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PAIRWISE CLOSURE-PRESERVING COLLECTIONS AND PAIRWISE PARACOMPACTNESS

M. K. Bose and Ajoy Mukharjee

Abstract. The notion of pairwise closure-preserving property of a collection of sets is introduced. Then some characterizations of pairwise paracompactness are obtained.

1. Introduction

The notion of pairwise paracompactness in a bitopological space was introduced and studied in Bose, Roy Choudhury and Mukharjee [1]. Some characterizations of pairwise paracompactness were obtained there. In this paper, we introduce the notion of pairwise closure-preserving collection of sets. Then we obtain some new characterizations of pairwise paracompactness which are analogous to the characterizations of paracompactness obtained by Michael [6].

2. Preliminaries

A collection $\mathcal{B}$ of subsets of a topological space $(X, \mathcal{T})$ is called a $(\mathcal{T})$closure-preserving collection if for any subcollection $\mathcal{D}$ of $\mathcal{B}$, $(\mathcal{T})\mathrm{cl}\big(\bigcup_{D\in\mathcal{D}} D\big) = \bigcup_{D\in\mathcal{D}}(\mathcal{T})\mathrm{cl}D$.

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two topologies on a set X. In the sequel, the bitopological space $(X, \mathcal{P}_1, \mathcal{P}_2)$ is denoted simply by X. The topology $\mathcal{P}_i$ is said to be regular with respect to $\mathcal{P}_j$, $i \neq j$, if for each $x \in X$ and $(\mathcal{P}_i)$closed set A with $x \notin A$, there exist $U \in \mathcal{P}_i$ and $V \in \mathcal{P}_j$ such that $x \in U$, $A \subset V$ and $U \cap V = \emptyset$. X is said to be pairwise regular (Kelly [5]) if $\mathcal{P}_i$ is regular with respect to $\mathcal{P}_j$ for both $i = 1$ and $i = 2$. X is said to be pairwise normal (Kelly [5]) if for any pair of a $(\mathcal{P}_i)$closed set A and a $(\mathcal{P}_j)$closed set B with $A \cap B = \emptyset$, $i \neq j$, there exist $U \in \mathcal{P}_j$ and $V \in \mathcal{P}_i$ such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$. X is said to be strongly pairwise regular (Bose, Roy Choudhury and Mukharjee [1]) if it is pairwise regular, and if both the topological spaces $(X, \mathcal{P}_1)$ and $(X, \mathcal{P}_2)$ are regular. A cover $\mathcal{U}$ of X is called a

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pairwise open cover (Fletcher, Hoyle III and Patty [4]) if $\mathcal{U} \subset \mathcal{P}_1 \cup \mathcal{P}_2$ and for each $i = 1, 2, \mathcal{U} \cap \mathcal{P}_i$ contains a nonempty set. A pairwise open cover $\mathcal{V}$ of X is said to be a parallel refinement (Datta [3]) of a pairwise open cover $\mathcal{U}$ of X if every $(\mathcal{P}_i)$open set of $\mathcal{V}$ is contained in some $(\mathcal{P}_i)$open set of $\mathcal{U}$. A subcollection $\mathcal{C}$ of a refinement $\mathcal{V}$ of a pairwise open cover $\mathcal{U}$ of X is said to be $\mathcal{U}$-locally finite (Bose, Roy Choudhury and Mukharjee [1]) if for each $x \in X$, there exists a neighbourhood of x intersecting a finite number of members of $\mathcal{C}$, the neighbourhood being $(\mathcal{P}_i)$open if x belongs to a $(\mathcal{P}_i)$open set of $\mathcal{U}$.

The bitopological space X is said to be pairwise paracompact (Bose, Roy Choudhury and Mukharjee [1]) if every pairwise open cover $\mathcal{U}$ of X has a $\mathcal{U}$-locally finite parallel refinement.

Throughout the paper, N and R denote the set of natural numbers and the set of real numbers respectively.

We require the following theorem.

Theorem 2.1. [1] *If the bitopological space X is strongly pairwise regular, then the following statements are equivalent.*

(a) *X is pairwise paracompact.*

(b) *Each pairwise open cover $\mathcal{U}$ of X has a parallel refinement $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$, where each $\mathcal{V}_n$ is $\mathcal{U}$-locally finite.*

(c) *Each pairwise open cover $\mathcal{U}$ of X has a $\mathcal{U}$-locally finite refinement.*

(d) *Each pairwise open cover $\mathcal{U}$ of X has a $\mathcal{U}$-locally finite refinement $\mathcal{B}$ such that if $B \subset U \in \mathcal{U}$, $B \in \mathcal{B}$, then $((\mathcal{P}_1)\mathrm{cl}B) \cup ((\mathcal{P}_2)\mathrm{cl}B) \subset U$.*

We introduce the following definitions:

Definition 2.2. X is said to be $(*)$pairwise normal if X is pairwise normal and if for every pair of a $(\mathcal{P}_j)$closed set A and a $(\mathcal{P}_i)$closed set B with $i \neq j$, $i, j = 1, 2$ and $A \cap B = \emptyset$, there exist $U, V \in \mathcal{P}_i$ such that

$$A \subset U, \quad B \subset V \quad \text{and} \quad U \cap V = \emptyset,$$

and there exist $G, H \in \mathcal{P}_j$ such that

$$A \subset G, \quad B \subset H \quad \text{and} \quad G \cap H = \emptyset.$$

It is easy to see that X is $(*)$pairwise normal if and only if it satisfies the following conditions:

For any $(\mathcal{P}_j)$closed set A and $(\mathcal{P}_i)$open set W with $A \subset W$,

(1) there exist $U \in \mathcal{P}_i$ such that $A \subset U \subset (\mathcal{P}_i)\mathrm{cl}U \subset W$,

(2) there exist $V \in \mathcal{P}_j$ such that $A \subset V \subset (\mathcal{P}_j)\mathrm{cl}V \subset W$,

(3) there exist $G \in \mathcal{P}_i$ such that $A \subset G \subset (\mathcal{P}_j)\mathrm{cl}G \subset W$.

