МАТЕМАТИЧКИ ВЕСНИК
63, 4 (2011), 285–294
December 2011

оригинални научни рад
research paper

A COMMON FIXED POINT THEOREM FOR WEAKLY COMPATIBLE MAPPINGS IN NON-ARCHIMEDEAN MENGER PM-SPACES

Amit Singh, R.C. Dimri and Sandeep Bhatt

Abstract. In the present paper we prove a unique common fixed point theorem for four weakly compatible self maps in non-Archimedean Menger PM-spaces without using the notion of continuity. Our result generalizes and extends the results of Khan and Sumitra [M.A. Khan, Sumitra, A common fixed point theorem in non-Archimedean Menger PM-space, Novi Sad J. Math. 39 (1) (2009), 81–87] and others.

1. Introduction

Non-Archimedean probabilistic metric spaces and some topological preliminaries on them were first studied by Istrăţescu and Crivăt [9] (see also [8]). Some fixed point theorems for mappings on non-Archimedean Menger spaces have been proved by Istrăţescu [6, 7] as a result of the generalizations of some of the results of Sehgal and Bharucha-Reid [16] and Sherwood [17]. Achari [1] studied the fixed points of quasi-contraction type mappings in non-Archimedean PM-spaces and generalized the results of Istrăţescu [7]. Recently Khan and Sumitra [13] proved a common fixed point theorem for three pointwise R-weakly commuting mappings in complete non-Archimedean Menger PM-spaces. In the present paper we prove a unique common fixed point theorem for four weakly compatible self maps in non-Archimedean Menger PM-spaces without using the notion of continuity. Our result generalizes and extends the results of Khan and Sumitra [13] and others.

2. Preliminaries

Definition 2.1. [7, 9] Let X be any non-empty set and D be the set of all left continuous distribution functions. An ordered pair (X, F) is said to be non-Archimedean probabilistic metric space (N.A. PM-space) if F is a mapping from $X \times X$ into D satisfying the following conditions, where the value of F at $(x, y) \in X \times X$ is represented by $F_{x,y}$ or $F(x, y)$ for all $x, y \in X$ such that

2010 AMS Subject Classification: 47H10, 54H25.

Keywords and phrases: Non-Archimedean Menger PM-spaces; weakly compatible maps; common fixed point.

(i) $F(x,y;t)=1$ for all $t>0$ if only if $x=y$;
(ii) $F(x,y;t)=F(y,x;t)$;
(iii) $F(x,y;0)=0$;
(iv) If $F(x,y;t_1)=F(y,z;t_2)=1$ then $F(x,z;\max\{t_1,t_2\})=1$ for all $x,y,z\in X$.

Definition 2.2. [14] A t-norm is a function $\Delta:[0,1]\times[0,1]\to[0,1]$ which is associative, commutative, non-decreasing in each coordinate and $\Delta(a,1)=a$ for all $a\in[0,1]$.

Definition 2.3. [8, 10] A non-Archimedean Menger PM-space is an ordered triplet (X,F,Δ), where Δ is a t-norm and (X,F) is an N.A. PM-space satisfying the following condition:

$$F(x,z;\max\{t_1,t_2\})\geq\Delta(F(x,y;t_1),F(y,z;t_2))\text{ for all }x,y,z\in X,t_1,t_2\geq 0.$$

For details of topological preliminaries on non-Archimedean Menger PM-spaces, we refer to Cho, Ha and Chang [3].

Definition 2.4. [2, 3] An N.A. Menger PM-space (X,F,Δ) is said to be of type $(C)_g$ if there exists a $g\in\Omega$ such that $g(F(x,z;t))\leq g(F(x,y;t))+g(F(y,z;t))$ for all $x,y,z\in X,t\geq 0$, where $\Omega=\{g\mid g:[0,1]\to[0,\infty)$ is continuous, strictly decreasing with $g(1)=0$ and $g(0)<\infty\}$.

Definition 2.5. [2, 3] An N.A. Menger PM-space (X,F,Δ) is said to be of type $(D)_g$ if there exists a $g\in\Omega$ such that $g(\Delta(t_1,t_2))\leq g(t_1)+g(t_2)$ for all $t_1,t_2\in[0,1]$.

