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THE DEPENDENCE OF THE EIGENVALUES OF THE STURM-LIOUVILLE PROBLEM ON BOUNDARY CONDITIONS

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Abstract. We prove a new asymptotic formula for the eigenvalues of Sturm-Liouville problem, which is a generalization of the known formulae and which takes into account the analytic dependence of the eigenvalues on boundary conditions.

1. Introduction and statement of the result

Let $L(q,\alpha,\beta)$ denote the Sturm-Liouville problem

$$\ell y \equiv -y'' + q(x)y = \mu y, \quad x \in (0,\pi),\ \mu \in \mathbf{C}, \tag{1.1}$$

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0, \quad \alpha \in (0,\pi], \tag{1.2}$$

$$y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \quad \beta \in [0,\pi), \tag{1.3}$$

where q is a real-valued, summable on $[0,\pi]$ function (we write $q \in L^1_{\mathbf{R}}[0,\pi]$). By $L(q,\alpha,\beta)$ we also denote the self-adjoint operator, generated by the problem (1.1)–(1.3), (see [1], [2]). It is known, that the spectra of the operator $L(q,\alpha,\beta)$ is discrete and consists of simple eigenvalues, which we denote by $\mu_n(q,\alpha,\beta)$, $n = 0,1,2,\ldots$, emphasizing the dependence of μ_n on q, α and β (concerning enumeration see §2).

The dependence of μ_n on q was investigated in [3]–[6] for $q \in L^2_{\mathbf{R}}[0,\pi]$. The dependence of μ_n on α and β is usually studied (see [1]–[3], [7]) in the following sense: the boundary conditions are separated into the three cases: 1) $\sin\alpha \neq 0$, $\sin\beta \neq 0$; 2) $\sin\alpha = 0$, $\sin\beta \neq 0$ or $\sin\alpha \neq 0$, $\sin\beta = 0$; 3) $\sin\alpha = \sin\beta = 0$, and results, in particular the asymptotics of the eigenvalues, are formulated separately for each case (more detailed list is in [8]), namely:

$$1)\ \ \mu_n(q,\alpha,\beta) = n^2 + \frac{2}{\pi}(\operatorname{ctg}\beta - \operatorname{ctg}\alpha) + [q] + r_n(q,\alpha,\beta),$$
$$\text{if} \quad \sin\alpha \neq 0,\ \sin\beta \neq 0, \tag{1.4}$$

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$$2)\ \mu_n(q,\pi,\beta) = \left(n+\frac{1}{2}\right)^2 + \frac{2}{\pi}\operatorname{ctg}\beta + [q] + r_n(q,\beta), \quad \text{if} \quad \sin\beta \neq 0, \tag{1.5}$$

$$2')\ \mu_n(q,\alpha,0) = \left(n+\frac{1}{2}\right)^2 - \frac{2}{\pi}\operatorname{ctg}\alpha + [q] + r_n(q,\alpha), \quad \text{if} \quad \sin\alpha \neq 0, \tag{1.6}$$

$$3)\ \mu_n(q,\pi,0) = (n+1)^2 + [q] + r_n(q), \tag{1.7}$$

where $[q] = \frac{1}{\pi}\int_0^\pi q(t)\,dt$ and $r_n = o(1)$ when $n \to \infty$, but this estimate is not uniform in $\alpha, \beta \in [0,\pi]$ and we cannot obtain 2), 2′) and 3) from 1) by passing to the limit when $\alpha \to \pi$ or $\beta \to 0$.

In the sequel we will prove, that the dependence of eigenvalues μ_n on α and β is smooth (analytic! see §2, Remark 3) and we want to obtain one formula instead of four, which takes this smooth dependence into account.