Example 2.3. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two topologies on R defined by

$$\mathcal{P}_1 = \{R, \emptyset, (-\infty, a], (a, \infty)\},$$
$$\mathcal{P}_2 = \{R, \emptyset, R - \{a\}, (-\infty, a), (-\infty, a], (a, \infty)\}.$$

where $a \in R$. The bitopological space $(X, \mathcal{P}_1, \mathcal{P}_2)$ is $(*)$pairwise normal.

Now we show that there exists a pairwise normal space which is not $(*)$pairwise normal.

EXAMPLE 2.4. Let X be any set with $a, b \in X$. Suppose

$$\mathcal{P}_1 = \{\emptyset, X\} \cup \{A \subset X \mid a \in A\},$$
$$\mathcal{P}_2 = \{\emptyset, X\} \cup \{A \subset X \mid a \notin A, b \in A\}.$$

Then the bitopological space $(X, \mathcal{P}_1, \mathcal{P}_2)$ is pairwise normal but it is not $(*)$pairwise normal.

DEFINITION 2.5. A collection of subsets of X is said to be pairwise closure-preserving if it is $(\mathcal{P}_i)$closure-preserving for both $i = 1$ and $i = 2$.

DEFINITION 2.6. [2] A collection $\mathcal{A}$ of subsets of X is hereditarily pairwise closure-preserving if any collection $\mathcal{B}$ containing subsets of sets belonging to $\mathcal{A}$ such that each set $A \in \mathcal{A}$ has one and only one subset belonging to $\mathcal{B}$, is pairwise closure-preserving.

DEFINITION 2.7. Let $\mathcal{U}$ be a pairwise open cover of X. A collection $\mathcal{C}$ of subsets of X is $\mathcal{U}$-discrete (resp. $\mathcal{U}$-locally finite) if for each $x \in X$ there exists a neighbourhood of x intersecting at most one set (resp. a finite number of sets) of $\mathcal{C}$, the neighbourhood being $(\mathcal{P}_i)$open if x belongs to a $(\mathcal{P}_i)$open set of $\mathcal{U}$.

For a subcollection $\mathcal{A}$ of a refinement of a pairwise open cover $\mathcal{U}$ of X, we denote by $\mathcal{A}_i$, the collection of sets in $\mathcal{A}$ which are subsets of $(\mathcal{P}_i)$open sets of $\mathcal{U}$. If a set A belonging to $\mathcal{A}$ is a subset of a $(\mathcal{P}_i)$open set of $\mathcal{U}$, then $\mathrm{cl}A$ denotes the $(\mathcal{P}_i)$closure of A. The collection $\{\mathrm{cl}A \mid A \in \mathcal{A}\}$ is denoted by $\overline{\mathcal{A}}$.

Throughout Section 3, we assume that the bitopological space $(X, \mathcal{P}_1, \mathcal{P}_2)$ satisfies the following two conditions:

$(*)$ For any pairwise open cover $\mathcal{U}$ of X

$$A \subset \bigcup\{E \mid E \in \mathcal{U}_i\} \Rightarrow (\mathcal{P}_i)\mathrm{cl}A \subset \bigcup\{E \mid E \in \mathcal{U}_i\}. \tag{2.1}$$

$(**)$ If $\mathcal{D}$ is $(\mathcal{P}_i)$closure-preserving, then $\mathcal{D}$ is $(\mathcal{P}_j)$closure-preserving, when $\mathcal{D}$ is a collection of subsets of a set belonging to $\mathcal{P}_1 \cup \mathcal{P}_2 - \{X\}$.

3. Lemmas

To prove the desired characterizations as anticipated in introduction, we require the following lemmas.

LEMMA 3.1. *Suppose $\mathcal{V}$ is a refinement of a pairwise open cover $\mathcal{U}$ of the bitopological space X. If a collection $\mathcal{A} \subset \mathcal{V}$ is $\mathcal{U}$-locally finite, then $\mathcal{A}$ is pairwise closure-preserving.*

Proof. Let $\mathcal{B}$ be a subcollection of $\mathcal{A}$ and let

$$x \in (\mathcal{P}_i)\mathrm{cl}\Big(\bigcup\nolimits_{B \in \mathcal{B}_i} B\Big). \tag{3.1}$$

By the condition $(*)$, x belongs to a $(\mathcal{P}_i)$open set of $\mathcal{U}$. Therefore there exists a $(\mathcal{P}_i)$open neighbourhood of x, which intersects a finite number of sets in $\mathcal{B}_i$, say $B_1, B_2, \ldots, B_n$. Again by (3.1), every $(\mathcal{P}_i)$open neighbourhood of x intersects $\bigcup_{B\in\mathcal{B}_i} B$. Hence it follows that every $(\mathcal{P}_i)$open neighbourhood of x intersects $B_1 \cup B_2 \cup \ldots \cup B_n$. So $x \in (\mathcal{P}_i)\mathrm{cl}(B_1 \cup B_2 \cup \ldots \cup B_n) = ((\mathcal{P}_i)\mathrm{cl}B_1) \cup ((\mathcal{P}_i)\mathrm{cl}B_2) \ldots \cup ((\mathcal{P}_i)\mathrm{cl}B_n)$. Therefore $x \in \bigcup_{B\in\mathcal{B}_i} (\mathcal{P}_i)\mathrm{cl}B$. Hence

$$(\mathcal{P}_i)\mathrm{cl}\Big(\bigcup_{B\in\mathcal{B}_i} B\Big) \subset \bigcup_{B\in\mathcal{B}_i} (\mathcal{P}_i)\mathrm{cl}B \;\Rightarrow\; (\mathcal{P}_i)\mathrm{cl}\Big(\bigcup_{B\in\mathcal{B}_i} B\Big) = \bigcup_{B\in\mathcal{B}_i} (\mathcal{P}_i)\mathrm{cl}B. \tag{3.2}$$

Therefore by the condition $(**)$,

$$(\mathcal{P}_j)\mathrm{cl}\Big(\bigcup_{B\in\mathcal{B}_i} B\Big) = \bigcup_{B\in\mathcal{B}_i} (\mathcal{P}_j)\mathrm{cl}B.$$