Remark 2.1. [2, 3] (i) If N.A. Menger PM-space is of type $(D)_g$ then (X,F,Δ) is of type $(C)_g$.

(ii) If (X,F,Δ) is N.A. Menger PM-space and $\Delta\geq\Delta(r,s)=\max(r+s-1,1)$, then (X,F,Δ) is of type $(D)_g$ for $g\in\Omega$ and $g(t)=1-t$.

Throughout this paper (X,F,Δ) is a complete N.A. Menger PM-space with a continuous strictly increasing t-norm Δ.

Let $\phi:[0,\infty)\to[0,\infty)$ be a function satisfying the condition

$$\phi\text{ is upper semi-continuous from the right and }\phi(t)<t\text{ for }t>0. \qquad (\Phi)$$

Definition 2.6. [2, 3] A sequence $\{x_n\}$ in the N.A. Menger PM-space (X,F,Δ) converges to x if and only if for each $\epsilon>0$, $\lambda>0$ there exists $M(\epsilon,\lambda)$ such that $g(F(x_n,x;\epsilon))<g(1-\lambda)$ for all $n>M$.

Definition 2.7. [3] A sequence $\{x_n\}$ in the N.A. Menger PM-space is a Cauchy sequence if and only if for each $\epsilon>0$, $\lambda>0$ there exists $M(\epsilon,\lambda)$ such that $g(F(x_n,x_{n+p};\epsilon))<g(1-\lambda)$ for all $n>M$ and $p\geq 1$.

Example 2.1. [3] Let X be any set with at least two elements. If we define $F(x,x;t)=1$ for all $x\in X$, $t>0$ and $F(x,y;t)=\{0$ if $t\leq 1$ and 1 if $t>1\}$, where $x, y \in X$, $x \neq y$, then (X, F, Δ) is the N.A. Menger PM-space with $\Delta(a, b) = \min(a, b)$ or $(a.b)$.

EXAMPLE 2.2. [3] Let $X = R$ be the set of real numbers equipped with metric defined as $d(x, y) = |x - y|$. Set $F(x, y; t) = \frac{t}{t+d(x,y)}$. Then (X, F, Δ) is an N.A. Menger PM-space with Δ as continuous t-norm satisfying $\Delta(r, s) = \min(r, s)$ or $(r.s)$.

LEMMA 2.1. [3] *If a function $\phi : [0, \infty) \to [0, \infty)$ satisfies the condition (Φ), then we get*

(i) for all $t \geq 0$, $\lim_{n\to\infty} \phi^n(t) = 0$, where $\phi^n(t)$ is the n-th iteration of $\phi(t)$,

(ii) if $\{t_n\}$ is a non-decreasing sequence of real numbers and $t_{n+1} \leq \phi(t_n)$, $n = 1, 2, \dots$, then $\lim_{n\to\infty} t_n = 0$. In particular, if $t \leq \phi(t)$, for each $t \geq 0$, then $t = 0$.

LEMMA 2.2. [3] *Let $\{y_n\}$ be a sequence in X such that $\lim_{n\to\infty} F(y_n, y_{n+1}; t) = 1$ for each $t > 0$. If $\{y_n\}$ is not a Cauchy sequence in X, then there exist $\epsilon_0 > 0$, $t_0 > 0$, and two sequences $\{m_i\}$ and $\{n_i\}$ of positive integers such that*

(i) $m_i > n_i + 1$ and $n_i \to \infty$ as $i \to \infty$.

(ii) $F(y_{m_i}, y_{n_i}; t_0) < 1 - \epsilon_0$ and $F(y_{m_i - 1}, y_{n_i}; t_0) \geq 1 - \epsilon_0$, $i = 1, 2, \dots$

DEFINITION 2.8. [10] Let $A, S : X \to X$ be mappings. A and S are said to be compatible if $\lim_{n\to\infty} g(F(ASx_n, SAx_n; t)) = 0$ for all $t > 0$, when $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = z = \lim_{n\to\infty} Sx_n$ for some $z \in X$.

DEFINITION 2.9. [11, 12] Let $A, S : X \to X$ be mappings. A and S are said to be weakly compatible if they commute at coincidence points. That is, if $Ax = Sx$ implies that $ASx = SAx$, for x in X.

