МАТЕМАТИЧКИ ВЕСНИК
61 (2009), 3–11

UDK 515.122
оригинални научни рад
research paper

SOME COVERING PROPERTIES FOR Ψ-SPACES

Maddalena Bonanzinga and Mikhail Matveev

Abstract. Menger, Hurewicz and Rothberger properties restricted to partitions into clopen sets and to covers by stars are considered. A question is discussed when a Ψ-space satisfies some of these properties.

1. Introduction

Recall (see for example [12], [13], [15], [16]) that a space X is:

Rothberger if for each sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of X one can pick $U_n \in \mathcal{U}_n$ so that $\{U_n : n \in \omega\}$ covers X;

Menger if for each sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of X one can pick finite $\mathcal{V}_n \subset \mathcal{U}_n$ so that $\bigcup_{n\in\omega} \mathcal{V}_n$ covers X;

Hurewicz if for each sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of X one can pick finite $\mathcal{V}_n \subset \mathcal{U}_n$ so that for each $x \in X$, $x \in \bigcup \mathcal{V}_n$ for all but finitely many n.

Below we use the abbreviations R, M, H for Rothberger, Menger and Hurewicz properties, respectively. One of our purposes in this paper is to consider these three properties restricted to partitions into clopen sets. We will see that these restricted versions of R, M, H properties may hold in zero-dimensional spaces that are very far from being Lindelöf (while R, M, H are in fact stronger forms of the Lindelöf property). Say that a space X is:

parR (*partition-Rothberger*) if for every sequence $(\mathcal{P}_n : n \in \omega)$ of partitions of X into clopen sets one can pick $V_n \in \mathcal{P}_n$, so that $\{V_n : n \in \omega\}$ covers X;

parM (*partition-Menger*) if for every sequence $(\mathcal{P}_n : n \in \omega)$ of partitions of X into clopen sets one can pick finite $\mathcal{V}_n \subset \mathcal{P}_n$, so that $\bigcup_{n\in\omega} \mathcal{V}_n$ covers X;

parH (*partition-Hurewicz*) if for every sequence $(\mathcal{P}_n : n \in \omega)$ of partitions of X into clopen sets one can pick finite $\mathcal{V}_n \subset \mathcal{P}_n$, so that for each $x \in X$, $x \in \bigcup \mathcal{V}_n$ for all but finitely many n.

AMS Subject Classification: 54D20.

Keywords and phrases: (Strongly) star-Rothberger, (strongly) star-Menger, (strongly) star-Hurewicz, partition-Rothberger, partition-Menger, partition-Hurewicz, Ψ-space.

Presented at the international conference *Analysis, Topology and Applications 2008 (ATA2008)*, held in Vrnjačka Banja, Serbia, from May 30 to June 4, 2008.

Recall that a family of sets is *almost disjoint* (a.d., for short) if the intersection of any two distinct elements is finite. Let $\mathcal{A}$ be an a.d. family of infinite subsets of ω. Put $\Psi(\mathcal{A}) = \mathcal{A} \cup \omega$ and topologize $\Psi(\mathcal{A})$ as follows: the points of ω are isolated and a basic neighbourhood of a point $a \in \mathcal{A}$ takes the form $\{a\} \cup (a \setminus F)$, where F is a finite set. $\Psi(\mathcal{A})$ is called a Ψ-*space* (see [8]). It is well known that $\mathcal{A}$ is a *maximal* almost disjoint family (m.a.d. family, for short) iff $\Psi(\mathcal{A})$ is pseudocompact. In general, when talking about Ψ spaces we will not assume the a.d. family to be maximal or the space pseudocompact. We will see that Ψ spaces help to distinguish (some of) parR, parM, parH properties and some properties defined in terms of stars.

First of all, parR, parM, parH properties can be interpreted in terms of ∞-*stars*: let X be a space, $\mathcal{U}$ be an open cover of X, $A \subset X$ and $n \in \omega$. Then $St(A,\mathcal{U}) = St^1(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$ and, inductively, $St^{n+1}(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap St^n(A,\mathcal{U}) \neq \emptyset\}$. Next, $St^\infty(A,\mathcal{U}) = \bigcup_{n\in\omega} St^n(A,\mathcal{U})$. Stars of level infinity with respect to an open cover provide a partition of the space into clopen sets (the classes of equivalence with respect to the relation given by $x \sim y$ iff $y \in St^\infty(\{x\},\mathcal{U})$, see, for example, [7], Lemma 5.3.8). On the other hand, a partition into clopen sets can be viewed as a cover by infinite stars with respect to itself. Thus, for example, a space X is parM iff for every sequence of $(\mathcal{U}_n : n \in \omega)$ of open covers one can pick finite $A_n \subset X$ so that $\{St^\infty(A_n,\mathcal{U}_n) : n \in \omega\}$ covers X.