Similarly, we get

$$(\mathcal{P}_i)\mathrm{cl}\Big(\bigcup_{B\in\mathcal{B}_j} B\Big) = \bigcup_{B\in\mathcal{B}_j} (\mathcal{P}_i)\mathrm{cl}B. \tag{3.3}$$

Now

$$\begin{aligned}(\mathcal{P}_i)\mathrm{cl}\Big(\bigcup_{B\in\mathcal{B}} B\Big) &= (\mathcal{P}_i)\mathrm{cl}\Big(\bigcup_{B\in\mathcal{B}_i} B\Big) \cup (\mathcal{P}_i)\mathrm{cl}\Big(\bigcup_{B\in\mathcal{B}_j} B\Big)\\ &= \Big(\bigcup_{B\in\mathcal{B}_i} (\mathcal{P}_i)\mathrm{cl}B\Big) \cup \Big(\bigcup_{B\in\mathcal{B}_j} (\mathcal{P}_i)\mathrm{cl}B\Big) \quad \text{(by (3.2) and (3.3))}\\ &= \bigcup_{B\in\mathcal{B}} (\mathcal{P}_i)\mathrm{cl}B. \quad \blacksquare\end{aligned}$$

LEMMA 3.2. *Let $\mathcal{V}$ be a refinement of a pairwise open cover $\mathcal{U}$ of X. Then a collection $\mathcal{A} \subset \mathcal{V}$ is pairwise closure-preserving iff $\overline{\mathcal{A}}$ is pairwise closure-preserving.*

Proof. Straightforward. ■

LEMMA 3.3. *If the pairwise open cover $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ of X has a pairwise closure-preserving refinement $\mathcal{B}$ such that*

$$((\mathcal{P}_2)\mathrm{cl}((\mathcal{P}_1)\mathrm{cl}B)) \cup ((\mathcal{P}_1)\mathrm{cl}((\mathcal{P}_2)\mathrm{cl}B)) \subset U_\alpha \tag{3.4}$$

where $B \subset U_\alpha$, $B \in \mathcal{B}$, then there exists a pairwise closure-preserving refinement $\mathcal{E} = \{E_\alpha \mid \alpha \in A\}$ of $\mathcal{U}$ such that

$$((\mathcal{P}_2)\mathrm{cl}((\mathcal{P}_1)\mathrm{cl}E_\alpha)) \cup ((\mathcal{P}_1)\mathrm{cl}((\mathcal{P}_2)\mathrm{cl}E_\alpha)) \subset U_\alpha \textit{ for each } \alpha \in A.$$

Proof. For each α, we write $E_\alpha = \bigcup\{B \in \mathcal{B} \mid B \subset U_\alpha\}$. Then

$$\begin{aligned}&((\mathcal{P}_2)\mathrm{cl}((\mathcal{P}_1)\mathrm{cl}E_\alpha)) \cup ((\mathcal{P}_1)\mathrm{cl}((\mathcal{P}_2)\mathrm{cl}E_\alpha))\\ &\quad= \Big((\mathcal{P}_2)\mathrm{cl}\Big((\mathcal{P}_1)\mathrm{cl}\Big(\bigcup_{B\subset U_\alpha} B\Big)\Big)\Big) \cup \Big((\mathcal{P}_1)\mathrm{cl}\Big((\mathcal{P}_2)\mathrm{cl}\Big(\bigcup_{B\subset U_\alpha} B\Big)\Big)\Big)\\ &\quad= \Big((\mathcal{P}_2)\mathrm{cl}\Big(\bigcup_{B\subset U_\alpha} (\mathcal{P}_1)\mathrm{cl}B\Big)\Big) \cup \Big((\mathcal{P}_1)\mathrm{cl}\Big(\bigcup_{B\subset U_\alpha} (\mathcal{P}_2)\mathrm{cl}B\Big)\Big)\\ &\qquad\qquad\qquad \text{(since } \mathcal{B} \text{ is pairwise closure-preserving).}\end{aligned}$$

Again we have for $i = 1, 2$,

$$(\mathcal{P}_i)\mathrm{cl}\Big(\bigcup_{B\subset U_\alpha}(\mathcal{P}_i)\mathrm{cl}B\Big) = \bigcup_{B\subset U_\alpha}(\mathcal{P}_i)\mathrm{cl}\left((\mathcal{P}_i)\mathrm{cl}B\right).$$

Therefore by the condition $(**)$, we get

$$(\mathcal{P}_j)\mathrm{cl}\Big(\bigcup_{B\subset U_\alpha}(\mathcal{P}_i)\mathrm{cl}B\Big) = \bigcup_{B\subset U_\alpha}(\mathcal{P}_j)\mathrm{cl}\left((\mathcal{P}_i)\mathrm{cl}B\right).$$

Hence

$$\begin{aligned}
&((\mathcal{P}_2)\mathrm{cl}\left((\mathcal{P}_1)\mathrm{cl}E_\alpha\right)) \cup ((\mathcal{P}_1)\mathrm{cl}\left((\mathcal{P}_2)\mathrm{cl}E_\alpha\right)) \\
&\quad = \Big(\bigcup_{B\subset U_\alpha}(\mathcal{P}_2)\mathrm{cl}\Big((\mathcal{P}_1)\mathrm{cl}B\Big)\Big) \cup \Big(\bigcup_{B\subset U_\alpha}(\mathcal{P}_1)\mathrm{cl}\Big((\mathcal{P}_2)\mathrm{cl}B\Big)\Big) \\
&\quad \subset U_\alpha \quad \text{(by (3.4))}.
\end{aligned}$$

Let us now consider a subcollection $\mathcal{D}$ of $\mathcal{E} = \{E_\alpha \mid \alpha \in A\}$. For $D \in \mathcal{D}$, there exists an $\alpha(D) \in A$ such that $D = E_{\alpha(D)}$. We write $\mathcal{C}_D = \{B \in \mathcal{B} \mid B \subset U_{\alpha(D)}\}$. Then $\mathcal{C} = \bigcup_{D\in\mathcal{D}}\mathcal{C}_D$ is a subcollection of $\mathcal{B}$, and

$$\bigcup_{C\in\mathcal{C}} C = \bigcup_{D\in\mathcal{D}}\Big(\bigcup_{C\in\mathcal{C}_D} C\Big) = \bigcup_{D\in\mathcal{D}} D. \tag{3.5}$$