3. Main results

THEOREM 3.1. *Let (X, F, Δ) be a complete N.A. Menger PM-space and $A, B, S, T \colon X \to X$ be mappings satisfying*

(i) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$,

(ii) the pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible and

(iii) $g(F(Ax, By; t)) \leq \phi[\max\{g(F(Sx, Ty; t)), g(F(Sx, Ax; t)), g(F(Ty, By; t)), \frac{1}{2}(g(F(Sx, By; t)) + g(F(Ty, Ax; t)))\}]$,

for every $x, y \in X$, where ϕ satisfies the condition (Φ). Then A, B, S and T have a unique common fixed point in X.

Proof. Since $A(X) \subseteq T(X)$, for any $x_0 \in X$, there exists a point $x_1 \in X$ such that $Ax_0 = Tx_1$. Since $B(X) \subseteq S(X)$, for this x_1, we can choose a point $x_2 \in X$ such that $Bx_1 = Sx_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that

$$y_{2n} = Ax_{2n} = Tx_{2n+1}, \quad y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \text{ for } n = 1, 2, \dots \tag{1}$$

Let $M_n = g(F(Ax_n, Bx_{n+1}; t)) = g(F(y_n, y_{n+1}; t))$ for $n = 1, 2, \ldots$. Then

$$\begin{aligned} M_{2n} &= g(F(Ax_{2n}, Bx_{2n+1}; t)) \\ &\leq \phi[\max\{g(F(Sx_{2n}, Tx_{2n+1}; t)), g(F(Sx_{2n}, Ax_{2n}; t)), g(F(Tx_{2n+1}, Bx_{2n+1}; t)), \\ &\qquad \tfrac{1}{2}(g(F(Sx_{2n}, Bx_{2n+1}; t)) + g(F(Tx_{2n+1}, Ax_{2n}; t))\}] \\ &\leq \phi[\max\{g(F(y_{2n-1}, y_{2n}; t)), g(F(y_{2n-1}, y_{2n}; t)), g(F(y_{2n}, y_{2n+1}; t)), \\ &\qquad \tfrac{1}{2}(g(F(y_{2n-1}, y_{2n+1}; t)) + g(F(y_{2n}, y_{2n}; t)))\}] \\ &\leq \phi[\max\{g(F(y_{2n-1}, y_{2n}; t)), g(F(y_{2n-1}, y_{2n}; t)), g(F(y_{2n}, y_{2n+1}; t)), \\ &\qquad \tfrac{1}{2}(g(F(y_{2n-1}, y_{2n}; t)) + g(F(y_{2n}, y_{2n+1}; t)))\}] \end{aligned}$$

i.e.

$$M_{2n} \leq \phi[\max\{M_{2n-1}, M_{2n-1}, M_{2n}, \tfrac{1}{2}(M_{2n-1} + M_{2n})\}] \tag{2}$$

If $M_{2n} > M_{2n-1}$ then by (2) $M_{2n} \geq \phi(M_{2n})$, a contradiction. If $M_{2n-1} > M_{2n}$ then by (2) $M_{2n} \leq \phi(M_{2n-1})$. So by Lemma 2.1, we have $\lim_{n\to\infty} M_{2n} = 0$, i.e.,

$$\lim_n g(F(Ax_{2n}, Bx_{2n+1}; t)) = 0 \ \text{ i.e. } \ \lim_n g(F(y_{2n}, y_{2n+1}; t)) = 0.$$

Similarly, we can show that

$$\lim_n g(F(Bx_{2n+1}, Ax_{2n+2}; t)) = 0 \ \text{ i.e. } \ \lim_n g(F(y_{2n+1}, y_{2n+2}; t)) = 0.$$

Thus we have $\lim_n g(F(Ax_n, Bx_{n+1}; t)) = 0$ for all $t > 0$, i.e.

$$\lim_n g(F(y_n, y_{n+1}; t)) = 0 \quad \text{for all } \ t > 0. \tag{3}$$

Before proceeding with the proof of the theorem, we first prove the following claim:

Claim. *Let A, B, S and $T\colon X \to X$ be maps satisfying (i), (ii) and (iii) and $\{y_n\}$ be defined by* (1) *such that*

$$\lim_n g(F(y_n, y_{n+1}; t)) = 0 \tag{4}$$

for all n. Then $\{y_n\}$ is a Cauchy sequence.

