and 2.7 give a complete description of the structure of the total graph $\mathrm{T}\Gamma(M)$ when $T(M)$ is a submodule. The question under what conditions $T(M)$ is a submodule of M and how is this related to the condition that $Z(R)$ is an ideal in R naturally arises. When $Z(R) = \{0\}$, i.e., if R is a domain, then $T(M)$ is obviously a submodule. We prove that a more general result holds.

Theorem 2.9. *If $Z(R) = (z)$ is a principal ideal of R and $z \in Nil(R)$, then $T(M)$ is a submodule of M.*

Proof. Let $Z(R) = (z)$ and assume that $T(M)$ is not a submodule of M. Then there exist $m_1, m_2 \in T(M)$ such that $m_1 + m_2 \notin T(M)$. Let $r_1, r_2 \in R^*$ be such that $r_1 m_1 = r_2 m_2 = 0$. Then $r_1 r_2(m_1 + m_2) = 0$ and $m_1 + m_2 \notin T(M)$; so we must have $r_1 r_2 = 0$, and thus $r_1, r_2 \in Z(R)$. Therefore, as $z \in Nil(R)$, we have that $r_1 = az^k$ and $r_2 = bz^m$, for some $a, b \notin Z(R)$. Without loss of generality, we may assume that $k \geq m$. Then for the element $br_1 \in R^*$ we have $br_1(m_1 + m_2) = 0$ which is contrary to the assumption that $m_1 + m_2$ is not torsion. ■

Remark. By the assumption of the previous theorem, it follows that $Z(R) = Nil(R)$. Let R be the polynomial ring $\mathbb{Z}[X]$. If we define a nonstandard multiplication in R by $p(X) * q(X) = p(0)q(0)$, then $Z(R) = Nil(R) = (X)$. Thus the conditions of Theorem 2.9 are fulfilled. It is also true in the rings $\mathbb{Z}_{p^n}$ for any integer $n \geq 2$ and prime p. Here $Z(R) = Nil(R) = (p)$.

When the zero-divisors form an ideal which is not principal, $T(M)$ need not be a submodule even if $Z(R) = Nil(R)$. Let us look at an example.

Example 2.10. Let R be the local ring $\mathbb{Z}_4[X]/(2X, X^2)$. Then $Z(R) = Nil(R) = (2, x)$, where x is the corresponding class. Let $M = R/2R \oplus R/xR$ and take $m_1 = (1 + 2R, xR)$, $m_2 = (2R, 1 + xR)$. Then $m_1, m_2 \in T(M)$ since $2m_1 = xm_2 = 0$, while $m_1 + m_2 = (1 + 2R, 1 + xR) \notin T(M)$.

3. $T(M)$ is not a submodule of M

We begin this section with an interesting result linking the total graph of a module to the zero-divisor graph.

Theorem 3.1. *Let R be a commutative ring with $diam(\Gamma(R)) = 3$. Then $T(R \oplus R)$ is not a submodule of the R-module $R \oplus R$.*

Proof. Since $\operatorname{diam}(\Gamma(R)) = 3$, there exist $r_1, r_2 \in Z(R)^*$ such that $d(r_1, r_2) = 3$. Let $r_1 — s — t — r_2$ be a path in $\Gamma(R)$. The elements $m_1 = (r_1, 0)$ and $m_2 = (0, r_2)$ belong to $T(R \oplus R)$ since $sm_1 = tm_2 = 0$, while $m_1 + m_2 = (r_1, r_2) \notin T(M)$. Namely, if $(r_1, r_2) \in T(M)$, there exists $r \in Z(R)^*$ such that $rr_1 = rr_2 = 0$; so we get the contradiction $d(r_1, r_2) \leq 2$. ∎