As the stars of level infinity are considered, it is natural to consider also stars of lower level. In Section 2 we do it for the case opposite to infinity: stars of level 1. The following properties were introduced by Kočinac in [10] and by Bonanzinga, Cammaroto and Kočinac in [4]. A space X is:

SR (*star-Rothberger*) if for every sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of X one can pick $U_n \in \mathcal{U}_n$ so that $\{St(U_n,\mathcal{U}_n) : n \in \omega\}$ covers X [10];

SM (*star-Menger*) if for every sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of X one can pick finite $\mathcal{V}_n \subset \mathcal{U}_n$ so that $\bigcup_{n\in\omega}\{St(V,\mathcal{U}_n) : V \in \mathcal{V}_n\}$ covers X [10];

SH (*star-Hurewicz*) if for every sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of X one can pick finite $\mathcal{V}_n \subset \mathcal{U}_n$ so that for every $x \in X$, $x \in St(\bigcup \mathcal{V}_n,\mathcal{U}_n)$ for all but finitely many n [4];

SSR (*strongly star-Rothberger*) if for every sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of X one can pick $x_n \in X$ so that $\{St(\{x_n\},\mathcal{U}_n) : n \in \omega\}$ covers X [10];

SSM (*strongly star-Menger*) if for every sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of X one can pick finite $A_n \subset X$ so that $\{St(A_n,\mathcal{U}_n) : n \in \omega\}$ covers X [10];

SSH (*strongly star-Hurewicz*) if for every sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of X one can pick finite $A_n \subset X$ so that each $x \in X$, $x \in St(A_n,\mathcal{U}_n)$ for all but finitely many n [4].

The following diagram shows the obvious implications between the properties we are going to discuss.

$$\begin{tikzcd}
\mathsf{R} \arrow[r] \arrow[d] & \mathsf{M} \arrow[d] & \mathsf{H} \arrow[l] \arrow[d] \\
\mathsf{SSR} \arrow[r] \arrow[d] & \mathsf{SSM} \arrow[d] & \mathsf{SSH} \arrow[l] \arrow[d] \\
\mathsf{SR} \arrow[r] \arrow[d] & \mathsf{SM} \arrow[d] & \mathsf{SH} \arrow[l] \arrow[d] \\
\mathsf{parR} \arrow[r] & \mathsf{parM} & \mathsf{parH} \arrow[l]
\end{tikzcd}$$

We will see that for Ψ-spaces, having star properties in this diagram depends mostly on cardinality while having partition properties depends also on the topology (i.e. the choice of particular a.d. family).

By a space we usually mean a Hausdorff topological space. In notation we follow [7].

Recall that for $f, g \in \omega^\omega$, $f \leq^* g$ means that $f(n) \leq g(n)$ for all but finitely many n (and $f \leq g$ means that $f(n) \leq g(n)$ for all n). A subset B of ω^ω is *bounded* if there is $g \in \omega^\omega$ such that $f \leq^* g$ for each $f \in B$. $D \subset \omega^\omega$ is *dominating* if for each $g \in \omega^\omega$ there is $f \in D$ such that $g \leq^* f$. The minimal cardinality of an unbounded subset of ω^ω is denoted by $\mathfrak{b}$, and the minimal cardinality of a dominating subset of ω^ω is denoted by $\mathfrak{d}$. The value of $\mathfrak{d}$ does not change if one considers the relation $\leq$ instead of $\leq^*$ ([5], Theorem 3.6). $\mathcal{M}$ denotes the family of all meager subsets of $\mathbb{R}$. $cov(\mathcal{M})$ is the minimum of the cardinalities of subfamilies $\mathcal{U} \subset \mathcal{M}$ such that $\bigcup \mathcal{U} = \mathbb{R}$. However, we will need another description of the cardinal $cov(\mathcal{M})$.

Theorem 1. ([1], Theorem 2.4.1 in [2]) $cov(\mathcal{M})$ *is the minimum cardinality of a family $F \subset \omega^\omega$ such that for every $g \in \omega^\omega$ there is $f \in F$ such that $f(n) \neq g(n)$ for all but finitely many n.*

Thus if $F \subset \omega^\omega$ and $|F| < cov(\mathcal{M})$, then there is $g \in \omega^\omega$ such that for every $f \in F$, $f(n) = g(n)$ for infinitely many n; it is often said that g *guesses* F.

2. Some star covering properties for Ψ-spaces

First, we consider strong forms of star properties. Let $X = \Psi(\mathcal{A})$ be a Ψ-space generated by an almost disjoint family $\mathcal{A}$.