Proof of Claim. Since $g \in \Omega$, it follows that $\lim_{n\to\infty} F(y_n, y_{n+1}; t) = 1$ for each $t > 0$ if and only if $\lim_{n\to\infty} g(F(y_n, y_{n+1}; t)) = 1$ for each $t > 0$.

By Lemma 2.2, if $\{y_n\}$ is not a Cauchy sequence in X, there exists $\epsilon_0 > 0$, $t_0 > 0$ and two sequences $\{m_i\}$ and $\{n_i\}$ of positive integers such that

(A) $m_i > n_i + 1$ and $n_i \to \infty$ as $i \to \infty$;

(B) $g(F(y_{m_i}, y_{n_i}; t_0)) > g(1 - \epsilon_0)$ and $g(F(y_{m_i - 1}, y_{n_i}; t_0)) \leq g(1 - \epsilon_0)$, $i = 1, 2, \ldots$.

Since $g(t) = 1 - t$, we have

$$\begin{aligned} g(1 - \epsilon_0) &< g(F(y_{m_i}, y_{n_i}; t_0)) \\ &\leq g(F(y_{m_i}, y_{m_i - 1}; t_0)) + g(F(y_{m_i - 1}, y_{n_i}; t_0)) \\ &\leq g(F(y_{m_i}, y_{m_i - 1}; t_0)) + g(1 - \epsilon_0). \end{aligned} \tag{5}$$

As i $\to \infty$ in (5) we have

$$\lim_{n\to\infty} g(F(y_{m_i}, y_{n_i}; t_0)) = g(1-\epsilon_0). \tag{6}$$

On the other hand, we have

$$\begin{aligned} g(1-\epsilon_0) &< g(F(y_{m_i}, y_{n_i}; t_0)) \\ &\leq g(F(y_{n_i}, y_{n_i+1}; t_0)) + g(F(y_{m_i}, y_{n_i+1}; t_0)) \end{aligned} \tag{7}$$

Now consider $g(F(y_{m_i}, y_{n_i+1}; t_0))$ in (7) and assume that both m_i and n_i are even. Then, by (iii), we have

$$\begin{aligned} &g(F(y_{m_i}, y_{n_i+1}; t_0)) = g(F(Ax_{m_i}, Bx_{n_i+1}; t_0)) \\ &\quad \leq \phi[\max\{g(F(Sx_{m_i}, Tx_{n_i+1}; t_0)), g(F(Sx_{m_i}, Ax_{m_i}; t_0)), g(F(Tx_{n_i+1}, Bx_{n_i+1}; t_0)), \\ &\qquad \tfrac{1}{2}(g(F(Sx_{m_i}, Bx_{n_i+1}; t_0)) + g(F(Tx_{n_i+1}, Ax_{m_i}; t_0)))\}] \\ &\quad \leq \phi[\max\{g(F(y_{m_i-1}, y_{n_i}; t_0)), g(F(y_{m_i-1}, y_{m_i}; t_0)), g(F(y_{n_i}, y_{n_i+1}; t_0)), \\ &\qquad \tfrac{1}{2}(g(F(y_{m_i-1}, y_{n_i+1}; t_0)) + g(F(y_{n_i}, y_{m_i}; t_0)))\}] \end{aligned}$$

Letting $i \to \infty$ in above equation, we have

$$g(1-\epsilon_0) \leq \phi[\max\{g(1-\epsilon_0), 0, 0, g(1-\epsilon_0)\}],$$

i.e. $g(1-\epsilon_0) \leq \phi(g(1-\epsilon_0))$, which is a contradiction. Hence the sequence $\{y_n\}$ defined by (1) is a Cauchy sequence, which concludes the proof of the claim.

Since X is complete, then the sequence $\{y_n\}$ converges to a point z in X and so the subsequences $\lim_{n\to\infty} Ax_{2n}$, $\lim_{n\to\infty} Bx_{2n+1}$, $\lim_{n\to\infty} Sx_{2n}$ and $\lim_{n\to\infty} Tx_{2n+1}$ of $\{y_n\}$ also converge to the limit z.