Now

$$\begin{aligned}
(\mathcal{P}_i)\mathrm{cl}\Big(\bigcup_{D\in\mathcal{D}} D\Big) &= (\mathcal{P}_i)\mathrm{cl}\Big(\bigcup_{C\in\mathcal{C}} C\Big) \quad \text{(by (3.5))} \\
&= \bigcup_{C\in\mathcal{C}}(\mathcal{P}_i)\mathrm{cl}C \quad \text{(since } \mathcal{B} \text{ is } (\mathcal{P}_i)\text{closure-preserving)} \\
&= \bigcup_{D\in\mathcal{D}}\Big(\bigcup_{C\in\mathcal{C}_D}(\mathcal{P}_i)\mathrm{cl}C\Big) = \bigcup_{D\in\mathcal{D}}(\mathcal{P}_i)\mathrm{cl}\Big(\bigcup_{C\in\mathcal{C}_D} C\Big) \\
&= \bigcup_{D\in\mathcal{D}}(\mathcal{P}_i)\mathrm{cl}D. \quad \blacksquare
\end{aligned}$$

LEMMA 3.4. *If any pairwise open cover $\mathcal{U}$ of X has a pairwise closure-preserving refinement $\mathcal{B}$ satisfying* (3.4)*, then X is $(*)$pairwise normal.*

Proof. Let A be a $(\mathcal{P}_i)$closed set and B be a $(\mathcal{P}_j)$closed set with $A \cap B = \emptyset$, $i \neq j$. Then $\{X - A, X - B\}$ is a pairwise open cover of X. So by Lemma 3.3, there exists a refinement $\{C, D\}$ of $\{X - A, X - B\}$ such that

$$((\mathcal{P}_1)\mathrm{cl}C) \cup ((\mathcal{P}_2)\mathrm{cl}C) \subset X - A$$
$$\text{and } ((\mathcal{P}_1)\mathrm{cl}D) \cup ((\mathcal{P}_2)\mathrm{cl}D) \subset X - B.$$

Then $A \subset X - (\mathcal{P}_i)\mathrm{cl}C$, $B \subset X - (\mathcal{P}_i)\mathrm{cl}D$, $X - (\mathcal{P}_i)\mathrm{cl}C$, $X - (\mathcal{P}_i)\mathrm{cl}D \in \mathcal{P}_i$

$$\text{and} \quad (X - (\mathcal{P}_i)\mathrm{cl}C) \cap (X - (\mathcal{P}_i)\mathrm{cl}D) = \emptyset.$$

Also $A \subset X - (\mathcal{P}_j)\mathrm{cl}C$, $B \subset X - (\mathcal{P}_j)\mathrm{cl}D$, $X - (\mathcal{P}_j)\mathrm{cl}C$, $X - (\mathcal{P}_j)\mathrm{cl}D \in \mathcal{P}_j$

$$\text{and} \quad (X - (\mathcal{P}_j)\mathrm{cl}C) \cap (X - (\mathcal{P}_j)\mathrm{cl}D) = \emptyset.$$

Moreover, $A \subset X - (\mathcal{P}_j)\mathrm{cl}C$, $B \subset X - (\mathcal{P}_i)\mathrm{cl}D$ and

$$(X - (\mathcal{P}_j)\mathrm{cl}C) \cap (X - (\mathcal{P}_i)\mathrm{cl}D) = \emptyset. \quad \blacksquare$$

Lemma 3.5. *Let the space X be $(*)$pairwise normal, $\mathcal{U}$ be a pairwise open cover of X and $\mathcal{V} = \{V_\gamma \mid \gamma \in \Gamma\}$ be a disjoint collection of sets belonging to $\mathcal{P}_1 \cup \mathcal{P}_2$ such that if V_γ is $(\mathcal{P}_i)$open, then it is a subset of a $(\mathcal{P}_i)$open set of $\mathcal{U}$ and let $\mathcal{D} = \{D_\gamma \mid \gamma \in \Gamma\}$ be a collection of subsets of X, which is pairwise closure-preserving and*

$$((\mathcal{P}_1)\mathrm{cl}D_\gamma) \cup ((\mathcal{P}_2)\mathrm{cl}D_\gamma) \subset V_\gamma. \tag{3.6}$$

Then there exists a $\mathcal{U}$-discrete collection $\{W_\gamma \mid \gamma \in \Gamma\}$ of subsets of X such that $D_\gamma \subset W_\gamma \subset V_\gamma$ and W_γ is $(\mathcal{P}_i)$open if V_γ is $(\mathcal{P}_i)$open.

Proof. We write $U_i = \bigcup_{U \in \mathcal{U}_i} U$. Let $S_i = \{x \in U_i \mid$some $(\mathcal{P}_i)$open neighbourhood of x intersects at most one $V_\gamma\}$. Then S_i is $(\mathcal{P}_i)$open and contains all $V \in \mathcal{V}_i$. By (3.6), we get

$$\bigcup_{D \in \mathcal{D}_i} (\mathcal{P}_j)\mathrm{cl}D \subset S_i \implies (\mathcal{P}_j)\mathrm{cl}\Big(\bigcup_{D \in \mathcal{D}_i} D\Big) \subset S_i.$$

Therefore by the $(*)$pairwise normality of X, there exist sets $G_i^1, G_i^2 \in \mathcal{P}_i$ such that

$$(\mathcal{P}_j)\mathrm{cl}\Big(\bigcup_{D \in \mathcal{D}_i} D\Big) \subset G_i^1 \subset (\mathcal{P}_i)\mathrm{cl}G_i^1 \subset S_i,$$
$$(\mathcal{P}_j)\mathrm{cl}\Big(\bigcup_{D \in \mathcal{D}_i} D\Big) \subset G_i^2 \subset (\mathcal{P}_j)\mathrm{cl}G_i^2 \subset S_i,$$

and there exist sets $H_j^1, H_j^2 \in \mathcal{P}_j$ such that

$$(\mathcal{P}_i)\mathrm{cl}\Big(\bigcup_{D \in \mathcal{D}_j} D\Big) \subset H_j^1 \subset (\mathcal{P}_i)\mathrm{cl}H_j^1 \subset S_j,$$
$$(\mathcal{P}_i)\mathrm{cl}\Big(\bigcup_{D \in \mathcal{D}_j} D\Big) \subset H_j^2 \subset (\mathcal{P}_j)\mathrm{cl}H_j^2 \subset S_j.$$

