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FIXED POINT THEOREMS FOR SOME GENERALIZED CONTRACTIVE MULTI-VALUED MAPPINGS AND FUZZY MAPPINGS

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Abstract. In this paper, first we give a theorem which generalizes the Banach contraction principle and fixed point theorems given by many authors, and then a fixed point theorem for a multi-valued (θ, L)-weak contraction. We extend the notion of (θ, L)-weak contraction to fuzzy mappings and obtain some fixed point theorems. A coincidence point theorem for a hybrid pair of mappings $f : X \to X$ and $T : X \to W(X)$ is established. Later on we prove a fixed point theorem for a different type of fuzzy mapping.

1. Introduction

Banach contraction principle plays a very important role in nonlinear analysis and has many generalizations (cf. [14] and the references therein). Recently, Suzuki gave a new type of generalization of the Banach contraction principle (cf. [20]). Then Kikkawa and Suzuki gave another generalization, which generalizes the work of Suzuki (cf. [20, Theorem 1]) and the Nadler fixed point theorem (cf. [16]). In [3], M. Berinde and V. Berinde extended the notion of weak contraction from single valued mappings to multi-valued mappings and obtained some convergence theorems for the Picard iteration associated with multi-valued weak contractions. As mentioned by Berinde and Berinde (cf. [3]), a lot of well-known contractive conditions considered in the literature contains (θ, L)-weak contraction as a special case. But this case, under consideration in this paper, is very general as unlike others the condition that $\theta + L < 1$ is not required. For details one is referred to [3]. In [12], Kamran further extended the notion of weak contraction and introduced the notion of multi-valued f-weak contraction and generalized multi-valued f-weak contraction. In this paper in Theorem 3.1, we generalize the work of Kikkawa and Suzuki (cf. [14, Theorem 2]), Nadler (cf. [16]), Kamran (cf. [12, Theorem 2.9]), and Berinde and Berinde (cf. [3, Theorem 3]. In Theorem 3.4, we proved a fixed point theorem for a multi-valued (θ, L)-weak contraction defined on a nonempty

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closed subset of a complete and convex metric space. In Theorem 4.1, a fixed point theorem for a (θ, L)-weak contractive fuzzy mapping is obtained which extends the result of Berinde and Berinde (cf. [3, Theorem 3]). In Theorem 4.2, a coincidence point theorem for a hybrid pair of mappings $f : X \to X$ and $T : X \to W(X)$; and in Theorem 4.3, a fixed point theorems for a (α, L)-weak contractive fuzzy mapping are obtained (definitions follow). Finally in Theorem 4.5 and in Theorem 4.7, we prove fixed point theorems for a different type of fuzzy mapping $T : X \to K(X)$.

2. Basic definitions and lemmas

In this section first we give the following basic definitions and lemmas for multi-valued mappings, and then that for the fuzzy mappings. (X, d) always represents a metric space, H represents the Hausdorff distance induced by the metric d, $CB(X)$ denotes the family of nonempty closed and bounded subsets of X, and $C(X)$ the family of nonempty compact subsets of X. Let $\mathcal{P}(X)$ be the family of all nonempty subsets of X, and let $T : X \to \mathcal{P}(X)$ be a multi-valued mapping. An element $x \in X$ such that $x \in T(x)$ is called a fixed point of T. We denote by $Fix(T)$ the set of all fixed points of T, i.e.,

$$Fix(T) = \{x \in X : x \in T(x)\}.$$

Note that, x is a fixed point of a multi-valued mapping T if and only if $d(x, T(x)) = 0$, whenever $T(x)$ is a closed subset of X.

Lemma 2.1. [16] *Let A and B be nonempty compact subsets of a metric space (X, d). If $a \in A$, then there exists $b \in B$ such that $d(a, b) \leq H(A, B)$.*

Definition 2.2. Let (X, d) be a complete metric space. X is said to be (metrically) convex if X has the property that for each $x, y \in X$ with $x \neq y$ there exists $z \in X$, $x \neq z \neq y$ such that

$$d(x, z) + d(z, y) = d(x, y).$$

Lemma 2.3. [5] *If K is a nonempty closed subset of a complete and metrically convex metric space (X, d), then for any $x \in K$, $y \notin K$, there exists a point $z \in \partial K$ (the boundary of K) such that*

$$d(x, z) + d(z, y) = d(x, y).$$

Definition 2.4. A multi-valued mapping $T : X \to CB(X)$ is said to be a multi-valued weak contraction or a multi-valued (θ, L)-weak contraction if and only if there exist two constants $\theta \in [0, 1)$ and $L \geq 0$ such that

$$H(T(x), T(y)) \leq \theta d(x, y) + L d(y, T(x)),$$

for all $x, y \in X$.

Definition 2.5. Let $f : X \to X$ and $T : X \to CB(X)$. The mapping T is said to be a multi-valued (f, θ, L)-weak contraction if and only if there exist two constants $\theta \in [0, 1)$ and $L \geq 0$ such that

$$H(T(x), T(y)) \leq \theta d(f(x), f(y)) + Ld(f(y), T(x)),$$

for all $x, y \in X$.

Lemma 2.6. [16] *If $A, B \in CB(X)$ and $x \in A$, then for each positive number α there exists $y \in B$ such that $d(x, y) \leq H(A, B) + \alpha$, i.e., $d(x, y) \leq qH(A, B)$ where $q > 1$.*

Lemma 2.7. [16] *Let $\{A_n\}$ be a sequence of sets in $CB(X)$, and suppose that $\lim_{n\to\infty} H(A_n, A) = 0$, where $A \in CB(X)$. Then if $x_n \in A_n$, $n = 1, 2, \ldots$, and if $\lim_{n\to\infty} x_n = x_0$, it follows that $x_0 \in A$.*

Definition 2.8. [21] Let (X, d) be a metric space, $f : X \to X$ be a self-mapping and $T : X \to CB(X)$ be a multi-valued mapping. The mappings f and T are called R-weakly commuting if for a given $x \in X$, $f(T(x)) \in CB(X)$ and there exists some real number R such that

$$H(f(T(x)), T(f(x))) \leq Rd(f(x), T(x)).$$

Definition 2.9. [11] The mappings $f : X \to X$ and $T : X \to CB(X)$ are weakly compatible if they commute at their coincidence points, i.e., if $f(T(x)) = T(f(x))$ whenever $f(x) \in T(x)$.

Definition 2.10. [13] Let $T : X \to CB(X)$. The mapping $f : X \to X$ is said to be T-weakly commuting at $x \in X$ if $f(f(x)) \in T(f(x))$.

Note that R-weakly commuting mappings commute at their coincidence points.

A real linear space X with a metric d is called a metric linear space if $d(x + z, y + z) = d(x, y)$ and $\alpha_n \to \alpha$, $x_n \to x \implies \alpha_n x_n \to \alpha x$. Let (X, d) be a metric linear space. A fuzzy set A in a metric linear space X is a function from X into $[0, 1]$. If $x \in X$, the function value $A(x)$ is called the grade of membership of x in A. The α-level set (or α-cut set) of A, denoted by A_α, is defined by

$$\begin{aligned} A_\alpha &= \{x : A(x) \geq \alpha\} \text{ if } \alpha \in (0, 1], \\ A_0 &= \overline{\{x : A(x) > 0\}}. \end{aligned}$$

Here $\overline{B}$ denotes the closure of the (non-fuzzy) set B.

Definition 2.11. A fuzzy set A is said to be an approximate quantity if and only if A_α is compact and convex in X for each $\alpha \in [0, 1]$ and $\sup_{x\in X} A(x) = 1$.

Let $\mathcal{F}(X)$ be the collection of all fuzzy sets in X and $W(X)$ be a sub-collection of all approximate quantities. When A is an approximate quantity and $A(x_0) = 1$ for some $x_0 \in X$, A is identified with an approximation of x_0. For $x \in X$, let

$\{x\} \in W(X)$ with membership function equal to the characteristic function χ_x of the set $\{x\}$.

Definition 2.12. Let $A, B \in W(X)$, $\alpha \in [0,1]$. Then we define

$$\begin{aligned} p_\alpha(A,B) &= \inf_{x \in A_\alpha,\, y \in B_\alpha} d(x,y), \\ p(A,B) &= \sup_\alpha p_\alpha(A,B), \\ D_\alpha(A,B) &= H(A_\alpha, B_\alpha), \\ D(A,B) &= \sup_\alpha D_\alpha(A,B). \end{aligned}$$

where H is the Hausdorff distance induced by the metric d.