Let M be an R-module. In the previous section, we have seen that the case when $Z(R)$ is an ideal of R may, but it need not imply that $T(M)$ is a submodule of M. The same holds if $Z(R)$ is not an ideal of R. For example, if we consider $M = \mathbb{Z}_6$ as the $\mathbb{Z}_6$-module, then clearly $T(M)(= Z(R))$ is not a submodule of M ($Z(R)$ is not an ideal of R). On the other hand, if we consider $M = \mathbb{Z}_6$ as a module over $R = \mathbb{Z}_6 \times \mathbb{Z}_6$, with the module operation given by $(a, b) \cdot m = am$, then $Z(R)$ is not an ideal, but $T(M)$ is a submodule since $Ann(M) = \{(0, r) \mid r \in \mathbb{Z}_6\}$. In [2], it has been proved that in the case when $Z(R)$ is not an ideal of R, $\mathrm{T}\Gamma(R)$ is connected if and only if $(Z(R)) = R$. If the ring R is additively generated by its zero-divisors, connectedness of the graph $\mathrm{T}\Gamma(R)$ has the essential role in the connectedness of the graph $\mathrm{T}\Gamma(M)$. We therefore discuss in this section the case when the torsion elements do not form a submodule of the R-module M nor do the zero-divisors form an ideal of R.

Lemma 3.2. *Suppose that M is an R-module. If the identity of the ring R is a sum of n zero-divisors, then every element of the module M is the sum of at most n torsion elements.*

Proof. Evidently, if $a \in Z(R)$ and $x \in M$, then $ax \in T(M)$; so for all $m \in M$:

$$1 = z_1 + \cdots + z_n \;\Rightarrow\; m = z_1 m + \cdots + z_n m\,. \quad ∎$$

Lemma 3.2 may be formulated in a slightly more general form: if R is generated (additively) by zero-divisors, then every R-module is generated by its torsion elements.

Theorem 3.3. *Let M be an R-module such that $T(M)$ is not a submodule. Then $\mathrm{T}\Gamma(M)$ is connected if and only if M is generated by its torsion elements.*

Proof. Let us first prove that the connectedness of the graph $\mathrm{T}\Gamma(M)$ implies that the module M is generated by its torsion elements. Suppose that this is not true. Then there exists $m \in M$ which does not have a representation of the form $m = m_1 + \cdots + m_n$, where $m_i \in T(M)$. Naturally, $m \neq 0$ since $0 \in T(M)$. We prove that there does not exist a path from 0 to m in $\mathrm{T}\Gamma(M)$. If $0 — s_1 — s_2 — \cdots — s_k — m$ is a path in $\mathrm{T}\Gamma(M)$, then s_1, $s_1 + s_2, \ldots, s_{k-1} + s_k$, $s_k + m$ are torsion elements and m may be represented as: $m = (s_k + m) - (s_{k-1} + s_k) + \cdots + (-1)^{k-1}(s_1 + s_2) + (-1)^k s_1$. This contradicts the assumption that m is not a sum of torsion elements. The reverse implication may be proved in a similar way as in [2, Theorem 3.3]. ∎

Theorem 3.4. *Let R be a commutative ring and M an R-module. If $\mathrm{T}\Gamma(R)$ is connected, then $\mathrm{T}\Gamma(M)$ is connected as well.*

Proof. Suppose that $\mathrm{T}\Gamma(R)$ is connected and let $m \in M$. Then there exists a path from from 0 to m in $\mathrm{T}\Gamma(M)$. Let

$$(A): \quad 0 - r_1 - r_2 - \cdots - r_k - 1$$

be a path from 0 to 1 in $\mathrm{T}\Gamma(R)$. Then $r_1,\ r_1 + r_2,\ \ldots,\ r_k + 1 \in Z(R)$. Since for $r \in Z(R)$ and $m \in M$, one has: $rm \in T(M)$. "Multiplying" the path (A) by m we obtain that

$$(B): \quad 0 - r_1 m - r_2 m - \cdots - r_k m - m$$

is a path from 0 to m in $\mathrm{T}\Gamma(M)$. Since all vertices may be connected via 0, $\mathrm{T}\Gamma(M)$ is connected. ∎

Remark. Let us observe that, according to the previous proof, the following property holds: if $d(0,1) = n$ in $\mathrm{T}\Gamma(R)$, then $d(0,m) \leq n$ in $\mathrm{T}\Gamma(M)$ for every $m \in M$.