Proposition 2. *The following conditions are equivalent:*

(1) $\Psi(\mathcal{A})$ *is SSM.* (2) $|\mathcal{A}| < \mathfrak{d}$.

Proof. Let $X = \Psi(\mathcal{A})$.

(1) ⇒ (2) Towards a contradiction assume $\mathcal{A} \geq \mathfrak{d}$. Let $\mathcal{F} = \{f_\alpha : \alpha < \mathfrak{d}\} \subset \omega^\omega$ be a family of functions which is dominating in ω^ω in the sense of $\leq$, i.e. for every

$f \in \omega^\omega$ there is f_α such that $f_\alpha(n) \geq f(n)$ for all $n \in \omega$. Choose distinct points $p_{\alpha,\beta} \in \mathcal{A}$ for all $\alpha, \beta < \mathfrak{d}$. Put $P = \{p_{\alpha,\beta} : \alpha, \beta < \mathfrak{d}\}$. For every $\alpha, \beta < \mathfrak{d}$ and $n \in \omega$ put $O_n(p_{\alpha,\beta}) = \{p_{\alpha,\beta}\} \cup \{m \in \omega : m > f_\alpha(n)\}$. Further, put $\mathcal{U}_n = \{O_n(p_{\alpha,\beta}) : \alpha, \beta < \mathfrak{d}\} \cup \{X \setminus P\}$. Then $\mathcal{U}_n$ is an open cover of X. The sequence $(\mathcal{U}_n : n \in \omega)$ witnesses that X is not SSM. Indeed, let $(K_n : n \in \omega)$ be a sequence of finite subsets of X. For $n \in \omega$, put

$$f^\star(n) = \begin{cases} \max(K_n \cap \omega) + 1, & \text{if } K_n \cap \omega \neq \emptyset \\ 1 & \text{otherwise.} \end{cases}$$

There is $\alpha < \mathfrak{d}$ such that $f_\alpha(n) \geq f^\star(n)$ for every $n \in \omega$. Further, there is $\beta < \mathfrak{d}$ such that $p_{\alpha,\beta} \notin \bigcup\{K_n : n \in \omega\}$. Then we have $O_n(p_{\alpha,\beta}) \cap K_n = \emptyset$ for every $n \in \omega$. Since $O_n(p_{\alpha,\beta})$ is the only element of $\mathcal{U}_n$ that contains the point $p_{\alpha,\beta}$, we conclude that $p_{\alpha,\beta} \notin \bigcup\{St(K_n, \mathcal{U}_n) : n \in \omega\}$.

(2) $\Rightarrow$ (1) Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. For each $n \in \omega$ and $a \in \mathcal{A}$, pick an element $U_{n,a}$ of $\mathcal{U}_n$ that contains a. For each $a \in \mathcal{A}$ define a function $g_a \in \omega^\omega$ by $g_a(n) = \min\{k \in \omega : k \in U_{n,a}\}$. Since $|\mathcal{A}| < \mathfrak{d}$, the set of functions $\{g_a : a \in \mathcal{A}\}$ is not dominating; so there is an $f^\star \in \omega^\omega$ such that $f^\star \not\leq^* g_a$ for every $a \in \mathcal{A}$. Put $A_n = [0, \max\{f^\star(n), n\}]$. Then $\{St(A_n, \mathcal{U}_n) : n \in \omega\}$ covers X. ∎

Proposition 3. *The following conditions are equivalent:*

(1) $\Psi(\mathcal{A})$ *is SSH.* (2) $|\mathcal{A}| < \mathfrak{b}$.

Proof. The proof is similar to the proof of Proposition 2. When proving (1) $\Rightarrow$ (2) we assume $\mathcal{A} \geq \mathfrak{b}$ and consider an unbounded family of functions $\mathcal{F} = \{f_\alpha : \alpha < \mathfrak{b}\} \subset \omega^\omega$.

In (2) $\Rightarrow$ (1) we use the fact that the family of functions $\{g_a : a \in \mathcal{A}\}$ is bounded by some function $f^\star$. ∎

Proposition 4. *If* $|\mathcal{A}| < cov(\mathcal{M})$, *then* $\Psi(\mathcal{A})$ *is* SSR.

Proof. The proof is similar to (2) $\Rightarrow$ (1) in the proof of Proposition 2. We put $g_a(n) = \min\{k \in \omega : k \in U_{2n,a}\}$, apply Theorem 1 to get the function $f^\star$ that guesses the family $\{g_a : a \in \mathcal{A}\}$, and put $x_{2n} = f^\star(n)$ and $x_{2n+1} = n$. ∎

Example 5. There is an a.d. family $\mathcal{A}$ of cardinality $cov(\mathcal{M})$ such that $\Psi(\mathcal{A})$ is not SSR.