The function $D_\alpha(A,B)$ is called an α-distance between $A, B \in W(X)$, and D a metric on $W(X)$. We note that p_α is a non-decreasing function of α and thus $p(A,B) = p_1(A,B)$. In particular if $A = \{x\}$, then $p(\{x\}, B) = p_1(x,B) = d(x, B_1)$. Next we define an order on the family $W(X)$, which characterizes the accuracy of a given quantity.

Definition 2.13. Let $A, B \in W(X)$. Then A is said to be more accurate than B, denoted by $A \subset B$ (or B includes A), if and only if $A(x) \le B(x)$ for each $x \in X$.

The relation $\subset$ induces a partial order on the family $W(X)$.

Definition 2.14. Let X be an arbitrary set and Y be any metric linear space. F is called a fuzzy mapping if and only if F is a mapping from the set X into $W(Y)$.

Definition 2.15. For $F : X \to W(X)$, we say that $u \in X$ is a fixed point of F if $\{u\} \subset F(u)$, i.e. if $u \in F(u)_1$.

Lemma 2.16. [10] *Let $x \in X$ and $A \in W(X)$. Then $\{x\} \subset A$ if and only if $p_\alpha(x, A) = 0$ for each $\alpha \in [0,1]$.*

Remark 2.17. Note that from the above lemma it follows that for $A \in W(X)$, $\{x\} \subset A$ if and only if $p(\{x\}, A) = 0$. If no confusion arises instead of $p(\{x\}, A)$, we will write $p(x, A)$.

Lemma 2.18. [10] *$p_\alpha(x, A) \le d(x,y) + p_\alpha(y, A)$ for each $x, y \in X$.*

Lemma 2.19. [10] *If $\{x_0\} \subset A$, then $p_\alpha(x_0, B) \le D_\alpha(A, B)$ for each $B \in W(X)$.*

Lemma 2.20. [15] *Let (X, d) be a complete metric linear space, $F : X \to W(X)$ be a fuzzy mapping and $x_0 \in X$. Then there exists $x_1 \in X$ such that $\{x_1\} \subset F(x_0)$.*

Remark 2.21. Let $f : X \to X$ be a self map and $T : X \to W(X)$ be a fuzzy mapping such that $\cup\{T(X)\}_\alpha \subseteq f(X)$ for each $\alpha \in [0,1]$. Then from Lemma 2.20, it follows that for any chosen point $x_0 \in X$ there exist points $x_1, y_1 \in X$ such that $y_1 = f(x_1)$ and $\{y_1\} \subset T(x_0)$. Here $T(x)_\alpha = \{y \in X : T(x)(y) \geq \alpha\}$.

Definition 2.22. Let $f : X \to X$ be a self mapping and $T : X \to W(X)$ be a fuzzy mapping. Then a point $u \in X$ is said to be a coincidence point of f and T if $\{f(u)\} \subset T(u)$, i.e. if $f(u) \in T(u)_1$.

Definition 2.23. A fuzzy mapping $T : X \to W(X)$ is said to be a weak contraction or a (θ, L)-weak contraction if and only if there exist two constants $\theta \in [0, 1)$ and $L \geq 0$ such that

$$D(T(x), T(y)) \leq \theta d(x, y) + Lp(y, T(x)),$$

for all $x, y \in X$.

Definition 2.24. A fuzzy mapping $T : X \to \mathcal{F}(X)$ is said to be a weak contraction or a (θ, L)-weak contraction if and only if there exist two constants $\theta \in [0, 1)$ and $L \geq 0$ such that

$$H(T(x)_{\alpha(x)}, T(y)_{\alpha(y)}) \leq \theta d(x, y) + Ld(y, T(x)_{\alpha(x)}),$$

for all $x, y \in X$ where $T(x)_{\alpha(x)}, T(y)_{\alpha(y)}$ are in $CB(X)$.

Definition 2.25. For a complete metric linear space X, let $f : X \to X$ be a self mapping and $F : X \to W(X)$ a fuzzy mapping. T is said to be a f-weak contraction or a (f, θ, L)-weak contraction if and only if there exist two constants $\theta \in [0, 1)$ and $L \geq 0$ such that

$$D(T(x), T(y)) \leq \theta d(f(x), f(y)) + Lp(f(y), T(x)).$$

Definition 2.26. A fuzzy mapping $T : X \to W(X)$ is said to be a generalized (α, L)-weak contraction if there exists a functions $\alpha : [0, +\infty) \to [0, 1)$ satisfying $\limsup_{r \to t^+} \alpha(r) < 1$ for every $t \in [0, +\infty)$, such that

$$D(T(x), T(y)) \leq \alpha(d(x, y))d(x, y) + Lp(y, T(x)),$$

for all $x, y \in X$ and $L \geq 0$.

Lemma 2.27. [17] *Let A be a subset of X. Let $\{A_\alpha : \alpha \in [0, 1]\}$ be a family of subsets of A such that*

(i) $A_0 = A$,

(ii) $\alpha \leq \beta$ implies $A_\beta \subseteq A_\alpha$,

(iii) $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$, $\lim_{n\to\infty} \alpha_n = \alpha$ implies $A_\alpha = \bigcup_{k=1}^{\infty} A_{\alpha_k}$.

Then the function $\phi : X \to I$ defined by $\phi(x) = \sup\{\alpha \in I : x \in A_\alpha\}$ has the property that $A_\alpha = \{x \in X : \phi(x) \geq \alpha\}$.

Conversely, in any fuzzy set μ in X the family of α-level sets of μ satisfies the above conditions from (i) to (iii).

The function ϕ in the above lemma is actually defined on the set A, but we can extend it to X by defining $\phi(x) = 0$ for all $x \in X - A$. This lemma is known as Negoite-Ralescu representation theorem.

3. Multi-valued mappings

In this section we prove all the main theorems of this paper regarding multi-valued mappings. Theorem 3.1 gives a generalization of Banach contraction principle. In Theorem 3.2 we have stated and proved a further generalization of Theorem 3.1 and Banach contraction theorem, and Theorem 3.4 concerns a multi-valued non-self weak contraction and its fixed point. In proving the existence of a fixed point of such a mapping, we follow the technique of Assad and Kirk (cf. [5]). Our theorems extend the results of several authors.

THEOREM 3.1. *Let (X, d) be a complete metric space and let $T : X \to CB(X)$. Suppose that there exists two constants $\theta \in [0, 1)$ and $L \geq 0$ such that*

$$\eta(\theta)d(x, T(x)) \leq d(x, y) \text{ implies } H(T(x), T(y)) \leq \theta d(x, y) + Ld(y, T(x))$$

for all $x, y \in X$, where $\eta : [0, 1) \to (\frac{1}{2+L}, \frac{1}{1+L}]$ defined by $\eta(\theta) = \frac{1}{1+\theta+L}$ is a strictly decreasing function. Then

(i) there exists $z \in X$ such that $z \in T(z)$, i.e., $Fix(T) \neq \emptyset$;

(ii) for any point $x_0 \in X$, there exists an orbit $\{x_n\}$ of T at x_0 with $x_{n+1} \in T(x_n)$ such that $\{x_n\}$ converges to a fixed point z of T for which the following estimates hold:

$$d(x_n, z) \leq \frac{h^n}{1-h} d(x_0, x_1) \text{ for } n = 0, 1, 2, \ldots$$

and

$$d(x_n, z) \leq \frac{h}{1-h} d(x_{n-1}, x_n) \text{ for } n = 1, 2, \ldots,$$

for some $h < 1$.

Proof. (i) Suppose $q > 1$. We select a sequence $\{x_n\}$ in X in the following way. Let $x_0 \in X$ and $x_1 \in T(x_0)$. Then we have $\eta(\theta)d(x_0, T(x_0)) \leq \eta(\theta)d(x_0, x_1) \leq d(x_0, x_1)$. Hence from the given hypothesis we have,

$$H(T(x_0), T(x_1)) \leq \theta d(x_0, x_1) + Ld(x_1, T(x_0)) = \theta d(x_0, x_1).$$

There exists a point $x_2 \in T(x_1)$ such that

$$d(x_1, x_2) \leq qH(T(x_0), T(x_1)) \leq q[\theta d(x_0, x_1) + Ld(x_1, T(x_0))] \leq q\theta d(x_0, x_1).$$

Since the above inequality is valid for any $q \geq 1$, we choose $q > 1$ such that $h = q\theta < 1$ for any $\theta \in [0, 1)$. Thus, $d(x_1, x_2) \leq hd(x_0, x_1)$.