Theorem 3.5. *If every element of a module M is a sum of at most n torsion elements, then $diam(T\Gamma(M)) \leq n$. If n is the smallest such number, then $diam(T\Gamma(M)) = n$.*

Proof. We first prove that, under the given conditions, the distance of any element m from 0 is at most n. Suppose that $m = m_1 + \cdots + m_n$, where $m_i \in T(M)$. Then, for $a_i = (-1)^{i+n}(m_1 + \cdots + m_i)$,

$$0 - a_1 - a_2 - \cdots - a_n$$

is a path from 0 to m of length n in $\mathrm{T}\Gamma(M)$. Let x and y be arbitrary elements of a module M. We prove that $d(x,y) \leq n$. In the proof, we use the path (A) from 0 to $x - y$ and the path (B) from 0 to $x + y$.

$$\begin{array}{ll} (A) & (x-y) - s_1 - s_2 - \cdots - s_{n-1} - 0 \\ (B) & (x+y) - t_1 - t_2 - \cdots - t_{n-1} - 0 \end{array}$$

From the previous discussion, the lengths of both paths are at most n. Depending on the fact whether n is even or odd, we get the paths (A') or (B') from x to y of length n.

$$\begin{array}{ll} (A') & x - (s_1 - y) - (s_2 + y) - \cdots - (s_{n-1} - y) - y \\ (B') & x - (t_1 + y) - (t_2 - y) - \cdots - (t_{n-1} - y) - y \end{array}$$

Suppose now that n is the smallest such number, and let $m = m_1 + \cdots + m_n$ be a shortest representation of the element m as a sum of torsion elements. We prove that $d(0,m) = n$. From the previous results we have $d(0,m) \leq n$. Suppose that $d(0,m) = k < n$ and let $0 - s_1 - s_2 - \cdots - s_{k-1} - m$ be a path in $\mathrm{T}\Gamma(M)$. By using the same argument as in the proof of Theorem 3.3, we arrive at a contradiction—a presentation of the element m as a sum of $k < n$ torsion elements. This concludes the proof of the theorem. ∎

Corollary 3.6. *Let R be a commutative ring such that $Z(R)$ is not an ideal of R and $(Z(R)) = R$. Let M be an R-module. If diam $(T\Gamma(R)) = n$, then diam $(T\Gamma(M)) \leq n$. In particular, if R is finite, then diam $(T\Gamma(M)) \leq 2$.*

Proof. It follows directly from Lemma 3.2 and Theorem 3.5. In particular, if R is a finite commutative ring such that $Z(R)$ is not an ideal of R, then diam $(\mathrm{T}\Gamma(R)) = 2$ [2, Theorem 3.4]. ■

For every $n \geq 2$, it is possible to construct a commutative ring R_n such that $Z(R_n)$ is not an ideal of R_n and diam $(\mathrm{T}\Gamma(R_n)) = n$ [2, Example 3.8]. This of course means that $\mathrm{T}\Gamma(M)$ may have diameter n for every n, since we may look at R_n as an R_n-module.

If the total graph $\mathrm{T}\Gamma(M)$ is connected and $(Z(R)) = R$, the diameter of $\mathrm{T}\Gamma(M)$ need not depend on the number of generators from $Z(R)$.

Example 3.7. Let $z_1, z_2 \in Z(R)$ be such that $z_1 + z_2 = 1$. Let $M = R/z_1R \oplus R/z_2R$.

If $z_1z_2 \neq 0$, then $z_1z_2 \in Ann(M)$, and it follows that M is a torsion module; so diam $(\mathrm{T}\Gamma(M)) = 1$.

If $z_1z_2 = 0$, then by multiplying equality $z_1 + z_2 = 1$ by z_1 one gets $z_1^2 = z_1$; so z_1 is an idempotent. Let $m_1 = (1 + z_1R, 0)$ and $m_2 = (0, 1 + z_2R)$ be elements from $T(M)$. We prove that $m_1 + m_2 \notin T(M)$. If $r(1 + z_1R, 1 + z_2R) = 0$, then $r \in z_1R \cap z_2R$, i.e., $r = z_1a = z_2b$. Multiplying the last equality by z_1, one gets $z_1^2a = 0$. Thus, since z_1 is an idempotent, $r = z_1a = 0$. Therefore, $T(M)$ is not a submodule of M, and from Theorem 3.5 we conclude that diam $(\mathrm{T}\Gamma(M)) = 2$.

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