We now have

$$(\mathcal{P}_i)\mathrm{cl}\left(G_i^1 \cup H_j^1\right) \cup (\mathcal{P}_j)\mathrm{cl}\left(G_i^2 \cup H_j^2\right) \subset S_i \cup S_j. \tag{3.7}$$

We write $G_i = G_i^1 \cap G_i^2, H_j = H_j^1 \cap H_j^2$ and,

$$\begin{aligned} W_\gamma &= V_\gamma \cap G_i \text{ if } V_\gamma \in \mathcal{P}_i, \\ &= V_\gamma \cap H_j \text{ if } V_\gamma \in \mathcal{P}_j. \end{aligned}$$

Then $D_\gamma \subset W_\gamma \subset V_\gamma$. Next we show that $\{W_\gamma \mid \gamma \in \Gamma\}$ is $\mathcal{U}$-discrete. Let x belongs to some $(\mathcal{P}_i)$open set of $\mathcal{U}$ i.e. $x \in U_i$. If $x \in S_i$, then there exists a $(\mathcal{P}_i)$open neighbourhood of x, intersecting at most one V_γ and hence intersecting at most one W_γ. If $x \notin S_i \cup S_j$, then by (3.7), $x \notin (\mathcal{P}_i)\mathrm{cl}\big(G_i^1 \cup H_j^1\big)$. Again since $G_i \subset G_i^1$ and $H_j \subset H_j^1$, we have $\bigcup_\gamma W_\gamma \subset G_i^1 \cup H_j^1$. Therefore there exists a $(\mathcal{P}_i)$open neighbourhood of x intersecting none of $\{W_\gamma \mid \gamma \in \Gamma\}$. Also we have $G_i \subset G_i^2$ and $H_j \subset H_j^2$, and so $\bigcup_\gamma W_\gamma \subset G_i^2 \cup H_j^2$. Thus if $x \in U_i \cap U_j$, and $x \notin S_i \cup S_j$, then considering $x \notin (\mathcal{P}_j)\mathrm{cl}\big(G_i^2 \cup H_j^2\big)$, we also get a $(\mathcal{P}_j)$open neighbourhood of x intersecting none of $\{W_\gamma \mid \gamma \in \Gamma\}$. ■

LEMMA 3.6. *Suppose* $\mathcal{U}$ *is a pairwise open cover of the space* X *and* $\{K_\alpha \mid \alpha \in A\}$ *is a* $\mathcal{U}$*-locally finite collection of subsets of* X *and suppose for each* $\alpha \in A$, $\mathcal{B}_\alpha$ *is a pairwise closure-preserving collection of subsets of* K_α *such that each member of* $\mathcal{B}_\alpha$ *is a subset of some set in* $\mathcal{U}$. *Then* $\mathcal{B} = \bigcup\{\mathcal{B}_\alpha \mid \alpha \in A\}$ *is also pairwise closure-preserving.*

Proof. Straightforward. ■

4. The characterizations of pairwise paracompactness

THEOREM 4.1. *If the bitopological space* X *is strongly pairwise regular and satisfies the conditions* $(*)$ *and* $(**)$, *then the following statements are equivalent.*

(a) X *is pairwise paracompact.*

(b) *Each pairwise open cover* $\mathcal{U}$ *of* X *has a hereditarily pairwise closure-preserving parallel refinement.*

(c) *Each pairwise open cover* $\mathcal{U}$ *of* X *has a parallel refinement* $\mathcal{V} = \bigcup_{n=1}^{\infty}\mathcal{V}_n$, *where each* $\mathcal{V}_n$ *is hereditarily pairwise closure-preserving.*

(d) *Each pairwise open cover* $\mathcal{U}$ *of* X *has a pairwise closure-preserving refinement.*

(e) *Each pairwise open cover* $\mathcal{U}$ *of* X *has a pairwise closure-preserving refinement* $\mathcal{B}$ *such that if* $B \subset U \in \mathcal{U}$, $B \in \mathcal{B}$, *then*

$$((\mathcal{P}_2)\mathrm{cl}\,((\mathcal{P}_1)\mathrm{cl}B)) \cup ((\mathcal{P}_1)\mathrm{cl}\,((\mathcal{P}_2)\mathrm{cl}B)) \subset U.$$

Proof. $(a) \Rightarrow (b)$: Follows from Lemma 3.1 and Theorem 2.1.

$(b) \Rightarrow (c)$: Obvious.

$(c) \Rightarrow (d)$: Let $\mathcal{U}$ be a pairwise open cover of X. By (c), $\mathcal{U}$ has a parallel refinement $\mathcal{V} = \bigcup_{n=1}^{\infty}\mathcal{V}_n$, where each $\mathcal{V}_n$ is hereditarily pairwise closure-preserving. Let

$$\begin{aligned} &V_n = \bigcup\{V \mid V \in \mathcal{V}_n\},\ n \in N,\\ &K_1 = X,\\ &K_n = X - \bigcup_{m=1}^{n-1} V_m,\ n = 2, 3, \dots . \end{aligned}$$

Then the class $\{K_n \mid n \in N\}$ is $\mathcal{U}$-locally finite.

We write $\mathcal{B}_n = \{V \cap K_n \mid V \in \mathcal{V}_n\}$, and $\mathcal{B} = \bigcup_{n=1}^{\infty}\mathcal{B}_n$. Then $\mathcal{B}$ is a refinement of $\mathcal{U}$. Since $\mathcal{V}_n$ is hereditarily pairwise closure-preserving, each $\mathcal{B}_n$ is pairwise closure-preserving. Since $\{K_n \mid n \in N\}$ is $\mathcal{U}$-locally finite, from Lemma 3.6, it follows that $\mathcal{B}$ is pairwise closure-preserving.