Let $x_3 \in T(x_2)$ be such that $d(x_2, x_3) \leq qH(T(x_1), T(x_2))$. Note that $\eta(\theta)d(x_1, T(x_1)) \leq \eta(\theta)d(x_1, x_2) \leq d(x_1, x_2)$, and so by the given hypothesis, $H(T(x_1), T(x_2)) \leq \theta d(x_1, x_2) + Ld(x_2, T(x_1)) = \theta d(x_1, x_2)$. Hence we have, $d(x_2, x_3) \leq hd(x_1, x_2)$. Proceeding in this way we can obtain a sequence $\{x_n\}$ in X such that $d(x_{n-1}, x_n) \leq hd(x_{n-2}, x_{n-1})$. It can easily be shown that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete let $x_n \to z \in X$.

Next we show that $d(z, T(x)) \leq \theta d(z, x) + Ld(x, z)$ for all $x \in X \backslash \{z\}$. Since $x_n \to z$, for $x \in X \backslash \{z\}$ there exists $\nu \in \mathbb{N}$ such that $d(z, x_n) \leq \frac{1}{3} d(z, x)$ for all $n \in \mathbb{N}$ with $n \geq \nu$. Then we have,

$$\begin{aligned}
\eta(\theta) d(x_n, T(x_n)) &\leq d(x_n, T(x_n)) \leq d(x_n, x_{n+1}) \leq d(x_n, z) + d(z, x_{n+1}) \\
&\leq \frac{1}{3} d(z, x) + \frac{1}{3} d(x, z) = \frac{2}{3} d(x, z) = d(x, z) - \frac{1}{3} d(x, z) \\
&\leq d(x, z) - d(x_n, z) \leq d(x, x_n) + d(x_n, z) - d(x_n, z) \\
&= d(x, x_n),
\end{aligned}$$

i.e., $\eta(\theta) d(x_n, T(x_n)) \leq d(x_n, x)$ for $n \geq \nu$, which implies $H(T(x_n), T(x)) \leq \theta d(x_n, x) + Ld(x, T(x_n))$ for $n \geq \nu$. For $n \geq \nu$, this implies

$$\begin{aligned}
d(x_{n+1}, T(x)) &\leq \theta d(x_n, x) + Ld(x, T(x_n)) \\
&\leq \theta d(x_n, x) + Ld(x, x_{n+1}).
\end{aligned}$$

Taking $n \to \infty$ we have, $d(z, T(x)) \leq \theta d(z, x) + Ld(x, z)$ for all $x \in X \backslash \{z\}$. Next we show that

$$H(T(x), T(z)) \leq \theta d(x, z) + Ld(z, T(x)) \text{ for all } x \in X. \tag{1}$$

Equation (1) is satisfied when $x = z$. Now we take $x \neq z$. For every $n \in \mathbb{N}$ there exists $y_n \in T(x)$ such that $d(z, y_n) \leq d(z, T(x)) + \frac{1}{n} d(x, z)$ as $d(z, T(x)) = \inf_{y \in T(x)} d(z, y)$. Consider the following

$$\begin{aligned}
d(x, T(x)) &\leq d(x, y_n) \leq d(x, z) + d(z, y_n) \\
&\leq d(x, z) + d(z, T(x)) + \frac{1}{n} d(x, z) \\
&\leq d(x, z) + (\theta + L) d(x, z) + \frac{1}{n} d(x, z) \\
&= (1 + \theta + L + \frac{1}{n}) d(x, z).
\end{aligned}$$

Dividing both sides by $1 + \theta + L$ we have,

$$\frac{1}{1 + \theta + L} d(x, T(x)) \leq (1 + \frac{1}{n(1 + \theta + L)}) d(x, z),$$

for any n, and hence $\eta(\theta) d(x, T(x)) \leq d(x, z)$. Then by the given hypothesis, $H(T(x), T(z)) \leq \theta d(x, z) + Ld(z, T(x))$ is satisfied for all $x \in X$. Now we have,

$$\begin{aligned}
d(z, T(z)) &= \lim_{n \to \infty} d(x_{n+1}, T(z)) \leq \lim_{n \to \infty} H(T(x_n), T(z)) \\
&\leq \lim_{n \to \infty} \{\theta d(x_n, z) + Ld(z, T(x_n))\} \\
&\leq \lim_{n \to \infty} \{\theta d(x_n, z) + L[d(z, x_{n+1}) + d(x_{n+1}, T(x_n))]\} = 0,
\end{aligned}$$

which implies $d(z, T(z)) = 0$, and hence $z \in T(z)$, i.e., $FixT \neq \emptyset$ as $T(z)$ is closed.

To prove (ii) let us proceed as follows: The sequence $\{x_n\}$ obtained in the proof of (i) are such that $x_{n+1} \in T(x_n)$ for $n \geq 0$ and satisfies

$$d(x_n, x_{n+1}) \leq hd(x_{n-1}, x_n) \leq h^2 d(x_{n-2}, x_{n-1}) \leq \cdots \leq h^n d(x_0, x_1).$$

Also we have

$$d(x_{n+k}, x_{n+k+1}) \leq h^{k+1} d(x_{n-1}, x_n) \text{ for any } k \geq 0.$$

Using the above inequalities we have

$$\begin{aligned} d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p}) \\ &\leq h^n d(x_0, x_1) + h^{n+1} d(x_0, x_1) + \cdots + h^{n+p-1} d(x_0, x_1) \\ &= h^n \frac{(1-h^p)}{1-h} d(x_0, x_1), \end{aligned} \tag{2}$$

and

$$\begin{aligned} d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p}) \\ &\leq hd(x_{n-1}, x_n) + h^2 d(x_{n-1}, x_n) + \cdots + h^p d(x_{n-1}, x_n) \\ &= \frac{h(1-h^p)}{1-h} d(x_{n-1}, x_n). \end{aligned} \tag{3}$$

Taking $p \to \infty$, and noting the fact that $\lim_{n\to\infty} d(x_n, x_{n+p}) = d(x_n, z)$ and $\lim_{p\to\infty} h^p = 0$, from (2) and (3) we obtain the assertion (ii) of the Theorem. ∎

The above theorem is a generalization of Theorem 2 of Kikkawa and Suzuki (cf. [14]) which is obtained when $L = 0$. It is also a generalization of Theorem 3 of Berinde and Berinde (cf. [3]).

Corollary 3.1.1. [14, Theorem 2] *Define a strictly decreasing function* η *from* $[0, 1)$ *onto* $(\frac{1}{2}, 1]$ *by* $\eta(r) = \frac{1}{1+r}$. *Let* (X, d) *be a complete metric space and let* T *be a mapping from* X *into* $CB(X)$. *Assume that there exists* $r \in [0, 1)$ *such that*

$$\eta(r) d(x, T(x)) \leq d(x, y) \textit{ implies } H(T(x), T(y)) \leq r d(x, y)$$

for all $x, y \in X$. *Then there exists* $z \in X$ *such that* $z \in T(z)$.

Corollary 3.1.2. (Nadler [16]) *Let* (X, d) *be a complete metric space and let* T *be a mapping from* X *into* $CB(X)$. *If there exists* $r \in [0, 1)$ *such that*

$$H(Tx, Ty) \leq r d(x, y) \textit{ for all } x, y \in X,$$

then there exists $z \in X$ *such that* $z \in Tz$.

Proof. Given that T satisfies the condition of Nadler's theorem, i.e.,

$$H(Tx, Ty) \leq r d(x, y) \text{ for all } x, y \in X \text{ and } r \in [0, 1), \tag{4}$$

we have to prove that there exists $z \in X$ such that $z \in T(z)$.

