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2-NSR LEMMA AND QUOTIENT SPACE IN 2-NORMED SPACE

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Abstract. In this paper we discuss properties of compactness and compact operator on 2-normed space. Also we consider a result which is similar to Riesz Lemma and its applications in 2-normed space. We introduce quotient space from the finite dimensional subspace of a 2-normed space.

1. Introduction

The concept of a linear 2-normed space was introduced as a natural 2-metric analogue of that of a normed space. In 1963, Gähler introduced the notion of a 2-metric space, a real valued function of point-triples on a set X, whose abstract properties were suggested by the area function for a triangle determined by a triple in Euclidean space. In [4] S. Gähler introduced the following definition of a 2-normed space.

2. Preliminaries

Definition 2.1. [4] Let X be a real linear space of dimension greater than 1. Suppose $\|\ ,\ \|$ is a real valued function on $X \times X$ satisfying the following conditions:

1. $\|x, y\| = 0$ if and only if x and y are linearly independent,
2. $\|x, y\| = \|y, x\|$,
3. $\|\alpha x, y\| = |\alpha| \|x, y\|$,
4. $\|x + y, z\| \leq \|x, z\| + \|y, z\|$.

Then $\|\ ,\ \|$ is called a 2-norm on X and the pair $(X, \|\ ,\ \|)$ is called a 2-normed space. Some of the basic properties of 2-norms, are that they are non-negative and $\|x, y + x\| = \|x, y\|$, $\forall x, y \in X$ and $\forall \alpha \in \mathbf{R}$.

Definition 2.2. [2] Let X and Y be two 2-normed spaces and $T\colon X \to Y$ be a linear operator. For any $e \in X$, we say that the operator T is e-bounded if there

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exists $M_e > 0$ such that $\|T(x), T(e)\| \leq M_e \|x, e\|$ for all $x \in X$. An e-bounded operator T for every e will be called bounded.

Definition 2.3. [4] A sequence $\{x_n\}$ in a 2-normed space X is said to be convergent if there exists an element $x \in X$ such that

$$\lim_{n\to\infty} \|\{x_n - x\}, y\| = 0$$

for all $x \in X$.

Definition 2.4. [4] Let X and Y be two 2-normed spaces and $T\colon X \to Y$ be a linear operator. The operator T is said to be sequentially continuous at $x \in X$ if for any sequence $\{x_n\}$ of X converging to x we have $T(\{x_n\}) \to T(x)$.

Definition 2.5. [2] The closure of a subset E of a 2-normed space X is denoted by $\overline{E}$ and defined as the set of all $x \in X$ such that there is a sequence $\{x_n\}$ of E converging to x. We say that E is closed if $E = \overline{E}$.

For a 2-normed space we consider the subsets

$$B_e(a, r) = \{x : \|x - a, e\| < r\}, \qquad B_e[a, r] = \{x : \|x - a, e\| \leq r\}.$$

Definition 2.6. [2] A subset A of a 2-normed space X is said to be locally bounded if there exist $e \in X - \{0\}$ and $r > 0$ such that $A \subseteq B_e(0, r)$.

Definition 2.7. [2] A subset B of a 2-normed space X is said to be compact if every sequence $\{x_n\}$ in B has a convergent subsequence in B.

Definition 2.8. [2] Let X and Y be two 2-normed spaces. A linear operator $T\colon X \to Y$ is called a compact operator if it maps every locally bounded sequence $\{x_n\}$ of X onto a sequence $\{T(x_n)\}$ in Y which has a convergent subsequence.