$(d) \Rightarrow (e)$: By strong pairwise regularity, there is a parallel refinement $\mathcal{V}$ of $\mathcal{U}$ such that for $V \in \mathcal{V}$, there exists a $U \in \mathcal{U}$ with

$$((\mathcal{P}_2)\mathrm{cl}\,((\mathcal{P}_1)\mathrm{cl}V)) \cup ((\mathcal{P}_1)\mathrm{cl}\,((\mathcal{P}_2)\mathrm{cl}V)) \subset U. \tag{4.1}$$

By (d), there is a pairwise closure-preserving refinement $\mathcal{B}$ of $\mathcal{V}$, and hence of $\mathcal{U}$. If $B \in \mathcal{B}$, then for some $V \in \mathcal{V}$ and $U \in \mathcal{U}$ satisfying (4.1), we have $B \subset V$ and so

$$((\mathcal{P}_2)\mathrm{cl}\,((\mathcal{P}_1)\mathrm{cl}B)) \cup ((\mathcal{P}_1)\mathrm{cl}\,((\mathcal{P}_2)\mathrm{cl}B)) \subset U.$$

$(e) \Rightarrow (a)$: Let $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ be a pairwise open cover of X and let the index set A be well-ordered. For each positive integer n, we construct a family $\mathcal{B}_n = \{B_{\alpha,n} \mid \alpha \in A\}$ of subsets of X satisfying the following conditions for all n:

(I) $\mathcal{B}_n = \{B_{\alpha,n} \mid \alpha \in A\}$ is a pairwise closure-preserving cover of X, and

$$((\mathcal{P}_2)\mathrm{cl}\,((\mathcal{P}_1)\mathrm{cl}B_{\alpha,n})) \cup ((\mathcal{P}_1)\mathrm{cl}\,((\mathcal{P}_2)\mathrm{cl}B_{\alpha,n})) \subset U_\alpha \text{ for all } \alpha.$$

(II) $((\mathcal{P}_i)\mathrm{cl}(\mathrm{cl}B_{\alpha,n+1})) \cap ((\mathcal{P}_i)\mathrm{cl}(\mathrm{cl}B_{\beta,n})) = \emptyset$ for all $\alpha > \beta$ if $U_\alpha \in \mathcal{P}_i$.

For $n = 1$, the cover can be obtained from Lemma 3.3.

Suppose for $n = 1, 2, \ldots, m$, the covers $\mathcal{B}_n$ have been constructed. For $U_\alpha \in \mathcal{P}_i$, we write

$$K_{\alpha,m} = \bigcup\nolimits_{\beta<\alpha} \{(\mathcal{P}_i)\mathrm{cl}(\mathrm{cl}B_{\beta,m})\}.$$

Since $\mathcal{B}_m$ is pairwise closure-preserving, by Lemma 3.2, it follows that the set $K_{\alpha,m}$ is $(\mathcal{P}_i)$closed. So the set $U_{\alpha,m+1} = U_\alpha - K_{\alpha,m}$ is $(\mathcal{P}_i)$open. If $x \in X$, then $x \in U_{\alpha,m+1}$ for the first α for which $x \in U_\alpha$. Therefore the collection $\mathcal{U}_{m+1} = \{U_{\alpha,m+1} \mid \alpha \in A\}$ forms a refinement of $\mathcal{U}$. By Lemma 3.3, it has a pairwise closure-preserving refinement $\{B_{\alpha,m+1} \mid \alpha \in A\}$ such that

$$((\mathcal{P}_2)\mathrm{cl}\,((\mathcal{P}_1)\mathrm{cl}B_{\alpha,m+1})) \cup ((\mathcal{P}_1)\mathrm{cl}\,((\mathcal{P}_2)\mathrm{cl}B_{\alpha,m+1})) \subset U_{\alpha,m+1} \text{ for all } \alpha. \tag{4.2}$$

Therefore the condition (I) is satisfied for $n = m+1$. From (4.2) and the definition of $U_{\alpha,m+1}$, it follows that (II) is satisfied for $n = m$. If $U_\alpha \in \mathcal{P}_i$, we define

$$V_{\alpha,n} = X - \bigcup\nolimits_{\beta\neq\alpha} \{(\mathcal{P}_i)\mathrm{cl}(\mathrm{cl}B_{\beta,n})\}.$$

We show that

(III) $\{V_{\alpha,n} \mid \alpha \in A, n \in N\}$ is a pairwise open cover of X such that for all $\alpha \in A$ and $n \in N$, $V_{\alpha,n} \subset U_\alpha$ and $V_{\alpha,n}$ is $(\mathcal{P}_i)$open if U_α is $(\mathcal{P}_i)$open.

(IV) $V_{\alpha,n} \cap V_{\beta,n} = \emptyset$ whenever $\alpha \neq \beta$.

Since $\mathcal{B}_n$ is pairwise closure-preserving, it follows that $V_{\alpha,n}$ is $(\mathcal{P}_i)$open. Also we have $V_{\alpha,n} \subset B_{\alpha,n} \subset U_\alpha$ for all $\alpha \in A$ and $n \in N$. Therefore from the definition of $V_{\alpha,n}$, (IV) follows. We consider a point $x \in X$. If $x \in \mathcal{U} \cap \mathcal{P}_i$, we define

$$\alpha_n = \min\{\alpha \in A \mid x \in (\mathcal{P}_i)\mathrm{cl}(\mathrm{cl}B_{\alpha,n}), n \in N\},$$

and $\alpha_l = \min\{\alpha_n \mid n \in N\}$. If $\alpha > \alpha_l$, from (II) we get

$$((\mathcal{P}_i)\mathrm{cl}(\mathrm{cl}B_{\alpha,l+1})) \cap ((\mathcal{P}_i)\mathrm{cl}(\mathrm{cl}B_{\alpha_l,l})) = \emptyset,$$

and therefore $x \notin (\mathcal{P}_i)\mathrm{cl}(\mathrm{cl}B_{\alpha,l+1})$, since $x \in (\mathcal{P}_i)\mathrm{cl}(\mathrm{cl}B_{\alpha_l,l})$. Also by the definition of α_l, $x \notin (\mathcal{P}_i)\mathrm{cl}(\mathrm{cl}B_{\alpha,l+1})$ for $\alpha < \alpha_l$. Therefore $x \in V_{\alpha_l,l+1}$. Thus the collection