Let $X \subset \omega^\omega$ be such that $|X| = cov(\mathcal{M})$ and X can not be guessed in the strong sense, that is for every $g \in \omega^\omega$ there is $f \in X$ such that $f(n) \neq g(n)$ for every n. Let $\mathcal{E}$ be the topology on X inherited from the Tychonoff product topology on ω^ω, and let $D \subset X$ be a dense countable subspace. Put $A = X \setminus D$, and for every $a \in A$, fix a discrete subset $D_a \subset D$ such that $\overline{D_a} = D_a \cup \{a\}$. Consider a finer topology $\mathcal{T} \supset \mathcal{E}$ in which points of D are isolated, and a basic neighborhood of a point $a \in A$ takes the form $\{a\} \cup (D_a \setminus F)$ where F is finite. Then $(X, \mathcal{T})$ is in fact a Ψ-space in which D and A take the role of ω and $\mathcal{A}$, respectively.

Since X cannot be guessed in strong sense, the space $(X,\mathcal{E})$ is not Rothberger. It is easy to see that for paracompact spaces $\mathsf{R} \Leftrightarrow \mathsf{SSR}$ (indeed, a space is paracompact iff every open cover has an open star-refinement, see [7], Theorem 5.1.12), so $(X,\mathcal{E})$ is not SSR, and therefore neither is the Ψ-space $(X,\mathcal{T})$. ■

Our next aim is to compare "strong" and "regular" forms of star covering properties for Ψ-spaces.

Question 6. Are properties SSM and SM equivalent for Ψ-spaces?

To provide a partial solution, we need some notation. Let K be a set such that $\omega \le |K| \le \mathfrak{c}$. Denote $\mathcal{F}_K$ the set of all functions from ω to $[K]^{<\omega}$. For $f,g \in \mathcal{F}_K$ we write $f \le g$ provided $f(n) \subseteq g(n)$ for every $n \in \omega$. Say that $\mathcal{F} \subset \mathcal{F}_K$ is *cofinal* in $\mathcal{F}_K$ provided for every $f \in \mathcal{F}_K$ there is $g \in \mathcal{F}$ such that $f \le g$. For a cardinal k such that $\omega \le k \le \mathfrak{c}$ put

$$\mathfrak{d}_k = \min\{\, |\mathcal{F}| : \mathcal{F} \text{ is cofinal in } \mathcal{F}_k \,\}.$$

Then obviously $\min\{|\mathcal{F}| : \mathcal{F} \text{ is cofinal in } \mathcal{F}_K\} = \mathfrak{d}_{|K|}$.

Lemma 7. (0) $\mathfrak{d}_0 = \mathfrak{d}$.

(1) *If* $\omega \le k \le \mathfrak{c}$ *then* $\max\{\mathfrak{d},k\} \le \mathfrak{d}_k \le \mathfrak{c}$.

(2) *If* $\omega \le k < \aleph_\omega$, *then* $\mathfrak{d}_k = \max\{\mathfrak{d},k\}$.

(3) $\mathfrak{d}_\mathfrak{c} = \mathfrak{c}$.

Proof. (0) and the first inequality in (1) are obvious. The second inequality in (1) follows from (3), and (3) follows from the obvious equality $|\mathcal{F}_\mathfrak{c}| = \mathfrak{c}$.

We prove (2) by induction on n where $k = \aleph_n$ and $0 \le n < \omega$. The case $n = 0$ follows from (0), so let $0 < n = n^* + 1 < \omega$ and suppose the fact has been proved for all $m < n$. Since $\mathrm{cf}(\aleph_n) > \omega$, $\mathcal{F}_{\aleph_n} = \bigcup\{\mathcal{F}_\alpha : 0 \le \alpha < \aleph_n\}$. By the inductive assumption for every α there is a subfamily $\mathcal{G}_\alpha \subset \mathcal{F}_\alpha$ such that $\mathcal{G}_\alpha$ is cofinal in $\mathcal{F}_\alpha$ and $|\mathcal{G}_\alpha| = \max\{\mathfrak{d},|\alpha|\} \le \max\{\mathfrak{d},\aleph_{n^*}\}$. Put $\mathcal{G} = \bigcup\{\mathcal{G}_\alpha : 0 \le \alpha < \aleph_n\}$. Then $\mathcal{G}$ is cofinal in $\mathcal{F}_{\aleph_n}$ and $|\mathcal{G}| \le \aleph_n \cdot \max\{\mathfrak{d},\aleph_{n^*}\} = \max\{\mathfrak{d},\aleph_n\}$. ■

Question 8. Is it true that $\mathfrak{d}_k = \max\{\mathfrak{d},k\}$ for every k such that $\omega \le k \le \mathfrak{c}$? In particular, is it true that $\mathfrak{d}_k = \mathfrak{d}$ for every k such that $\omega \le k \le \mathfrak{d}$?[1]

Proposition 9. *If* $|\mathcal{A}| = k$ *and* $\mathfrak{d}_k = k$, *then* $X = \Psi(\mathcal{A})$ *is not* SM.