Lemma 2.9. [2] *Let X and Y be two 2-normed spaces. If $T\colon X \to Y$ is a surjective bounded linear operator then T is sequentially continuous.*

Corollary 2.10. [2] *Let X and Y be two 2-normed spaces. Then every compact operator $T : X \to Y$ is bounded.*

3. Main results

Lemma 3.1. *Let X be a 2-normed space. If $B_e[a, r]$ is compact in X for some $a, e \in X$ and $r > 0$ then X is of finite dimension.*

Proof. Suppose that $B_e[a, r]$ is compact. The quotient space $X/\langle e\rangle$ is a normed space equipped with the norm

$$\begin{aligned}\|x + \langle e\rangle\|_Q = \inf&\left\{\frac{\|x, e\|}{\|e, e'\|} : e' \notin \langle e\rangle \;\&\; \|e, e'\| \leq 1\right\}\\ &+ \sup\left\{\frac{\|x, e\|}{\|e, e'\|} : e' \notin \langle e\rangle \;\&\; \|e, e'\| > 1\right\}\\ &= \|x, e\| + \sup\left\{\frac{\|x, e\|}{\|e, e'\|} : e' \notin \langle e\rangle \;\&\; \|e, e'\| > 1\right\}.\end{aligned}$$

Define $A'_e = \{x + \langle e\rangle : \|x - a + \langle e\rangle\|_Q \leq \frac{r}{\|e,e'\|}\}$ and let $A = \bigcap\{A_{e'} :$ e and e' are linearly independent$\}$. Then A is a closed ball in the normed space $X/\langle e\rangle$. We aim to show that A is a compact set in the normed space $X/\langle e\rangle$. For that let $\{x_n + \langle e\rangle\}$ be any sequence in A. Then

$$\|x_n + \langle e\rangle - (a + \langle e\rangle)\|_Q = \|x_n - a + \langle e\rangle\|_Q \leq \frac{r}{\|e,e'\|};\ \forall\ e' \notin \langle e\rangle \text{ and } \forall\ n$$
$$\Rightarrow \|x_n - a, e\| \leq \frac{r}{\|e,e'\|};\ \forall\ e' \notin \langle e\rangle \text{ and } \forall\ n.$$

In particular,

$$\|x_n - a, e\| \leq r\ ;\ \forall\ n \Rightarrow x_n\ \in B_e[a, r].$$

Hence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ converges to a point x_0. We have

$$\begin{aligned}
\|x_{n_k} + \langle e\rangle - (x_0 + \langle e\rangle)\|_Q &= \|x_{n_k} - x_0 + \langle e\rangle\|_Q \\
&= \|x_{n_k} - x_0, e\| + \sup\left\{\frac{\|x_{n_k} - x_0, e\|}{\|e,e'\|} : e' \notin \langle e\rangle \text{ and } \|e,e'\| > 1\right\} \\
&= \|x_{n_k} - x_0, e\|\left[1 + \sup\left\{\frac{1}{\|e,e'\|} : e' \notin \langle e\rangle \text{ and } \|e,e'\| > 1\right\}\right] \\
&\Rightarrow \lim_{k\to\infty} \|x_{n_k} + \langle e\rangle - (x_0 + \langle e\rangle)\|_Q = 0.
\end{aligned}$$

Hence $\{x_{n_k} + \langle e\rangle\}$ is a convergent subsequence of $\{x_n + \langle e\rangle\}$. This implies that A is compact and so $X/\langle e\rangle$ is of finite dimension. Hence X is of finite dimension. ∎

Here we introduce a result which is similar to Riesz Lemma.

Lemma 3.2. [2-NSR Lemma] *Let X be a 2-normed space and let $0 \neq e \in X$. Let r be any number such that $0 < r < 1$. Then there exists some $x_r \in X$ such that $\|x_r, e\| = 1$ and $r < \|x_r, e\|$.*

Proof. Since $\|x, e\| > 0,\ \forall\ x \notin \langle e\rangle$, we have

$$\begin{aligned}
\|x + \langle e\rangle\|_Q &= \|x, e\| + \sup\left\{\frac{\|x,e\|}{\|e,e'\|}\ : e' \notin \langle e\rangle \text{ and } \|e,e'\| > 1\right\} \\
&> 0; \quad \forall\ x \notin \langle e\rangle.
\end{aligned}$$

Also, as $r < 1$, $\|x, e\| \leq \|x + \langle e\rangle\|_Q < \frac{\|x+\langle e\rangle\|_Q}{r}$, $\forall\ x \notin \langle e\rangle$. Put $x_r = x/\|x, e\|$, so that $\|x_r, e\| = 1$ and

$$\|x_r + \langle e\rangle\|_Q = \|\frac{x}{\|x,e\|} + \langle e\rangle\|_Q > r.$$