For any $x \in X$, $y \in T(x)$ we have $d(y, T(y)) \leq H(T(x), T(y))$. Hence by (4) we have,

$$d(y, T(y)) \leq H(T(x), T(y)) \leq r d(x, y),$$

i.e., $\eta(r) d(y, T(y)) \leq r d(x, y)$, i.e.

$$\eta(r) d(x, T(x)) \leq d(y, x) = d(x, y) \tag{5}$$

as $r < 1$ and $\eta(r) < 1$. Hence by (4), (5) and Theorem 3.1 for $L = 0$, it follows that there exists $z \in X$ such that $z \in T(z)$. ■

THEOREM 3.2. *Let (X, d) be a metric space, $T : X \to CB(X)$ and $f : X \to X$. Suppose that there exists two constants $\theta \in [0, 1)$ and $L \geq 0$ such that*

$$\eta(\theta) d(f(x), T(x)) \leq d(f(x), f(y)) \quad \textit{implies}$$
$$H(T(x), T(y)) \leq \theta d(f(x), f(y)) + L d(f(y), T(x))$$

for all $x, y \in X$, where $\eta : [0, 1) \to (\frac{1}{2+L}, \frac{1}{1+L}]$ defined by $\eta(\theta) = \frac{1}{1+\theta+L}$ is a strictly decreasing function, $T(X) \subset f(X)$ and $f(X)$ is complete. Then

(i) the set of coincidence point of f and T, $C(f, T)$ is nonempty.

(ii) for any $x_0 \in X$, there exists an f-orbit $O_f(x_0) = \{f(x_n) : n = 1, 2, 3 \dots\}$ of T at the point x_0 such that $f(x_n) \to f(u)$, where u is a coincidence point of f and T, for which the following estimates hold:

$$d(f(x_n), f(u)) \leq \frac{h^n}{1-h} d(f(x_0), f(x_1)), \ n = 0, 1, 2, \dots,$$
$$d(f(x_n), f(u)) \leq \frac{h}{1-h} d(f(x_{n-1}), f(x_n)), \ n = 1, 2, \dots.$$

for a certain constant $h < 1$. Further, if f is R-weakly commuting at u and $f(f(u)) = f(u)$, then f and T have a common fixed point.

Proof. Let $x_0 \in X$, and $x_1 \in X$ such that $f(x_1) \in T(x_0)$. Then $\eta(\theta) d(f(x_0), T(x_0)) \leq d(f(x_0), f(x_1))$, and so by the given hypothesis we have

$$H(T(x_0), T(x_1)) \leq \theta d(f(x_0), f(x_1)) + L d(f(x_1), T(x_0)) = \theta d(f(x_0), f(x_1)).$$

Let $x_2 \in X$ be such that $f(x_2) \in T(x_1)$ and then $d(f(x_1), f(x_2)) \leq q H(T(x_0), T(x_1))$ $\leq h d(f(x_0), f(x_1))$, where $q > 1$ and q is chosen in such a way that $h = q\theta < 1$. Now

$$\eta(\theta) d(f(x_1), T(x_1)) \leq \eta(\theta) d(f(x_1), f(x_2)) \leq d(f(x_1), f(x_2)),$$

which implies

$$H(T(x_1), T(x_2)) \leq \theta d(f(x_1), f(x_2)) + L d(f(x_2), T(x_1)) = \theta d(f(x_1), f(x_2)).$$

Let $x_3 \in X$ be such that $f(x_3) \in T(x_2)$ and then $d(f(x_2), f(x_3)) \leq q H(T(x_1), T(x_2))$ $\leq h d(f(x_1), f(x_2)) \leq h^2 d(f(x_0), f(x_1))$. Proceeding in this way we obtain a sequence $\{f(x_n)\}$ in X. It can easily be shown that the sequence $\{f(x_n)\}$ is a Cauchy sequence in X. Since $f(X)$ is complete, the sequence converges to some point $f(u) \in f(X)$. So there exists a positive integer ν such that for all $x \in X \setminus \{u\}$ we have $d(f(x_n), f(u)) \leq \frac{1}{3} d(f(x), f(u))$ for $n \geq \nu$. Then for $n \geq \nu$ we can write

$$\begin{aligned}
\eta(\theta) d(f(x_n), T(x_n)) &\leq d(f(x_n), T(x_n)) \leq d(f(x_n), f(x_{n+1})) \\
&\leq d(f(x_n), f(u)) + d(f(u), f(x_{n+1})) \\
&\leq \frac{1}{3} d(f(x), f(u)) + \frac{1}{3} d(f(x), f(u)) \\
&= d(f(x), f(u)) - \frac{1}{3} d(f(x), f(u)) \\
&\leq d(f(x), f(u)) - d(f(x_n), f(u)) \\
&\leq d(f(x), f(x_n)) + d(f(x_n), f(u)) - d(f(x_n), f(u)) \\
&= d(f(x_n), f(x)).
\end{aligned}$$

Hence from the given hypothesis it follows that

$$H(T(x_n), T(x)) \leq \theta d(d(x_n), f(x)) + L d(f(x), T(x_n)).$$

This implies for any $n \geq \nu$,

$$\begin{aligned}
d(f(x_{n+1}), T(x)) &\leq H(T(x_n), T(x)) \\
&\leq \theta d(f(x_n), f(x)) + L d(f(x), T(x_n)) \\
&\leq \theta d(f(x_n), f(x)) + L d(f(x), f(x_{n+1})) + L d(f(x_{n+1}), T(x_n)) \\
&= \theta d(f(x_n), f(x)) + L d(f(x), f(x_{n+1})).
\end{aligned}$$

Hence taking $n \to \infty$ we have $d(f(u), T(x) \leq \theta d(f(u), f(x)) + L d(f(x), f(u)) = (\theta + L) d(f(x), f(u))$ for $x \in X \setminus \{u\}$. Next we show

$$H(T(x), T(u)) \leq \theta d(f(x), f(u)) + L d(f(u), T(x)) \tag{6}$$

for all $x \in X$. It is true if $x = u$. Suppose $x \neq u$. Since $d(f(u), T(x)) = \inf_{v \in T(x)} d(f(u), v)$, for each $n \in \mathbb{N}$ we can obtain a sequence $\{v_n\}$ in $T(x)$ such that $d(f(u), v_n) \leq d(f(u), T(x)) + \frac{1}{n} d(f(x), f(u))$ for each $n \in \mathbb{N}$. Hence for $x \neq u$ we have

$$\begin{aligned}
d(f(x), T(x)) &\leq d(f(x), v_n) \leq d(f(x), f(u)) + d(f(u), v_n) \\
&\leq d(f(x), f(u)) + d(f(u), T(x)) + \frac{1}{n} d(f(x), f(u)) \\
&\leq d(f(x), f(u)) + (\theta + L) d(f(x), f(u)) + \frac{1}{n} d(f(x), f(u)) \\
&= (1 + \theta + L + \frac{1}{n}) d(f(x), f(u)).
\end{aligned}$$

and so $\frac{1}{1+\theta+L} d(f(x), T(x)) \leq (1 + \frac{1}{(1+\theta+L)n}) d(f(x), f(u))$ for any n, and hence

$$\eta(\theta) d(f(x), T(x)) \leq d(f(x), f(u)),$$

which implies $H(T(x),T(u)) \leq \theta d(f(x),f(u)) + Ld(f(u),T(x))$, and hence (6) is proved. Now

$$\begin{aligned}
d(f(u),T(u)) &= \lim_{n\to\infty} d(f(x_{n+1}),T(u)) \leq \lim_{n\to\infty} H(T(x_n),T(u)) \\
&\leq \lim_{n\to\infty}[\theta d(f(x_n),f(u)) + Ld(f(u),T(x_n))] \\
&\leq \lim_{n\to\infty}[\theta d(f(x_n),f(u)) + Ld(f(u),f(x_{n+1}) + d(f(x_{n+1}),T(x_n))] \\
&= 0,
\end{aligned}$$

and hence $f(u) \in T(u)$ which completes the proof of (i).