Proof. Since $\mathfrak{d}_k = k$, $\mathcal{F}_\mathcal{A}$ contains a cofinal subset $\mathcal{F}$ of cardinality k; let $\mathcal{F} = \{f_\alpha : \alpha < k\}$. For $\alpha < k$, $\bigcup f_\alpha(\omega)$ is a countable subset of $\mathcal{A}$. Since $|\mathcal{A}| = k \ge \mathfrak{d}$, for all $\alpha < k$ and $\beta < \mathfrak{d}$ we can pick distinct points $p_{\alpha,\beta} \in \mathcal{A} \setminus \bigcup f_\alpha(\omega)$. Put $P = \{p_{\alpha,\beta} : \alpha < k, \beta < \mathfrak{d}\}$. Let $\mathcal{G} = \{g_\beta : \beta < \mathfrak{d}\} \subset \omega^\omega$ be dominating (in the sense of $\le$). For $n \in \omega$, $\alpha < k$ and $\beta < \mathfrak{d}$ put $H_{n,\alpha,\beta} = \bigcup\{\bigcup f_\alpha(i) : 0 \le i \le n\} \cup [0, \max g_\beta([0,n])]$. Then $H_{n,\alpha,\beta} \subset \omega$, and $H_{n,\alpha,\beta}$ has at most finite intersection with the set $p_{\alpha,\beta}$.

[1] One of the referees forwarded the authors the following information: Both questions were recently answered in the negative by C. Chis, M. Ferrer, S. Hernandez and B. Tsaban, in their forthcoming paper *Bounded sets in topological groups*.

Put $O_n(p_{\alpha,\beta}) = \{p_{\alpha,\beta}\} \cup (p_{\alpha,\beta} \setminus H_{n,\alpha,\beta})$. For $n \in \omega$, put $\mathcal{U}_n = \{O_n(p_{\alpha,\beta}) : \alpha < k, \beta < \mathfrak{d}\} \cup \{\{p\} \cup p : p \in \mathcal{A} \setminus P\} \cup \{\{m\} : m \in \omega\}$; then $\mathcal{U}_n$ is an open cover of X. The sequence $(\mathcal{U}_n : n \in \omega)$ witnesses that X is not SM. Indeed, let $(\mathcal{V}_n : n \in \omega)$ be a sequence of finite subfamilies $\mathcal{V}_n \subset \mathcal{U}_n$. For $n \in \omega$ put $H_n = (\bigcup \mathcal{V}_n) \cap \omega$. Consider the function: $f : \omega \to [\mathcal{A}]^{<\omega}$ defined by $f(n) = \{p_{\alpha,\beta} \in P : O_n(p_{\alpha,\beta}) \in \mathcal{V}_n\} \bigcup \{p \in \mathcal{A} \setminus P : \{p\} \cup p \in \mathcal{V}_n\}$. Since $\mathcal{F}$ is cofinal in $\mathcal{F}_{\mathcal{A}}$, there exists $\alpha^* < k$ such that $f \leq f_{\alpha^*}$. Then, for each $n \in \omega$, $\{p_{\alpha,\beta} \in P : O_n(p_{\alpha,\beta}) \in \mathcal{V}_n\} \bigcup \{p \in \mathcal{A} \setminus P : \{p\} \cup p \in \mathcal{V}_n\} = f(n) \subseteq f_{\alpha^*}(n)$. Consider the function $g : \omega \to \omega$ defined by $g(n) = \max\{m \in \omega : \{m\} \in \mathcal{V}_n\}$. Since $\mathcal{G}$ is a dominating family, there exists $\beta^* < \mathfrak{d}$ such that $g \leq g_{\beta^*}$. Then $\max\{m \in \omega : \{m\} \in \mathcal{V}_n\} = g(n) \leq g_{\beta^*}(n)$, for all $n \in \omega$. Further, $H_{n,\alpha^*,\beta^*} = \bigcup\{\bigcup f_{\alpha^*}(i) : 0 \leq i \leq n\} \cup [0, \max g_{\beta^*}([0,n])] \supseteq H_n$ for each $n \in \omega$. Then $O_n(p_{\alpha^*,\beta^*}) = \{p_{\alpha^*,\beta^*}\} \cup (p_{\alpha^*,\beta^*} \setminus H_{n,\alpha^*,\beta^*}) \subseteq \{p_{\alpha^*,\beta^*}\} \cup (p_{\alpha^*,\beta^*} \setminus H_n)$, for each $n \in \omega$. Since $O_n(p_{\alpha^*,\beta^*})$ is the only element of $\mathcal{U}_n$ containing p_{α^*,β^*}, $p_{\alpha^*,\beta^*} \notin \bigcup \mathcal{V}_n$, ω is dense in X, and $O_n(p_{\alpha^*,\beta^*}) \cap H_n = \emptyset$, by (*) it follows that $p_{\alpha^*,\beta^*} \notin \bigcup\{St(\bigcup \mathcal{V}_m, \mathcal{U}_m) : m \in \omega\}$. ∎