Since $B(X) \subseteq S(X)$, there exists a point $u \in X$ such that $z = Su$. Then, using (iii), we have

$$\begin{aligned} g(F(Au, z; t)) &\leq g(F(Au, Bx_{2n-1})) + g(F(Bx_{2n-1}, z)) \\ &\leq \phi[\max\{g(F(Su, Tx_{2n-1}; t)), g(F(Su, Au; t)), g(F(Tx_{2n-1}, Bx_{2n-1}; t)), \\ &\quad \tfrac{1}{2}(g(F(Su, Bx_{2n-1})) + g(F(Tx_{2n-1}, Au)))\}] \end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned} g(F(Au, z; t)) &\leq \phi[\max\{g(z, z; t)), g(F(z, Au; t)), g(F(z, z; t)), \\ &\qquad \tfrac{1}{2}(g(F(z, z; t)) + g(F(z, Au; t)))\}] \\ &= \phi[\max\{0, g(F(z, Au; t)), 0, \tfrac{1}{2}(0 + g(F(z, Au; t)))\}] \\ &\leq \phi(g(F(Au, z; t))) \end{aligned}$$

for all $t > 0$, which implies that $g(F(Au, z; t)) = 0$ for all $t > 0$ by Lemma 2.1. Therefore $Au = Su = z$. Since $A(X) \subseteq T(X)$, there exists a point v in X such that $z = Tv$. Again using (iii), we have

$$g(F(z, Bv; t)) = g(F(Au, Bv; t))$$

$$\begin{aligned}
&\le \phi[\max\{g(Su,Tv;t)), g(F(Su,Au;t)), g(F(Tv,Bv;t)),\\
&\quad \tfrac{1}{2}(g(F(Su,Bv;t)) + g(F(Tu,Au;t)))\}]\\
&\le \phi[\max\{g(z,z;t)), g(F(z,z;t)), g(F(z,Bv;t)),\\
&\quad \tfrac{1}{2}(g(F(z,Bv;t)) + g(F(z,z;t)))\}]\\
&= \phi[\max\{0, 0, g(F(z,Bv;t)), \tfrac{1}{2}(g(F(z,Bv;t)) + 0)\}]\\
&\le \phi(g(F(Bv,z;t))) \text{ for all } t > 0,
\end{aligned}$$

which implies that $g(F(Bv,z;t)) = 0$ for all $t > 0$ by Lemma 2.1. Therefore $Bv = Tv = z$. Since A and S are weakly compatible mappings, $ASz = SAz$ i.e. $Az = Sz$. Now we show that z is a fixed point of A. If Az $\neq$ z, then by (iii), we have

$$\begin{aligned}
g(F(Az,z;t)) = g(F(Az,Bv;t)) &\le \phi[\max\{g(F(Sz,Tv;t)), g(F(Sz,Az;t)),\\
&\quad g(F(Tv,Bv;t)), \tfrac{1}{2}(g(F(Sz,Bv)) + g(F(Tv,Az)))\}]\\
&\le \phi[\max\{g(F(Az,z;t)), 0, 0, \tfrac{1}{2}(g(F(Az,z)) + g(F(z,Az)))\}]\\
&\le \phi(g(F(Az,z;t))) \text{ for all } t > 0,
\end{aligned}$$

which implies that $g(F(Az,z;t)) = 0$ for all $t > 0$ by Lemma 2.1. Therefore $Az = z$. Hence $Az = Sz = z$.

Similarly, as B and T are weakly compatible mappings, we have $Bz = Tz = z$, since by (iii), we have

$$\begin{aligned}
g(F(z,Bz;t)) = g(F(Az,Bz;t)) &\le \phi[\max\{g(F(Sz,Tz;t)), g(F(Sz,Az;t)),\\
&\quad g(F(Tz,Bz;t)), \tfrac{1}{2}(g(F(Sz,Bz)) + g(F(Tz,Az)))\}]\\
&\le \phi[\max\{g(F(z,Bz;t)), 0, 0, \tfrac{1}{2}(g(F(z,Bz)) + g(F(Bz,z)))\}]\\
&\le \phi(g(F(Bz,z;t))) \text{ for all } t > 0,
\end{aligned}$$

which implies that $g(F(Bz,z;t)) = 0$ for all $t > 0$ by Lemma 2.1. Therefore $Bz = z$. Hence $Bz = Tz = z$.