Thus there exist $x_r \in X$ such that $\|x_r, e\| = 1$ and $r < \|x_r, e\|$. ∎

Remark. Let X be a 2-normed space ad let Y be a finite dimensional subspace of X generated by $\{e_1, e_2, \ldots, e_n\}$. Then X/Y is a normed space equipped with the norm

$$\|x + Y\|_Q = \sum_{k=1}^{n} \|x + \langle e_k\rangle\|_Q.$$

Corollary 3.3. *Let X be a 2-normed space and let Y be a finite dimensional subspace of X. Let r be any number such that $0 < r < 1$. Then there exists some $x_r \in X$ such that $r < \|x_r + Y\|_Q \leq 2$.*

Proof. Let $\{e_1, e_2, \dots, e_n\}$ be a basis for Y. Then for any $x \notin Y$,

$$r < \| \frac{x}{\|x, e_k\|} + \langle e_k \rangle \|_Q \ \leq 2; \text{ for } k = 1, 2, \dots, n$$

$$\Rightarrow \ r\|x, e_k\| < \|x + \langle e_k \rangle\|_Q$$

$$\leq 2\|x, e_k\|; \text{ for } k = 1, 2, \dots, n \text{ and for every } x \notin Y.$$

$$\Rightarrow \ r \sum_{k=1}^{n} \|x, e_k\| < \|\|x + Y\|_Q \ \leq 2 \sum_{k=1}^{n} \|x, e_k\|.$$

Put $x_r = x / \sum_{k=1}^{n} \|x, e_k\|$ so that $r < \|x_r + Y\| \leq 2$. ∎

Lemma 3.4. *Every finite dimensional subspace Y of a 2-normed space X is complete.*

Proof. To prove the completeness of Y, we use mathematical induction on the dimension m of Y. Let $m = 1$. Then $Y = \{ke \ : \ k \in \mathbf{R}\}$ with $e \neq 0$. If $\{x_n\}$ is a Cauchy sequence in Y with $x_n = k_n e$ then for every $x \in X$, $\|x_n - x_m, x\| \to 0$.

$$\|x_n - x_m, x\| = \|(k_n - k_m)e, x\| = |k_n - k_m|\|e, x\|; \ \forall \ x \ \in X.$$

$$\Rightarrow |k_n - k_m| = \frac{\|x_n - x_m, x\|}{\|x, e\|}, \ \forall \ x \ \notin \langle e \rangle.$$

It follows that $\{k_n\}$ is a Cauchy sequence in $\mathbf{R}$ which is complete.

If $k_n \to k$ in $\mathbf{R}$ then $x_n \to ke$ in Y. Thus Y is complete. Now assume that every $m - 1$ dimensional subspace of X is complete. Let $\dim Y = m$ and let $\{x_n\}$ be a Cauchy sequence in Y. Let $\{e_1, e_2, \dots, e_n\}$ be a basis for Y and let $Z = \operatorname{span}\{e_2, e_3, \dots, e_n\}$. Now for each $n = 1, 2, 3, \dots$, $x_n = k_n e_1 + z_n$ for some $k_n \in \mathbf{R}$ and $z_n \in Z$.

$$\|x_n - x_m, x\| = \|(k_n - k_m)e_1 + (z_n - z_m), x\|$$

$$= |k_n - k_m|\|e_1 + \frac{(z_n - z_m)}{k_n - k_m)}, x\|; \ \forall \ x \in X.$$

In particular,

$$\|x_n - x_m, e_2\| = |k_n - k_m|\|e_1 + \frac{(z_n - z_m)}{k_n - k_m)}, e_2\| > \frac{|k_n - k_m|}{2}\|e_1 + Z\|_Q.$$

It follows that $\{k_n\}$ a Cauchy sequence in $\mathbf{R}$ which is complete. As $z_n = x_n - k_n e_1$, it follows that $\{z_n\}$ is a Cauchy sequence in Z which is complete. If $k_n \to k$ in R and $z_n \to z$ in Z, then $x_n \to ke_1 + z$ in Y. Hence Y is complete. ∎

Theorem 3.5. *Let X be a 2-normed space and let T be a surjective compact operator on X and $0 \neq k \in \mathbf{R}$. If $\{x_n\}$ is a locally bounded sequence in X such* *that* $T(x_n) - kx_n \to y$ *in* X *then there is a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $\{x_{n_k}\}$ *converges to* x *in* X *and* $T(x) - kx = y$.