Corollary 10. *If* $|\mathcal{A}| < \aleph_\omega$, *then* $\Psi(\mathcal{A})$ *is* SM *iff* $\Psi(\mathcal{A})$ *is* SSM.

Proof. If $|\mathcal{A}| < \mathfrak{d}$, then $\Psi(\mathcal{A})$ is SMM and therefore SM. If $|\mathcal{A}| \geq \mathfrak{d}$, then $\mathfrak{d}_{|\mathcal{A}|} = |\mathcal{A}|$, and thus $\Psi(\mathcal{A})$ is not SM and therefore it is not SSM. ∎

The next corollary follows from Lemma 7, part (3).

Corollary 11. *If* $|\mathcal{A}| = \mathfrak{c}$, *then* $\Psi(\mathcal{A})$ *is not* SM.

3. Some partition properties for Ψ-spaces

Proposition 12. *Every σ-pseudocompact space is* parH.

Proof. Let $X = \bigcup\{X_n : n \in \omega\}$ where each X_n is pseudocompact. Without loss of generality we can assume that $X_n \subset X_{n+1}$ for all n. Let $(\mathcal{P}_n : n \in \omega)$ be a sequence of partitions of X into clopen sets. Since a partition of a pseudocompact space into clopen sets is finite, for every $n \in \omega$ the family $\mathcal{F}_n = \{P \in \mathcal{P}_n : P \cap X_n \neq \emptyset\}$ is finite. Then every point $x \in X$ is in all but finitely many X_n and thus in all but finitely many sets $\bigcup \mathcal{F}_n$. ∎

Thus, if $\mathcal{A}$ is a m.a.d. family, then $\Psi(\mathcal{A})$ is parH and hence parM. Our purpose in this section is to show that for some m.a.d. families $\mathcal{A}$, $\Psi(\mathcal{A})$ is parR while for some others it is not. Moreover, for some (necessarily non maximal) a.d. families $\mathcal{A}$, $\Psi(\mathcal{A})$ is not parM. We begin with an easy proof of this last fact.

Example 13. A Ψ-space of cardinality $\mathfrak{c}$ which is not parM.

The construction is similar to Example 5 only this time $X = \omega^\omega$. It is well known that $(X, \mathcal{E})$ (where like in Example 5 $\mathcal{E}$ is the topology of Tychonoff product) is not Menger; moreover, this fact can be witnessed by a sequence of partitions into clopen sets. Then the same sequence of partitions witnesses that the Ψ-space $(X, \mathcal{T})$ is not parM. ∎

Example 14. There is a m.a.d. family $\mathcal{A}$ such that the Ψ-space $Y = \Psi(\mathcal{A})$ is not parR.

The construction is similar to the previous example, only now we start with $X = 2^\omega$. Let D, A, and D_a be like in the previous example, and let $\mathcal{D} = \{D_a : a \in A\}$. Denote by $\mathcal{B}$ the family of all infinite subsets of D and by $\mathcal{C}$ the family of all discrete subsets $S \subset D$ such that $|\overline{S} \setminus S| = 1$. Then $\mathcal{D} \subset \mathcal{C} \subset \mathcal{B}$. Extend $\mathcal{D}$ to a m.a.d. subfamily $\mathcal{A} \subset \mathcal{C}$. We claim that $\mathcal{A}$ is not only maximal among the a.d. subfamilies of $\mathcal{C}$, but also maximal among the a.d. subfamilies of $\mathcal{B}$. Indeed, let $B \in \mathcal{B} \setminus \mathcal{A}$. One easily derives from the compactness and first countability of $(X, \mathcal{E})$ that there is $C \in \mathcal{C}$ such that $C \subset B$. By maximality of $\mathcal{A}$ among the a.d. subfamilies of $\mathcal{C}$ there is $C' \in \mathcal{C}$ such that $C \cap C'$ is infinite. Then $C \cap C'$ is infinite as well, and thus B can not be added to $\mathcal{A}$ without loosing almost disjointness.