Thus $Az = Bz = Sz = Tz = z$, that is, z is a common fixed point of A, B, S and T.

Finally, in order to prove the uniqueness of z, suppose that w is another common fixed point of A, B, S and T. Then by (iii), we have

$$\begin{aligned}
g(F(z,w;t)) = g(F(Az,Bw;t)) &\le \phi[\max\{g(F(Sz,Tw;t)), g(F(Sz,Az;t)),\\
&\quad g(F(Tw,Bw;t)), \tfrac{1}{2}(g(F(Sz,Bw;t)) + g(Tw,Az;t)))\}]\\
&\le \phi(g(F(z,w;t))) \text{ for all } t > 0,
\end{aligned}$$

which implies that $g(F(z,w;t)) = 0$ for all $t > 0$ by Lemma 2.1. Hence $z = w$. Therefore z is a unique common fixed point of A, B, S and T. ∎

COROLLARY 3.1. *Let $A, S, T : X \to X$ be the mappings satisfying*

(i) $A(X) \subseteq S(X) \cap T(X)$,

(ii) *the pairs* $\{A, S\}$ *and* $\{A, T\}$ *are weakly compatible and*

(iii) $g(F(Ax, Ay; t)) \leq \phi[\max\{g(F(Sx, Ty; t)), g(F(Sx, Ax; t)), g(F(Ty, Ay; t))$
$\frac{1}{2}(g(F(Sx, Ay; t)) + g(F(Ty, Ax; t)))\}]$,

for every $x, y \in X$, where ϕ satisfies the condition (Φ). Then A, S and T have a unique common fixed point in X.

COROLLARY 3.2. *Let $A, S\colon X \to X$ be the mappings satisfying*

(i) $A(X) \subseteq S(X)$,

(ii) *the pair* $\{A, S\}$ *is weakly compatible and*

(iii) $g(F(Ax, Ay; t)) \leq \phi[\max\{g(F(Sx, Sy; t)), g(F(Sx, Ax; t)), g(F(Sy, Ay; t))$
$\frac{1}{2}(g(F(Sx, Ay; t)) + g(F(Sy, Ax; t)))\}]$,

for every $x, y \in X$, where ϕ satisfies the condition (Φ). Then A and S have a unique common fixed point in X.

We can also derive the following results from Theroem 3.1.

COROLLARY 3.3. *Let S and T be two continuous self-maps of a complete N.A. Menger PM-space (X, F, Δ). Let A be a self-map satisfying*

(i) $\{A, S\}$ *and* $\{A, T\}$ *are pointwise R-weakly commuting and* $A(X) \subseteq S(X) \cap T(X)$,

(ii) $g(F(Ax, Ay; t)) \leq \phi[\max\{g(F(Sx, Ty; t)), g(F(Sx, Ax; t)), g(F(Sx, Ay; t)),$
$g(F(Ty, Ay; t))\}]$,

for every $x, y \in X$, where ϕ satisfies the condition (Φ). Then A, S and T have a unique common fixed point in X.

Taking $T = S$ in Corollary 3.3 we get the following corollary unifying Vasuki's theorem [20], which in turn also generalizes the result of Pant [15].

COROLLARY 3.4. *Let (X, F, Δ) be a complete N.A. Menger PM-space and S be a continuous self-mapping of X. Let A be another self-mapping of X satisfying that*

(i) $\{A, S\}$ *is R-weakly commuting with* $A(X) \subseteq S(X)$,

(ii) $g(F(Ax, Ay, a; t)) \leq \phi[\max\{g(F(Sx, Sy; t)), g(F(Sx, Ax; t)), g(F(Sx, Ay; t)),$
$g(F(Sy, Ay; t))\}]$,

for each $x, y \in X$ and ϕ satisfies the condition (Φ). Then A and S have a unique common fixed point.