To prove (ii) let us proceed as follows: The sequence $\{f(x_n)\}$ obtained in the proof of (i) are such that $f(x_{n+1}) \in T(x_n)$ for $n \geq 0$ and satisfies

$$\begin{aligned}
d(f(x_n),f(x_{n+1})) &\leq hd(f(x_{n-1}),f(x_n)) \leq h^2 d(f(x_{n-2}),f(x_{n-1})) \\
&\leq \cdots \leq h^n d(f(x_0),f(x_1)).
\end{aligned}$$

Also we have

$$d(f(x_{n+k}),f(x_{n+k+1})) \leq h^{k+1} d(f(x_{n-1}),f(x_n)) \text{ for any } k \geq 0 \text{ and } n \geq 1.$$

Using the above inequalities we have

$$\begin{aligned}
&d(f(x_n),f(x_{n+p})) \\
&\qquad \leq d(f(x_n),f(x_{n+1})) + d(f(x_{n+1}),f(x_{n+2})) + \cdots + d(f(x_{n+p-1}),f(x_{n+p})) \\
&\qquad \leq h^n d(f(x_0),f(x_1)) + h^{n+1} d(f(x_0),f(x_1)) + \cdots + h^{n+p-1} d(f(x_0),f(x_1)) \\
&\qquad = h^n \frac{(1-h^p)}{1-h} d(f(x_0),f(x_1)), \qquad (7)
\end{aligned}$$

and

$$\begin{aligned}
&d(f(x_n),f(x_{n+p})) \\
&\qquad \leq d(f(x_n),f(x_{n+1})) + d(f(x_{n+1}),f(x_{n+2})) + \cdots + d(f(x_{n+p-1}),f(x_{n+p})) \\
&\qquad \leq hd(f(x_{n-1}),f(x_n)) + h^2 d(f(x_{n-1}),f(x_n)) + \cdots + h^p d(f(x_{n-1}),f(x_n)) \\
&\qquad = \frac{h(1-h^p)}{1-h} d(f(x_{n-1}),f(x_n)). \qquad (8)
\end{aligned}$$

Taking $p \to \infty$, and noting the fact that $\lim_{n\to\infty} d(f(x_n),f(x_{n+p})) = d(f(x_n),f(u))$ and $\lim_{p\to\infty} h^p = 0$, from (7) and (8) we obtain the assertion (ii) of the Theorem.

If f is R-weakly commuting at u we have $H(f(T(u)),T(f(u))) \leq Rd(f(u),T(u))$. As $f(u) \in T(u)$ this implies $f(T(u)) = T(f(u))$. Again $f(f(u)) = f(u)$, and so $f(u) \in T(u)$ implies $f(f(u)) \in f(T(u)) = T(f(u))$, i.e., $f(u) \in T(f(u))$. Hence, $f(u)$ is a fixed point of both f and T, i.e., f and T have a common fixed point. ∎

Remark 3.3. In Definition 2.9 we need $f(T(x)) \in CB(X)$. If f is continuous and $T(x) \in C(X)$, then $f(T(X))$ also belongs to $C(X)$. The above theorem is a generalization of Theorem 3.1, since by taking f as the identity mappings in Theorem 3.2 we obtain Theorem 3.1. It is easy to see that the map $f : X \to X$ is T-weakly commuting at a coincidence point of f and T. Hence Theorem 3.2 is generalization of Theorem 2.9 of Kamran (cf. [12]). In some sense the above theorem is also a generalization of Theorem 3 of Kikkawa and Suzuki (cf. [14]) in two directions: The mapping T is multi-valued and we have an additional term in the second inequality. If we take $L = 0$ and $T : X \to X$ (single-valued), then we get Theorem 3 of [14] without the continuity condition on the mapping f, but with an additional condition that $f(X)$ is complete. If X is assumed to be compact then $f(X)$ is compact when f is continuous.

THEOREM 3.4. *Let K be a nonempty closed subset of a complete and convex metric space (X, d) and $T : K \to CB(X)$ be a multi-valued (θ, L)-weak contraction (see Definition 2.4). If $T(x) \subset K$ for each $x \in \partial K$ (the boundary of K), then T has a fixed point.*

Proof. We select a sequence $\{x_n\}$ in the following way. Let $x_0 \in K$ and $x_1' \in T(x_0)$. If $x_1' \in K$ let $x_1 = x_1'$; otherwise select a point $x_1 \in \partial K$ s.t. $d(x_0, x_1) + d(x_1, x_1') = d(x_0, x_1')$. Thus $x_1 \in K$ and by Lemma 2.6 we can choose a point $x_2' \in T(x_1)$ so that $d(x_1', x_2') \le H(T(x_0), T(x_1)) + \theta$, where $\theta < 1$. Now put $x_2' = x_2$ if $x_2' \in K$, otherwise let x_2 be a point of ∂K such that $d(x_1, x_2) + d(x_2, x_2') = d(x_1, x_2')$. By induction we can obtain a sequence $\{x_n\}$, $\{x_n'\}$ such that for $n = 1, 2, 3, \ldots$

(*i*) $x_{n+1}' \in T(x_n)$
(*ii*) $d(x_n', x_{n+1}') \le H(T(x_{n-1}), T(x_n)) + \theta^n$ where
(*iii*) $x_{n+1}' = x_{n+1}$ if $x_{n+1}' \in K$, or
(*iv*) $d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+1}') = d(x_n, x_{n+1}')$ if $x_{n+1}' \notin K$. Now let

$$P = \{x_i \in \{x_n\} : x_i = x_i', i = 1, 2, \ldots\}$$
$$Q = \{x_i \in \{x_n\} : x_i \neq x_i', i = 1, 2, \ldots\}.$$

Observe that if $x_n \in Q$ for some n, then $x_{n+1} \in P$. Now for $n \ge 2$ we estimate the distance $d(x_n, x_{n+1})$. There arises three cases:

Case 1. The case that $x_n \in P$ and $x_{n+1} \in P$. In this case we have,

$$\begin{aligned} d(x_n, x_{n+1}) = d(x_n', x_{n+1}') &\le H(T(x_{n-1}), T(x_n)) + \theta^n \\ &\le \theta d(x_{n-1}, x_n) + Ld(x_n, T(x_{n-1})) + \theta^n \\ &\le \theta d(x_{n-1}, x_n) + \theta^n. \end{aligned}$$

Case 2. The case that $x_n \in P$ and $x_{n+1} \in Q$. In this case we use (*iv*) and proceeding in the same way as Case 1 we obtain,

$$d(x_n, x_{n+1}) \le d(x_n, x_{n+1}') = d(x_n', x_{n+1}') \le H(T(x_{n-1}), T(x_n)) \le \theta d(x_{n-1}, x_n) + \theta^n.$$

Case 3. The case that $x_n \in Q$ and $x_{n+1} \in P$. From the construction of the sequence $\{x_n\}$ it is clear that two consecutive terms of $\{x_n\}$ can not be in Q, and hence $x_{n-1} \in P$ and $x_{n-1}' = x_{n-1}$. Using this below we obtain,

$$d(x_n, x_{n+1}) \le d(x_n, x_n') + d(x_n', x_{n+1})$$

$$
\begin{aligned}
&= d(x_n, x'_n) + d(x'_n, x'_{n+1}) \\
&\le d(x_n, x'_n) + H(T(x_{n-1}), T(x_n)) + \theta^n \\
&\le d(x_n, x'_n) + \theta d(x_{n-1}, x_n) + \theta^n \qquad (\text{ as in Case 1}) \\
&\le d(x_n, x'_n) + d(x_{n-1}, x_n) + \theta^n \\
&= d(x_{n-1}, x'_n) + \theta^n = d(x'_{n-1}, x'_n) + \theta^n \\
&\le H(T(x_{n-2}), T(x_{n-1})) + \theta^{n-1} + \theta^n \qquad (\text{ as in Case 2}) \\
&\le \theta d(x_{n-2}, x_{n-1}) + \theta^{n-1} + \theta^n.
\end{aligned}
$$

The only other possibility, $x_n \in Q$, $x_{n+1} \in Q$ can not occur. Thur for $n \ge 2$ we have

$$
d(x_n, x_{n+1}) \le \begin{cases} \theta d(x_{n-1}, x_n) + \theta^n, \text{ or} \\ \theta d(x_{n-2}, x_{n-1}) + \theta^{n-1} + \theta^n \end{cases} \tag{9}
$$

Let $\delta = \theta^{-1/2} \max\{d(x_0, x_1), d(x_1, x_2)\}$. Now as in [5], it can be we proved that for $n \ge 1$,

$$
d(x_n, x_{n+1}) \le \theta^{n/2}(\delta + n). \tag{10}
$$

From (10) it follows that

$$
d(x_k, x_N) \le \delta \sum_{i=N}^{\infty} (\theta^{1/2})^i + \sum_{i=N}^{\infty} i(\theta^{1/2})^i, \quad k > N \ge 1.
$$