Consider the Ψ-space $Y = D \bigcup \mathcal{A}$ in which D plays the role of ω. It is well known that 2^ω is not Rothberger, and this fact is witnessed by a sequence of partitions into clopen sets. It follows from the construction of $\mathcal{A}$ that these partitions can be extended to Y, so Y is not parR. ■

It follows from Example 14 that parH does not imply parR even for Ψ-spaces constructed by m.a.d. families.

Say that a space X is *partition-trivial* if every partition of X into clopen sets contains a co-countable element. Recall that families $\mathcal{B}, \mathcal{C}$ of infinite subsets of ω are *separated* if there is a subset $S \subset \omega$ such that for every $B \in \mathcal{B}$, $B \subseteq^* S$ and for every $C \in \mathcal{C}$, $C \subseteq^* \omega \setminus S$ (we will say that S separates $\mathcal{B}$ from $\mathcal{C}$).

Proposition 15. (1) *Every partition-trivial space is* parR *and* parH.

(2) *If* $X = \Psi(\mathcal{A})$ *is a* Ψ*-space, and for every uncountable subfamilies* $\mathcal{B}, \mathcal{C} \subset \mathcal{A}$, $\mathcal{B}$ *and* $\mathcal{C}$ *can not be separated, then* X *is partition-trivial.*

(3) *If* X *is a* Ψ*-space, and* $\beta X = aX$, *then* X *is partition-trivial.*

Proof. (1) Let X be partition-trivial, and let $(\mathcal{P}_n : n \in \omega)$ be a sequence of partitions of X into clopen sets. There is $P_0 \in \mathcal{P}_0$ such that $X_0 = X \setminus P_0$ is countable. Enumerate $X_0 = \{x_n : n \in \omega \setminus \{0\}\}$. For every $n \in \omega \setminus \{0\}$ there is $P_n \in \mathcal{P}_n$ such that $x_n \in P_n$. Then $\bigcup\{P_n : n \in \omega\} = X$ which proves that X is parR.

Now, for every $n \in \omega$ there is $Q_n \in \mathcal{P}_n$ such that $X \setminus Q_n$ is countable. Then $Y = X \setminus \bigcap\{Q_n : n \in \omega\}$ is countable as well; enumerate $Y = \{y_m : m \in \omega\}$. For every $n, m \in \omega$, let $R_{n,m}$ be the element of $\mathcal{P}_n$ that contains y_m. For $n \in \omega$ put $\mathcal{F}_n = \{Q_n\} \cup \{R_{n,m} : 0 \le m \le n\}$. Then $\mathcal{F}_n$ is a finite subset of $\mathcal{P}_n$. To prove that X is parH it suffices to show that every point $x \in X$ is contained in all but finitely many sets $\bigcup \mathcal{F}_n$. If $x \in X \setminus Y$, then x is contained in Q_n for all n and thus in $\bigcup \mathcal{F}_n$ for all n. If $x \in Y$, then $x = y_m$ for some m, and then x is contained in $\bigcup \mathcal{F}_n$ for all $n \ge m$.

(2) Suppose $X = \Psi(\mathcal{A})$ is not partition trivial which is witnessed by a partition $\mathcal{P}$ which does not have a co-countable element. Then there is $P \in \mathcal{P}$ such that both P and $Q = X \setminus P$ are uncountable. Put $\mathcal{B} = P \cap \mathcal{A}$, $\mathcal{C} = Q \cap \mathcal{A}$ and $S = P \cap \omega$. Then S separates $\mathcal{B}$ from $\mathcal{C}$.

(3) Suppose $X = \Psi(\mathcal{A})$ is not partition trivial, and let $\mathcal{P}$, P, Q, $\mathcal{B}$ and $\mathcal{C}$ be like in the proof of (2). Put $f(x) = 0$ for $x \in P$ and $f(x) = 1$ for $x \in Q$. Then f is a continuous function from X to I. Extend it to a continuous function $\tilde{f} : \beta X \to I$. Suppose $\beta X = aX = X \cup \{p\}$. Then $\mathcal{A} \cup \{p\}$ is the one-point compactification of the discrete space $\mathcal{A}$ and thus every neighborhood of p contains both points from $\mathcal{B}$ and points from $\mathcal{C}$, that is both points x with $\tilde{f}(x) = 0$ and points y with $\tilde{f}(y) = 1$. So $\tilde{f}$ can not be continuous at p, a contradiction. ■

There is an a.d. family $\mathcal{A}$ of infinite subsets of ω such that $|\mathcal{A}| = \omega_1$ and for every uncountable subfamilies $\mathcal{B}, \mathcal{C} \subset \mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ can not be separated ([5], Theorem 4.1). Also for some m.a.d. families $\mathcal{A}$, the Ψ-space $X = \Psi(\mathcal{A})$ satisfies the condition $\beta X = aX$ [14]. Together with Proposition 15 this provides the next two examples.