REMARK 3.1. In Theorem 3.1, if S and T are continuous and pairs $\{A, S\}$ and $\{B, T\}$ are compatible instead of condition (ii), the theorem remains true.

REMARK 3.2. In our generalization the inequality condition (iii) satisfied by the mappings A, B, S and T is stronger than that of Theorem 2 of Khan and Sumitra [13] and Theorem 1.9 of Vasuki [20].

Example 3.1. Let $X = R$ and $A, S, T \colon X \to X$ be mappings such that $S(x) = 2x - 1$,

$$T(x) = \begin{cases} -1 - x, & x < 0 \\ 2x - 1, & 0 \le x < 1 \\ \frac{x+1}{2}, & x \ge 1 \end{cases} \quad \text{and} \quad A(x) = \begin{cases} 0, & x = -1 \\ x^2, & x \ne -1 \end{cases}$$

Then we see that

(i) $\{A, S\}$ and $\{A, T\}$ are point-wise R-weakly commuting.

(ii) $A(X) \subseteq S(X) \cap T(X)$.

(iii) 1 is the unique common fixed point of A, S and T.

(iv) $g(F(Ax, Ay; t)) \le \phi[\max\{g(F(Sx, Ty; t)), g(F(Sx, Ax; t)), g(F(Sx, Ay; t)), g(F(Ty, Ay; t))\}]$, for every $x, y \in X$ is also true.

4. An application

Theorem 4.1. *Let (X, F, Δ) be a complete N. A. Menger PM-space and A, B, S and T be mappings from the product $X \times X$ to X such that*

$$\begin{gathered} A(X \times \{y\}) \subseteq T(X \times \{y\}), \qquad B(X \times \{y\}) \subseteq S(X \times \{y\}), \\ g(F(A(T(x, y), y), T(A(x, y), y); t)) \le g(F(A(x, y), T(x, y); t)), \\ g(F(B(S(x, y), y), S(B(x, y), y); t)) \le g(F(B(x, y), S(x, y); t)), \end{gathered} \tag{8}$$

for all $t > 0$. If S and T are continuous with respect to their direct argument and

$$\begin{aligned} g(F(A(x, y), B(x', y'); t)) \le \phi[\max\{&g(F(S(x, y), T(x', y'); t)), \\ &g(F(S(x, y), A(x, y); t)), g(F(T(x', y'), B(x', y'); t)), \\ &\tfrac{1}{2}(g(F(S(x, y), B(x', y'); t)) + g(F(T(x', y'), A(x, y); t)))\}] \end{aligned} \tag{9}$$

for all $t > 0$ and x, y, x', y' in X, then there exists only one point b in X such that

$$A(b, y) = S(b, y) = B(b, y) = T(b, y) \quad \forall y \in X.$$

Proof. By (8) and (9),

$$\begin{aligned} g(F(A(x, y), B(x', y'); t)) \le \phi[\max\{&g(F(S(x, y), T(x', y'); t)), \\ &g(F(S(x, y), A(x, y); t)), g(F(T(x', y'), B(x', y'); t)), \\ &\tfrac{1}{2}(g(F(S(x, y), B(x', y'); t)) + g(F(T(x', y'), A(x, y); t)))\}] \end{aligned}$$

for all $t > 0$, therefore by Theorem 3.1, for each y in X, there exists only one $x(y)$ in X such that

$$A(x(y), y) = S(x(y), y) = B(x(y), y) = T(x(y), y) = x(y),$$

for every y, y' in X and

$$\begin{aligned} g(F(x(y),x(y');t)) &= g(F(A(x(y),y),A(x(y'),y');t)) \\ &\le \phi[\max\{g(F(A(x,y),A(x',y');t)), g(F(A(x,y),A(x,y);t)), \\ &\quad g(F(T(x',y'),A(x',y');t)), \\ &\quad \tfrac{1}{2}(g(F(A(x,y),A(x',y');t)) + g(F(A(x',y'),A(x,y);t)))\}] \\ &= g(F(x(y),x(y');t)). \end{aligned}$$

This implies that $x(y) = x(y')$ and hence $x(\cdot)$ is some constant $b \in X$ so that

$$A(b,y) = b = T(b,y) = S(b,y) = B(b,y) \quad \forall y \in X. \quad \blacksquare$$

REFERENCES

[**1**] J. Achari, *Fixed point theorems for a class of mappings on non-Archimedean probablistic metric spaces*, Mathematica **25** (1983), 5–9.