Proof. Since T is compact, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $T(x_{n_k})$ converges to some $z \in X$. Then $kx_{n_k} = [kx_{n_k} - T(x_{n_k})] + T(x_{n_k})$ converges to $-y + z$ and so $\{x_{n_k}\} \to \frac{z-y}{k} = x$. Since T is sequentially continuous [2.9], $T(x_{n_k}) \to T(x)$. It follows that

$$T(x) - kx = \lim_{k\to\infty} [T(x_{n_k}) - kx_{n_k}] = z - (-y + z) = y. \quad \blacksquare$$

THEOREM 2.6. *Let* X *be a 2-normed space and* $T\colon X \to X$. *Let* $0 \neq k \in \mathbf{R}$ *and* $0 \neq e \in X$ *such that* $(T - kI)X \subseteq \langle e\rangle$. *Then there is some* $x_0 \in X$ *such that* $\|x_0, e\| = 1$ *and for every* $y \in \langle e\rangle$, $\|T(x_0) - T(y), e\| > \frac{|k|}{4}$.

Proof. Let $Y = \langle e\rangle$. Then $T(y) = (T(y) - ky) + ky \subseteq Y + Y = Y$, $\forall\, y \in Y$. It follows that $T(Y) \subseteq Y$. Choose some $x_0 \in X$ such that $\|x_0, e\| = 1$ and $\frac{1}{2} < \|x_0 + \langle e\rangle\|_Q$. For any $y \in Y$,

$$\begin{aligned}
\|T(x_0) - T(y), e\| &= \|kx_0 - [kx_0 - T(x_0) + T(y)], e\| \\
&= |k| \|x_0 - \frac{1}{k}[kx_0 - T(x_0) + T(y)], e\| \\
&\geq \frac{|k|}{2} \|x_0 - \frac{1}{k}[kx_0 - T(x_0) + T(y)] + \langle e\rangle\|_Q \\
&= \frac{|k|}{2} \|x_0 + \langle e\rangle\|_Q > \frac{|k|}{4}. \quad \blacksquare
\end{aligned}$$

COROLLARY 2.7. *Let* X *be a 2-normed space and* $T\colon X \to X$. *Let* $0 \neq k \in \mathbf{R}$ *and let* Y *be a finite dimensional proper subspace of* X *such that* $(T - kI)X \subseteq Y$. *Then there is some* $x_0 \in X$ *such that for every* $x_0, y_0 \in Y$, $\|T(x_0) - T(y), x\| > \frac{|k|}{4}$.

Proof. Let $\{e_1, e_2, \ldots, e_m\}$ be a basis for Y. Then $T(y) = (T(y) - ky) + ky \subseteq Y + Y = Y$, $\forall\, y \in Y$ and so $T(Y) \subseteq Y$. Choose some $x_0 \in X$ such that $\frac{1}{2} < \|x_0 + Y\|_Q$. For any $x, y \in Y$,

$$\begin{aligned}
\|T(x_0) - T(y), x\| &= \|kx_0 - [kx_0 - T(x_0) + T(y)], x\| \\
&= |k| \|x_0 - \frac{1}{k}[kx_0 - T(x_0) + T(y)], x\| \\
&\geq \frac{|k|}{2} \|x_0 - \frac{1}{k}[kx_0 - T(x_0) + T(y)] + Y\|_Q \\
&= \frac{|k|}{2} \|x_0 + Y\|_Q > \frac{|k|}{4}.
\end{aligned}$$

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