This implies $\{x_n\}$ is a Cauchy sequence in K, and since X is complete and K is closed, $\{x_n\}$ converges to a point in K. Let $u = \lim_{n\to\infty} x_n$. Hence there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ each of whose terms is in the set P (i.e., $x_{n_k} = x'_{n_k}$ for $k = 1, 2, \dots$). Thus by (i), $x'_{n_k} \in T(x_{n_k-1})$ for $k = 1, 2, \dots$, and since $x_{n_k-1} \to u$ as $k \to \infty$ we have $T(x_{n_k-1}) \to T(u)$ as $k \to \infty$ in the Hausdorff metric. Hence it follows from Lemma 2.7 that $u \in T(u)$, i.e., T has a fixed point, which completes the proof. ∎

4. Fuzzy mappings

Many authors considered the class of fuzzy sets with nonempty compact α-cut sets in a metric space or nonempty compact convex α-cut sets in a metric linear space, but some have given attention to class of fuzzy sets with nonempty closed and bounded α-cut sets in a metric space. Theorems 4.1–4.3 deal with fuzzy mappings with α-cut sets as nonempty, compact and convex subsets of X. Next following the work in [2, 7, 22], we present Theorem 4.5 and Theorem 4.7 concerning a different kind of fuzzy mappings with special α-cut sets as nonempty, closed and bounded subsets of X.

THEOREM 4.1. *Let (X, d) be a complete metric linear space and $T : X \to W(X)$ be a (θ, L)-weak contractive fuzzy mapping (see Definition 2.24). Then*

(i) $Fix(T) \ne \emptyset$;

(ii) For any $x_0 \in X$, there exists an orbit $\{x_n\}_{n=0}^{\infty}$ of T at the point x_0 that converges to a fixed point u of T, for which the following estimates hold:

$$d(x_n, u) \le \frac{\theta^n}{1-\theta} d(x_0, x_1),\ n = 0, 1, 2, \dots,$$
$$d(x_n, u) \le \frac{\theta}{1-\theta} d(x_{n-1}, x_n),\ n = 1, 2, \dots.$$

Proof. Let $x_0 \in X$. Then there exists $x_1 \in X$ such that $\{x_1\} \subset T(x_0)$. If $D(T(x_0), T(x_1)) = 0$, then $T(x_0) = T(x_1)$, i.e., $\{x_1\} \subset T(x_1)$, which actually means that $Fix(T) \neq \emptyset$. Let $D(T(x_0), T(x_1)) \neq 0$. Then by Lemmas 2.20 and 2.21, we can find $x_2 \in X$ such that $\{x_2\} \subset T(x_1)$ and

$$\begin{aligned} d(x_1, x_2) &\le H(T(x_0)_1, T(x_1)_1) = D_1(T(x_0), T(x_1)) \\ &\le D(T(x_0), T(x_1)) \le \theta d(x_0, x_1) + Lp(x_1, T(x_0)) \\ &\le \theta d(x_0, x_1). \end{aligned}$$

If $D(T(x_1), T(x_2)) = 0$ then $T(x_1) = T(x_2)$, i.e., $\{x_2\} \subset T(x_2)$. Otherwise, we assume $D(T(x_1), T(x_2)) \neq 0$ and $x_3 \in X$ such that $\{x_3\} \subset T(x_2)$ and

$$\begin{aligned} d(x_2, x_3) &\le H(T(x_1)_1, T(x_2)_1) = D_1(T(x_1), T(x_2)) \\ &\le D(T(x_1), T(x_2)) \le \theta d(x_1, x_2) + Lp(x_2, T(x_1)) \\ &\le \theta d(x_1, x_2). \end{aligned}$$

In this manner, we obtain an orbit $\{x_n\}_{n=0}^{\infty}$ at x_0 for T satisfying

$$d(x_n, x_{n+1}) \le \theta d(x_{n-1}, x_n),\ n = 1, 2, \dots. \tag{11}$$

From (11) we obtain inductively,

$$d(x_n, x_{n+1}) \le \theta^n d(x_0, x_1) \text{ and } d(x_{n+k}, x_{n+k+1}) \le \theta^{k+1} d(x_{n-1}, x_n) \tag{12}$$

for $k \in \mathbb{N}$, $n \ge 1$. Now from (12) we have,

$$\begin{aligned} d(x_n, x_{n+p}) &\le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p}) \\ &= (\theta^n + \theta^{n+1} + \cdots + \theta^{n+p-1}) d(x_0, x_1) \\ &= \frac{\theta^n (1-\theta^p)}{1-\theta} d(x_0, x_1), \end{aligned} \tag{13}$$

which in view of $0 < \theta < 1$ shows that $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, it follows that $\{x_n\}_{n=0}^{\infty}$ converges to some point in X. Let $u = \lim_{n\to\infty} x_n$. Then we have,

$$\begin{aligned} p(u, T(u)) &\le d(u, x_{n+1}) + p(x_{n+1}, T(u)) \\ &\le d(u, x_{n+1}) + D(T(x_n), T(u)) \\ &\le d(u, x_{n+1}) + \theta d(x_n, u) + Lp(u, T(x_n)) \\ &\le d(u, x_{n+1}) + \theta d(x_n, u) + Ld(u, x_{n+1}) + Lp(x_{n+1}, T(x_n)). \end{aligned}$$

Noting that $p(x_{n+1}, T(x_n)) = 0$ and taking $n \to \infty$ we have, $p(u, T(u)) \leq 0 \implies p(u, T(u)) = 0 \implies \{u\} \subset T(u)$.

From (13) taking $p \to \infty$ we have

$$d(x_n, u) \leq \frac{\theta^n}{1-\theta} d(x_0, x_1), n = 0, 1, 2, \ldots$$

Again by (12) we have

$$\begin{aligned} d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p}) \\ &= (\theta + \theta^2 + \cdots + \theta^p) d(x_{n-1}, x_n) \\ &= \frac{\theta(1-\theta^p)}{1-\theta} d(x_{n-1}, x_n). \end{aligned}$$

Taking $p \to \infty$ we have,

$$d(x_n, u) \leq \frac{\theta}{1-\theta} d(x_{n-1}, x_n).$$

Hence the proof is complete. ■

THEOREM 4.2. *Let (X, d) be a complete metric linear space, $f : X \to X$ be a self mapping, and $T : X \to W(X)$ be a (f, θ, L)-weak contractive fuzzy mapping (see Definition 2.25). Suppose $\cup\{T(X)\}_\alpha \subseteq f(X)$ for $\alpha \in [0, 1]$ and $f(X)$ is complete. Then there exists $u \in X$ such that u is a coincidence point of f and T, that is $\{f(u)\} \subset T(u)$. Here $T(x)_\alpha = \{y \in X : (T(x))(y) \geq \alpha\}$.*

Proof. Let $x_0 \in X$ and $y_0 = f(x_0)$. Since $\cup\{T(X)\}_\alpha \subset f(X)$ for each $\alpha \in [0, 1]$, by Remark 2.21 for $x_0 \in X$ there exist points $x_1, y_1 \in X$ such that $y_1 = f(x_1)$ and $\{y_1\} \subset T(x_0)$. By Remark 2.21 and Lemma 2.1, for $x_1 \in X$ there exist points $x_2, y_2 \in X$ such that $y_2 = f(x_2)$ and $\{y_2\} \subset T(x_1)$, and

$$\begin{aligned} d(y_1, y_2) &\leq H(T(x_0)_1, T(x_1)_1) \leq D(T(x_0), T(x_1)) \\ &\leq \theta d(f(x_0), f(x_1)) + Lp(f(x_1), T(x_0)) = \theta d(y_0, y_1). \end{aligned}$$

By repeating this process we can select points $x_k, y_k \in X$ such that $y_k = f(x_k)$ and $\{y_k\} \subset T(x_{k-1})$, and hence

$$\begin{aligned} d(y_k, y_{k+1}) &\leq H(T(x_{k-1})_1, T(x_k)_1) \leq D(T(x_{k-1}), T(x_k)) \\ &\leq \theta d(f(x_{k-1}), f(x_k)) + Lp(f(x_k), T(x_{k-1})) = \theta d(y_{k-1}, y_k). \end{aligned} \tag{14}$$

From (14) we obtain inductively,

$$d(y_n, y_{n+1}) \leq \theta^n d(y_0, y_1) \text{ and } d(y_{n+k}, y_{n+k+1}) \leq \theta^{k+1} d(y_{k-1}, y_k) \tag{15}$$

for all $k \in \mathbb{N}$, $n \geq 1$.