Example 16. There is an a.d. family $\mathcal{A}$ such that $|\mathcal{A}| = \omega_1$ and the Ψ-space $X = \Psi(\mathcal{A})$ is parR and parH.

Example 17. There is an uncountable m.a.d. family $\mathcal{A}$ such that the Ψ-space $X = \Psi(\mathcal{A})$ is parR (and parH).

We conclude with an example that distinguishes parM and parH.

Example 18. ($\mathfrak{b} < \mathfrak{d}$) There are a.d. families $\mathcal{A}$ such that the Ψ-space $X = \Psi(\mathcal{A})$ is parM but not parH.

The construction is similar to Example 13, only now we start with $(X, \mathcal{E})$ where $X \subset \omega^\omega$ is an unbounded subset of cardinality $\mathfrak{b}$ (it is well known that an unbounded subspace of ω^ω does not have the Hurewicz property). Since $|X| < \mathfrak{d}$, $(X, \mathcal{T})$ is SSM hence parM. ■

Similarly one gets

Example 19. ($cov(\mathcal{M}) < \mathfrak{d}$) There are a.d. families $\mathcal{A}$ such that the Ψ-space $X = \Psi(\mathcal{A})$ is parR but not parH.

Question 20. Can one construct within ZFC:

– a Ψ-space which is parM but not parH?

– a Ψ-space which is parR but not parH?

Acknowledgement. The authors express gratitude to R. Levy for useful discussions and to the referees for careful reading and suggestions that helped to improve the paper.

References

[1] T. Bartoszyński, *Combinatorial aspects of measure and category*, Fund. Math. **127**, 3 (1987), 225–239.

[2] T. Bartoszyński and H. Judah, *Set Theory. On the Structure of the Real Line*, A. K. Peters, 1995.

[3] J. Baumgartner and M. Weese, *Partition algebras for almost disjoint families*, Trans. Amer. Math. Soc. **274**, 2 (1982), 619–630.

[4] M. Bonanzinga, F. Cammaroto and Lj. D. R. Kočinac, *Star-Hurewicz and related properties*, Applied General Topology **5** (2004), 79–89.

[**5**] E. K. van Douwen, *The integers in topology*, in: Handbook of Set-Theoretic Topology. K. Kunen and J.E. Vaughan Eds., Elsevier Sci. Pub., (1984), 111–167.

[**6**] E. K. van Douwen, G. M. Reed, A. W. Roscoe and I. J. Tree, *Star covering properties*, Topol. and Appl. **39** (1991), 71–103.

[**7**] R. Engelking, *General Topology*, Heldermann Verlag, Berlin, Sigma Series in Pure Mathematics, 6, 1989.

[**8**] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, New York 1960.

[**9**] W. Hurewicz, *Über eine Verallgemeinerung des Borelschen Theorems*, Math. Z. **24** (1925) 401–421.

[**10**] Lj. D. R. Kočinac, *Star-Menger and related spaces*, Publ. Math. Debrecen **55** (1999), 421–431.

[**11**] Lj. D. R. Kočinac, *Star-Menger and related spaces* II, Filomat (Niš) **13** (1999), 129–140.

[**12**] Lj. D. R. Kočinac, *Selected results on selection principles*, in: Proceedings of the 3rd Seminar on Geometry and Topology (Sh. Rezapour, ed.), July 15–17, Tabriz, Iran (2004), 71–104.

[**13**] A. W. Miller, D. H. Fremlin *On some properties of Hurewicz, Menger and Rothberger*, Fund. Math. **129** (1988) 17–33.

[**14**] S. Mrówka, *Some set-theoretic constructions in topology*, Fund. Math. **94** (1977), 83–92.

[**15**] M. Scheepers, *Selection principles and covering properties in topology*, Note di Matematica, **22** (2003), 3–41.

[**16**] B. Tsaban, *Some new directions in infinite-combinatorial topology*, in: Set Theory (J. Bagaria and S. Todorčević, eds.), Trends in Mathematics, Birkhäuser 2006, 225–255.

(received 27.5.2008; in revised form 14.10.2008)

Maddalena Bonanzinga, Dipartimento di Matematica, Università di Messina, 98166 Messina, Italia

E-mail: `milena_bonanzinga@hotmail.com`

Mikhail Matveev, Department of Mathematical Sciences, George Mason University, 4400 University Drive, Fairfax, Virginia, 22030 USA

E-mail: `misha_matveev@hotmail.com`