$\mathcal{V} = \{V_{\alpha,n} \mid \alpha \in A, n \in N\}$ forms a pairwise open cover of X. By Lemma 3.3, we get a pairwise closure-preserving cover $\mathcal{D} = \{D_{\alpha,n} \mid \alpha \in A, n \in N\}$ of X such that

$$((\mathcal{P}_1)\mathrm{cl}D_{\alpha,n}) \cup ((\mathcal{P}_2)\mathrm{cl}D_{\alpha,n}) \subset V_{\alpha,n}$$

for all α and n. By Lemma 3.4, X is $(*)$pairwise normal and so applying Lemma 3.5, for each n, we get a $\mathcal{U}$-discrete collection $\mathcal{W}_n = \{W_{\alpha,n} \mid \alpha \in A\}$ such that $W_{\alpha,n}$ is $(\mathcal{P}_i)$open if $V_{\alpha,n}$ is $(\mathcal{P}_i)$open and

$$D_{\alpha,n} \subset W_{\alpha,n} \subset V_{\alpha,n}$$

for all α. Then the collection $\mathcal{W} = \bigcup_{n=1}^{\infty} \mathcal{W}_n$ is a parallel refinement of $\mathcal{U}$ where each $\mathcal{W}_n$ is $\mathcal{U}$-discrete and hence $\mathcal{U}$-locally finite. Therefore by Theorem 2.1, X is pairwise paracompact. ∎

5. Some examples

In this section, $\mathcal{T}$ denotes the usual topology on R, and for a set $A \subset R, \mathcal{T}_A$ denotes the subspace topology on A in $(R, \mathcal{T})$. Firstly we give an example of a strongly pairwise regular pairwise paracompact space.

Example 5.1. Let Q be the set of rational numbers. If $\mathcal{E}^1$ is the collection of the singleton sets $\{r\}, r \in Q$ and $\mathcal{E}^2$ is the collection of the singleton sets $\{r\}, r \in R - Q$, then for $i = 1, 2$, we define $\mathcal{P}_i$ to be the topology generated by the base $\mathcal{T} \cup \mathcal{E}^i$. Then the topological spaces $(R, \mathcal{P}_i)$ are regular (Steen and Seebach [7, p. 90]). We now consider the bitopological space $(R, \mathcal{P}_1, \mathcal{P}_2)$. Let F be a $(\mathcal{P}_i)$closed set and $x \in R - F \in \mathcal{P}_i$. If x belongs to some $(\mathcal{T})$open set, then there exist a $(\mathcal{T})$open set U and a $(\mathcal{T})$open set V such that $x \in U, F \subset V$ and $U \cap V = \emptyset$. If $x \in \{r\} = U \in \mathcal{E}^i$, then $F \subset R - \{r\} = V$. So in any case $x \in U \in \mathcal{P}_i$ and $F \subset V \in \mathcal{P}_j$ and $U \cap V = \emptyset$. Thus $(R, \mathcal{P}_1, \mathcal{P}_2)$ is pairwise regular and hence strongly pairwise regular. Now let $\mathcal{U}$ be a pairwise open cover of R. Then there exists a parallel refinement $\mathcal{V}$ containing sets belonging to $\mathcal{T}$ and sets belonging to $\mathcal{E}^1 \cup \mathcal{E}^2$. We may assume that no element of $\mathcal{V} \cap (\mathcal{E}^1 \cup \mathcal{E}^2)$ belongs to any element of $\mathcal{V} \cap \mathcal{T}$, since otherwise we can delete the corresponding singleton sets from $\mathcal{V}$. Let $V = \bigcup\{G \in \mathcal{V} \cap \mathcal{T}\}$. The $(\mathcal{T}_V)$open cover $\{G \in \mathcal{V} \cap \mathcal{T}\}$ of the subspace $(V, \mathcal{T}_V)$ has a $(\mathcal{T}_V)$locally finite $(\mathcal{T}_V)$open refinement $\mathcal{W}$. Let $\mathcal{E}^i_{\mathcal{V}} = \{\{r\} \in \mathcal{E}^i \cap \mathcal{V}\}$. If $x \notin V$, then $x \in \bigcup\{\{r\} \in \mathcal{E}^1_{\mathcal{V}} \cup \mathcal{E}^2_{\mathcal{V}}\}$ and $\{x\}$ can intersect only $\{x\} \in \mathcal{V}$. Again no element of $\mathcal{W}$ can intersect any element of $\mathcal{E}^1_{\mathcal{V}} \cup \mathcal{E}^2_{\mathcal{V}}$. Thus it follows that $\mathcal{W} \cup \mathcal{E}^1_{\mathcal{V}} \cup \mathcal{E}^2_{\mathcal{V}}$ is a $\mathcal{U}$-locally finite parallel refinement of $\mathcal{U}$. Therefore $(R, \mathcal{P}_1, \mathcal{P}_2)$ is pairwise paracompact.

Now we give an example of a bitopological space satisfying both the conditions $(*)$ and $(**)$.

Example 5.2. For each $i = 1, 2$, let $\{a^i_n\}_{n=1}^{\infty}$ be a strictly decreasing sequence of real numbers with $\lim a^i_n = -\infty$ and $\{b^i_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of real numbers with $\lim b^i_n = \infty$ such that $a^i_1 < b^i_1$. Let $\mathcal{P}_i$ be the topology on R generated by the base

$$\mathcal{B}_i = \{\emptyset\} \cup \{(a^i_1, b^i_1)\} \cup \{\, (a^i_{n+1}, a^i_n), (b^i_n, b^i_{n+1}) \mid n \in N \,\} \cup \{\, \{a^i_n\}, \{b^i_n\} \mid n \in N \,\}.$$

Then each $(\mathcal{P}_i)$open set is $(\mathcal{P}_i)$closed. Therefore the bitopological space $(R, \mathcal{P}_1, \mathcal{P}_2)$ satisfies both the conditions $(*)$ and $(**)$. Obviously the space $(R, \mathcal{P}_1, \mathcal{P}_2)$ is pairwise paracompact.