Now from (15), we have for any $p \geq 1$,

$$\begin{aligned} d(y_n, y_{n+p}) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{n+p-1}, y_{n+p}) \\ &\leq (\theta^n + \theta^{n+1} + \cdots + \theta^{n+p-1}) d(y_0, y_1) \\ &= \frac{\theta^n(1-\theta^p)}{1-\theta} d(y_0, y_1). \end{aligned}$$

In view of $0 < \theta < 1$, we see that $\{y_n\}$ is a Cauchy sequence. Since $f(X)$ is complete, $\{y_n\}$ converges to some point in $f(X)$. Let $y = \lim_{n\to\infty} y_n$ and $u \in X$ be such that $y = f(u)$. Now

$$\begin{aligned} p(f(u), T(u)) = p(y, T(u)) &\leq d(y, y_{k+1}) + p(y_{k+1}, T(u)) \\ &\leq d(y, y_{k+1}) + D(T(x_k), T(u)) \\ &\leq d(y, y_{k+1}) + \theta d(f(x_k), f(u)) + Lp(f(u), T(x_k)) \\ &\leq d(y, y_{k+1}) + \theta d(y_k, y) + L[d(y, y_{k+1}) + p(y_{k+1}, T(x_k))]. \end{aligned}$$

Noting that $p(y_{k+1}, T(x_k)) = 0$ and the fact that $y_k \to y$ as $k \to \infty$ we have, $p(f(u), T(u)) = 0$, i.e., $\{f(u)\} \subset T(u)$. ■

THEOREM 4.3. *Let* (X, d) *be a complete metric linear space and* $T : X \to W(X)$ *be a generalized* (α, L)*-weak contraction (see Definition* 2.26*). Then* T *has a fixed point.*

Proof. Let $x_0 \in X$. Then by Lemma 2.20, there exists $x_1 \in X$ such that $\{x_1\} \subset T(x_0)$. Now by Lemma 2.20 and Lemma 2.1, there exists a point $x_2 \in X$ such that $\{x_2\} \subset T(x_1)$ and

$$\begin{aligned} d(x_1, x_2) &\leq H(T(x_0)_1, T(x_1)_1) \leq D(T(x_0), T(x_1)) \\ &\leq \alpha(d(x_0, x_1)) d(x_0, x_1) + Lp(x_1, T(x_0)) \leq d(x_0, x_1). \end{aligned}$$

By repeating this process we can select a point $x_{k+1} \in X$ such that $\{x_{k+1}\} \in T(x_k)$ and

$$\begin{aligned} d(x_k, x_{k+1}) &\leq H(T(x_{k-1})_1, T(x_k)_1) \leq D(T(x_{k-1}), T(x_k)) \\ &\leq \alpha(d(x_{k-1}, x_k)) d(x_{k-1}, x_k) + Lp((x_{k-1}, T(x_k)) \leq d(x_{k-1}, x_k). \end{aligned} \tag{16}$$

Let $d_k = d(x_{k-1}, x_k)$. Since d_k is a non-increasing sequence of nonnegative real numbers, therefore $\lim_{k\to\infty} d_k = c \geq 0$. By hypothesis we get $\limsup_{t\to c+} \alpha(t) < 1$. Therefore there exists k_0 such that $k \geq k_0$ implies that $\alpha(d_k) < h$, where $\limsup_{t\to c+} \alpha(t) < h < 1$. Now by (16) we deduce that the sequence $\{d_k\}$ satisfies the following recurrence inequality:

$$\begin{aligned} d_{k+1} &\leq \alpha(d_k) d_k \leq \alpha(d_k)\alpha(d_{k-1}) d_{k-1} \cdots \\ &\leq \prod_{i=1}^{k} \alpha(d_i) d_1 \leq \prod_{i=1}^{k_0-1} \alpha(d_i) \prod_{i=k_0}^{k} \alpha(d_i) d_1 \leq \prod_{i=1}^{k_0-1} \alpha(d_i) h^{k-k_0+1} d_1 \\ &= Ch^k, \quad \text{(where } C \text{ is a generic positive constant).} \end{aligned}$$

Hence for $p \geq 1$ we have,

$$\begin{aligned} d(x_k, x_{k+p}) &\leq d(x_k, x_{k+1}) + d(x_{k+1}, x_{k+2}) + \cdots + d(x_{k+p-1}, x_{k+p}) \\ &= d_{k+1} + d_{k+2} + \cdots + d_{k+p-1} \leq C(h^k + h^{k+1} + \cdots + h^{k+p-1}) \\ &= C\frac{h^k(1-h^p)}{1-h}. \end{aligned}$$

which in view of $0 < h < 1$ shows that $\{x_k\}$ is a Cauchy sequence. Since X is complete, the sequence $\{x_k\}$ converges to some point in X. Let $u = \lim_{k\to\infty} x_k$. Now we have,

$$\begin{aligned} p(u, T(u)) &\leq d(u, x_{k+1}) + p(x_{k+1}, T(u)) \\ &\leq d(u, x_{k+1}) + D(T(x_k), T(u)) \\ &\leq d(u, x_{k+1}) + \alpha(d(x_k, u))d(x_k, u) + Lp(u, T(x_k)) \\ &\leq d(u, x_{k+1}) + \alpha(d(x_k, u))d(x_k, u) + L[d(u, x_{k+1}) + p(x_{k+1}, T(x_k))]. \end{aligned}$$

Using the fact that $x_{k+1} \in T(x_k)$ and the fact that $x_k \to u$ we have, $p(u, T(u)) \leq 0$, i.e., $p(u, T(u)) = 0$, i.e., $\{u\} \subset T(u)$. Hence the proof is complete. ■

NOTE. Du first introduced the concept of Reich-functions as follows (cf. [8]):

DEFINITION 4.4. A function $\phi : [0,\infty) \to [0,1)$ is called to be a Reich-function (R-function for short) if for each $t \in [0,\infty)$ there exists $r_t \in [0,1)$ and $\epsilon_t > 0$ such that $\phi(s) \leq r_t$ for all $s \in [t, t+\epsilon_t)$.

EXAMPLES. Let $\phi : [0,\infty) \to [0,1)$ be a function.

(i) Obviously, if ϕ is defined by $\phi(t) = c$, where $c \in [0,1)$, then ϕ is a R-function;

(ii) If ϕ is nondecreasing function, then ϕ is a R-function;

(iii) It is easy to see that if ϕ satisfies $\limsup_{s\to t+} \phi(s) < 1$ for all $t \in [0,\infty)$, then ϕ is a R-function.

Note that in Theorem 4.3, α is a R-function and so the proof of showing the sequence $\{x_k\}$ is Cauchy can be done in another way as follows. Since α is a R-function, there exists $r_c \in [0,1)$ and $\epsilon_c > 0$ such that $\phi(s) \leq r_c$ for all $s \in [c, c+\epsilon_c)$. Again $\{d_k\}$ being non-increasing and $d_k \to c$ as $k \to \infty$, there exists k_0 such that for all $k \geq k_0$ we have $d_k \in [c, c+\epsilon_c)$. Hence, by (16) we have

$$d_{k+1} \leq \alpha(d_k)d_k \leq r_c d_k \leq r_c^2 d_{k-1} \leq \cdots \leq r_c^{k-k_0+1} d_{k_0} \leq r_c^k \frac{d_{k_0}}{r_c^{k_0-1}}.$$

Hence, for $p \geq 1$ we have

$$\begin{aligned} d(x_k, x_{k+p}) &\leq d(x_k, x_{k+1}) + d(x_{k+1}, x_{k+2}) + \cdots + d(x_{k+p-1}, x_{k+p}) \\ &\leq (r_c^k + r_c^{k+1} + \cdots + r_c^{k+p-1}) \frac{d_{k_0}}{r_c^{k_0-1}} \\ &= \frac{r_c^k(1-r_c^p)}{1-r_c} \frac{d_{k_0}}{r_c^{k_0-1}}, \end{aligned}$$

which in view of $r_c \in [0,1)$ shows that $\{x_k\}$ is a Cauchy sequence.