[**2**] S.S. Chang, *Fixed point theorems for single-valued and multivalued mappings in non-archimedean menger probabilistic metric spaces*, Math. Japon. **35** (5) (1990), 875–885.

[**3**] Y.J. Cho, K.S. Ha, S.S. Chang, it Common fixed point theorems for compatible mappings of type(A) in non-Archimedean Menger PM-spaces, Math. Japon. **46** (1) (1997), 169–179.

[**4**] Y.J. Cho, H.K. Pathak, S.M. Kang, *Remarks on R-weakly commuting maps and common fixed point theorems*, Bull. Korean Math. Soc. **34** (1997), 247–257.

[**5**] R.C. Dimri, B.D. Pant, *Fixed point theorems in non-Archimedean Menger spaces*, Kyungpook Math. J. **31** (1) (1991), 89–95.

[**6**] I. Istrătescu, *On some fixed point theorems with applications to the non-Archimedean Menger spaces*, Attidella Acad. Naz. Lincei **58** (1975), 374–379.

[**7**] I. Istrătescu, F̂ixed point theorems for some classes of contraction mappings on non-Archimedean probablistic spaces, Publ. Math. (Debrecen) **25** (1978), 29–34.

[**8**] I. Istrătescu, Gh. Babescu, *On the completion on non-Archimedean probabilistic metric spaces*, Seminar de spatii metrice probabiliste, Universitatea Timisoara, **17**, 1979.

[**9**] I. Istrătescu, N. Crivat, *On some classes of non-Archimedean probabilistic metric spaces*, Seminar de spatii metrice probabiliste, Universitatea Timisoara, **12**, 1974.

[**10**] G. Jungck, *Compatible mappings and common fixed points*, Internat. J. Math. Math. Sci. **9** (1986), 771–779.

[**11**] G. Jungck *Common fixed points for noncontinuous nonself maps on non-metric spaces*, Far East J. Math. Sci. **4** (1996), 199–215.

[**12**] G. Jungck, B.E. Rhoades, *Fixed point for setvalued functions without continuty*, Indian J. Pure Appl. Math. **29** (3) (1998), 227–238.

[**13**] M.A. Khan, Sumitra, *A common fixed point theorem in non-Archimedean Menger PM-space*, Novi Sad J. Math. **39** (1) (2009), 81–87.

[**14**] K. Menger, *Statistical matrices*, Proc. Nat. Acad. Sci. USA **28** (3) (1942), 535–537.

[**15**] R.P. Pant, *Common fixed points of non-commuting mappings*, J. Math. Anal. Appl. **188** (2) (1994), 436–440.

[**16**] V.M. Sehgal, A.T. Bharucha-Reid, *Fixed points of contraction mappings on probabilistic metric spaces*, Math. Systems Theory, **6** (1972), 97–102.

[**17**] H. Sherwood, *Complete probabilistic metric spaces*, Z. Wahrsch, Verw Gebiete, **20** (1971), 117–128.

[**18**] A. Singh, R.C. Dimri, U.C. Gairola, *A fixed point theorem for near-hybrid contraction*, J. Nat. Acad. Math. **22** (2008), 11–22.

[**19**] A. Singh, R.C. Dimri, S. Joshi, *Some fixed point theorems for pointwise R-weakly commuting hybrid mappings in metrically convex spaces*, Armenian J. Math. **2** (4) (2009), 135–145.

[**20**] R. Vasuki, *Common fixed points for R-weakly commuting maps in fuzzy metric spaces*, Indian J. Pure Appl. Math. **30** (1999), 419–423.

(received 27.07.2010; in revised form 24.09.2010)

Post Box-100, Department of Mathematics, H.N.B. Garhwal University, Srinagar (Garhwal), Uttarakhand-246174, INDIA

E-mail: `singhamit841@gmail.com, dimrirc@gmail.com, bhattsandeep1982@gmail.com`