In the space considered above, for $i = 1, 2$, any collection of subsets of R is $(\mathcal{P}_i)$closure-preserving. Next we give an example of a bitopological space $(X, \mathcal{P}_1, \mathcal{P}_2)$ in which for $i = 1, 2$, there are collections of sets which are not $(\mathcal{P}_i)$closure-preserving, but if a collection is $(\mathcal{P}_1)$closure-preserving, then it is $(\mathcal{P}_2)$closure-preserving, and conversely.

Example 5.3. Let $a \in R$ and let us consider the infinite intervals $(-\infty, a]$ and (a, ∞). We write $A = (-\infty, a]$. Suppose $\{b_n^1\}_{n=1}^{\infty}$ and $\{b_n^2\}_{n=1}^{\infty}$ are two strictly increasing sequences of real numbers with $a = b_1^i$ and $\lim b_n^i = \infty$ for $i = 1, 2$. Let $\mathcal{P}_i$ be the topology on R generated by the base

$$\mathcal{B}_i = \mathcal{T}_A \cup \{\, (b_n^i, b_{n+1}^i] \mid n \in N \,\}.$$

We now consider a $(\mathcal{P}_i)$closure-preserving collection $\mathcal{A}$ of subsets of R. Let $\mathcal{D}$ be a subcollection of $\mathcal{A}$. We write

$$\begin{aligned}
\mathcal{D}_1 &= \{D \in \mathcal{D} \mid D \subset (-\infty, a]\},\\
\mathcal{D}_2 &= \{D \in \mathcal{D} \mid D \subset (a, \infty)\},\\
\mathcal{D}^a &= \{D \in \mathcal{D} \mid D \cap (-\infty, a] \neq \emptyset, D \cap (a, \infty) \neq \emptyset\},\\
\mathcal{D}_1^a &= \{D \cap (-\infty, a] \mid D \in \mathcal{D}^a\},\\
\mathcal{D}_2^a &= \{D \cap (a, \infty) \mid D \in \mathcal{D}^a\}.
\end{aligned}$$

Then

$$\begin{aligned}
&(\mathcal{P}_j)\mathrm{cl}\Big(\bigcup\{D \in \mathcal{D}\}\Big)\\
&\quad = (\mathcal{P}_j)\mathrm{cl}\Big(\bigcup\{D \in \mathcal{D}_1\}\Big) \cup (\mathcal{P}_j)\mathrm{cl}\Big(\bigcup\{D \in \mathcal{D}_2\}\Big) \cup (\mathcal{P}_j)\mathrm{cl}\Big(\bigcup\{D \in \mathcal{D}^a\}\Big)\\
&\quad = (\mathcal{P}_j)\mathrm{cl}\Big(\bigcup\{D \in \mathcal{D}_1\}\Big) \cup (\mathcal{P}_j)\mathrm{cl}\Big(\bigcup\{D \in \mathcal{D}_2\}\Big) \cup (P_j)\mathrm{cl}\Big(\bigcup\{D \in \mathcal{D}_1^a\}\Big)\\
&\qquad\qquad \cup (P_j)\mathrm{cl}\Big(\bigcup\{D \in \mathcal{D}_2^a\}\Big). \qquad (5.1)
\end{aligned}$$

The $(\mathcal{P}_j)$closure of any set contained in $(-\infty, a]$, is identical with its $(\mathcal{P}_i)$closure, and any collection of sets contained in (a, ∞) are both $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$closure-preserving. Since $\mathcal{A}$ is $(\mathcal{P}_i)$closure-preserving, it follows from (5.1) that $\mathcal{A}$ is $(\mathcal{P}_j)$closure-preserving. Thus the bitopological space $(R, \mathcal{P}_1, \mathcal{P}_2)$ satisfies the condition $(**)$. It is also clear that the bitopological space $(R, \mathcal{P}_1, \mathcal{P}_2)$ is pairwise paracompact.

Note 5.4. The space $(R, \mathcal{P}_1, \mathcal{P}_2)$ of Example 5.3, does not satisfy the condition $(*)$ but satisfies a slightly weaker condition. To explain this, let us consider a pairwise open cover $\mathcal{U}$ of R containing only one $(\mathcal{P}_1)$open set and suppose it is of the form $(\alpha, \beta) \cup (b_n^1, b_{n+1}^1]$ such that $(\alpha_1, \alpha_2) \cup (b_m^1, b_{m+1}^1]$ is the only $(\mathcal{P}_2)$open set belonging to $\mathcal{U}$ with $\alpha_1 < \alpha < \alpha_2 \leq a$. Then $\mathcal{U}$ does not satisfy (2.1). Hence the space $(R, \mathcal{P}_1, \mathcal{P}_2)$ does not satisfy the condition $(*)$. But replacing the sets of type $G \cup (\bigcup_{n\in N_0}(b^i_n, b^i_{n+1}])$ by the sets G and $\bigcup_{n\in N_0}(b^i_n, b^i_{n+1}]$, where $G \in \mathcal{T}_A$ and $N_0 \subset N$, we can have a parallel refinement $\mathcal{U}_0$ of $\mathcal{U}$ such that $\mathcal{U}_0$ satisfies (2.1). It is clear from the context that it is sufficient to have (2.1) satisfied by a parallel refinement of $\mathcal{U}$. So we can relax the condition $(*)$ in this manner. In that case the Lemma 3.1 is required to change slightly according to our requirements.

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Department of Mathematics, University of North Bengal, Siliguri, W. Bengal-734013, India

E-mail: manojkumarbose@yahoo.com

Department of Mathematics, St. Joseph's College, North Point, Darjeeling, W. Bengal-734104, India

E-mail: ajoyjee@yahoo.com