THEOREM 4.5. *Let (X,d) be a complete metric space and $T : X \to \mathcal{F}(X)$ be a (θ, L)-weak contractive fuzzy mapping satisfying the condition that for each $x \in X$ there is $\alpha(x) \in (0,1]$ such that $T(x)_{\alpha(x)}$ is a nonempty closed bounded subset of X. Then*

(i) There exists a point $u \in X$ such that $u \in T(u)_{\alpha(u)}$;

(ii) For any $x_0 \in X$, there exists an orbit $\{x_n\}_{n=0}^{\infty}$ of T at the point x_0 that converges to a point $u \in X$, for which the following estimates hold:

$$d(x_n, u) \leq \frac{h^n}{1-h} d(x_0, x_1),\ n = 0, 1, 2, \ldots,$$
$$d(x_n, u) \leq \frac{h}{1-h} d(x_{n-1}, x_n),\ n = 1, 2, \ldots.$$

for a certain constant $h < 1$, such that $u \in T(u)_{\alpha(u)}$.

Proof. Let $F : X \to \mathcal{F}(X)$ be a fuzzy mapping. By assumption, there exists $\alpha(x) \in (0,1]$ such that $F(x)_{\alpha(x)}$ is a nonempty closed and bounded subset of X. Let us now construct a sequence $\{x_n\}$ $(n \geq 0)$ as follows. By α_{n+1} we denote $\alpha_{n+1} = \alpha(x_n)$ for $n \geq 0$. Let $x_0 \in X$. Let $x_1 \in T(x_0)_{\alpha_1}$ and $q > 1$. Then there exists a point $x_2 \in T(x_1)_{\alpha_2}$ such that

$$\begin{aligned} d(x_1, x_2) &\leq qH(T(x_0)_{\alpha_1}, T(x_1)_{\alpha_2}) \\ &\leq q\theta d(x_0, x_1) + qLd(x_1, T(x_0)_{\alpha_1}) \leq q\theta d(x_0, x_1). \end{aligned}$$

Let us choose $q > 1$ in such a way that $h = q\theta < 1$ for any $\theta \in [0,1)$, and then $d(x_1, x_2) \leq hd(x_0, x_1)$. Now for $x_2 \in T(x_1)_{\alpha_2}$, there exists a point $x_3 \in T(x_2)_{\alpha_3}$ such that

$$\begin{aligned} d(x_2, x_3) &\leq qH(T(x_1)_{\alpha_2}, T(x_2)_{\alpha_3}) \\ &\leq q\theta d(x_1, x_2) + qLd(x_2, T(x_1)_{\alpha_2}) \leq hd(x_1, x_2). \end{aligned}$$

In this manner, we obtain an orbit $\{x_n\}_{n=0}^{\infty}$ at x_0 for T satisfying

$$d(x_n, x_{n+1}) \leq hd(x_{n-1}, x_n),\ n = 1, 2, \ldots. \tag{17}$$

From (17) we obtain inductively,

$$d(x_n, x_{n+1}) \leq h^n d(x_0, x_1) \text{ and } d(x_{n+k}, x_{n+k+1}) \leq h^{k+1} d(x_{n-1}, x_n) \tag{18}$$

for $k \in \mathbb{N}$, $n \geq 1$. Now from (18) we have

$$\begin{aligned} d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p}) \\ &= (h^n + h^{n+1} + \cdots + h^{n+p-1}) d(x_0, x_1) \\ &= \frac{h^n(1-h^p)}{1-h} d(x_0, x_1), \end{aligned} \tag{19}$$

which in view of $0 < h < 1$ shows that $\{x_n\}$ is a Cauchy sequence. Since (X,d) is complete, it follows that $\{x_n\}_{n=0}^{\infty}$ converges to some point in X. Let $u = \lim_{n\to\infty} x_n$. Then we have,

$$d(u, T(u)_{\alpha(u)}) \leq d(u, x_{n+1}) + d(x_{n+1}, T(u)_{\alpha(u)})$$

$$\begin{aligned}
&\leq d(u,x_{n+1}) + d(T(x_n)_{\alpha_{n+1}}, T(u)_{\alpha(u)}) \\
&\leq d(u,x_{n+1}) + hd(x_n,u) + qLd(u,T(x_n)_{\alpha_{n+1}}) \\
&\leq d(u,x_{n+1}) + hd(x_n,u) + qLd(u,x_{n+1}) + qLd(x_{n+1},T(x_n)_{\alpha_{n+1}}).
\end{aligned}$$

Noting that $d(x_{n+1},T(x_n)_{\alpha_{n+1}}) = 0$ and taking $n \to \infty$ we have, $d(u,T(u)_{\alpha(u)}) \leq 0 \implies d(u,T(u)_{\alpha(u)}) = 0 \implies u \in T(u)_{\alpha(u)}$.

From (19) taking $p \to \infty$ we have

$$d(x_n,u) \leq \frac{h^n}{1-h} d(x_0,x_1), n = 0,1,2,\ldots$$

Again by (18) we have,

$$\begin{aligned}
d(x_n,x_{n+p}) &\leq d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \cdots + d(x_{n+p-1},x_{n+p}) \\
&= (h + h^2 + \cdots + h^p) d(x_{n-1},x_n) \\
&= \frac{h(1-h^p)}{1-h} d(x_{n-1},x_n).
\end{aligned}$$

Taking $p \to \infty$ we have, $d(x_n,u) \leq \dfrac{h}{1-h} d(x_{n-1},x_n)$. This completes the proof. ∎

Now we discuss a different type of fuzzy mapping. As defined earlier we know a fuzzy set in X is a function with domain X and range in $[0,1]$, $\mathcal{F}(X)$ denotes the collection of all fuzzy set in X and $CB(X)$ represents the nonempty closed and bounded subsets of X. Let $K(X)$ be the set of all fuzzy sets $\mu : X \to [0,1]$ such that $\hat{\mu} \in CB(X)$ where $\hat{\mu} = \{x \in X : \mu(x) = \max_{y \in X} \mu(y)\}$, i.e., $K(X) = \{\mu \in \mathcal{F}(X) : \hat{\mu} \in CB(X)\}$.

A fuzzy mapping T is a mapping from X into $K(X)$. For a fuzzy mapping $T : X \to K(X)$ and a mapping $\Lambda : K(X) \to CB(X)$, the composition $\hat{T} = \Lambda \circ T : X \to CB(X)$ is defined as $(\Lambda \circ T)(x) = \hat{T}(x) = \{y \in X : T(x)(y) = \max_{z \in X} T(x)(z)\}$. A point $x^* \in X$ is called a fixed point of a fuzzy mapping $T : X \to K(X)$ if $T(x^*)(x^*) \geq T(x^*)(x)$ for all $x \in X$, i.e., $T(x^*)(x^*) = \max_{y \in X} T(x^*)(y)$.

Lemma 4.6. [2] *A point $x^* \in X$ is a fixed point of a fuzzy mapping $T : X \to K(X)$ if and only if x^* is a fixed point of the induced mapping $\Lambda \circ T : X \to CB(X)$.*

Now we define $\alpha(x) = \max_{y \in X} T(x)(y)$, and then $T(x)_{\alpha(x)} = \{y \in X : T(x)(y) = \max_{z \in X} T(x)(z)\} = \{y \in X : T(x)(y) \geq \alpha\}$. Then using Theorem 4.5 and Lemma 4.6, we get the following result.

Theorem 4.7. *Let the fuzzy mapping $T : X \to K(X)$ be a (θ, L)-weak contractive fuzzy mapping. Then T has a fixed point.*

Proof. By Theorem 4.5, there exists $u \in X$ such that $u \in T(u)_{\alpha(u)}$. But here $\alpha(u)$ by definition is $\max_{y \in X} T(u)(y)$, i.e. $u \in \hat{T}(u)$, i.e., u is a fixed point of the induced mapping $\hat{T}$. Then by Lemma 4.6, u is a fixed point of the fuzzy mapping $T : X \to K(X)$. ∎

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