Theorem 1. *The lowest eigenvalue $\mu_0(q,\alpha,\beta)$ has the property: $\lim\limits_{\alpha\to 0}\mu_0(q,\alpha,\beta) = -\infty$, $\lim\limits_{\beta\to\pi}\mu_0(q,\alpha,\beta) = -\infty$. For eigenvalues $\mu_n(q,\alpha,\beta)$, $n = 1,2,\dots,$ the following formula*

$$\mu_n(q,\alpha,\beta) = [n+\delta_n(\alpha,\beta)]^2 + [q] + r_n(q,\alpha,\beta), \tag{1.8}$$

holds, where $[q] = \frac{1}{\pi}\int_0^\pi q(t)\,dt$,

$$\delta_n(\alpha,\beta) = \frac{1}{\pi}\arccos\frac{\cos\alpha}{\sqrt{[n+\delta_n(\alpha,\beta)]^2\sin^2\alpha+\cos^2\alpha}} - \frac{1}{\pi}\arccos\frac{\cos\beta}{\sqrt{[n+\delta_n(\alpha,\beta)]^2\sin^2\beta+\cos^2\beta}}, \tag{1.9}$$

and $r_n = r_n(q,\alpha,\beta) = o(1)$, when $n\to\infty$, uniformly in $\alpha,\beta\in[0,\pi]$ and q from the bounded subsets of $L^1_{\mathbf{R}}[0,\pi]$ (we will write $q \in BL^1_{\mathbf{R}}[0,\pi]$).

Remark 1. Although (1.9) is not a representation of $\delta_n(\alpha,\beta)$, but only an (transcendental) equation, we will see that it is sufficiently convenient for investigation of the functions $\delta_n(\alpha,\beta)$ and the asymptotics of the eigenvalues. In particular, all previous formulae (1.4)–(1.7) are consequences of (1.8) and (1.9) (see below §3), and in (1.8) we can pass to the limit, when $\alpha\to\pi$ or $\beta\to 0$.

We start from the formula

$$\mu_n(q,\alpha,\beta) = \mu_n(0,\alpha,\beta) + \int_0^1\left[\int_0^\pi q(x)h_n^2(x,tq,\alpha,\beta)\,dx\right]dt, \quad n = 0,1,\dots, \tag{1.10}$$

where $h_n(x,tq,\alpha,\beta)$ are the normalized eigenfunctions $\left(\int_0^\pi h_n^2(x,tq,\alpha,\beta)\,dx = 1\right)$ of the problem $L(tq,\alpha,\beta)$ and where t is a real parameter. Formula (1.10) was proved in [6] in the case $\alpha=\pi$, $\beta=0$. In the general case ($\alpha\in(0,\pi]$, $\beta\in[0,\pi)$) the proof is similar and we omit it. Below we prove two lemmas.

Lemma 1. $\mu_n(0,\alpha,\beta) = [n+\delta_n(\alpha,\beta)]^2$, $n = 1,2,\dots$.

Lemma 2. $\int_0^1 \left[\int_0^\pi q(x)h_n^2(x,tq,\alpha,\beta)\,dx\right] dt = [q] + r_n(q,\alpha,\beta)$, *where* $r_n(q,\alpha,\beta) = o(1)$, $n \to \infty$, *uniformly by* $\alpha,\beta \in [0,\pi]$ *and* $q \in BL^1_{\mathbf{R}}[0,\pi]$.

From these lemmas formula (1.8) of Theorem 1 will follow. For more detailed investigation of the dependence of eigenvalues on the parameters α and β (in particular for the properties of $\mu_0(q,\alpha,\beta)$ and the analytic dependence μ_n on α and β), in §2 we introduce the concept of "the eigenvalues function" and study its properties. This study reduces us, in particular, to the study of functions $\delta_n(\alpha,\beta)$ and we do it in §3. In §4 we will prove Lemma 2.

2. The eigenvalues function

Let us consider firstly the problem $L(q,\pi,0)$ and enumerate its eigenvalues in increasing order

$$\mu_0(q,\pi,0) < \mu_1(q,\pi,0) < \cdots < \mu_n(q,\pi,0) < \cdots \, . \tag{2.1}$$

According to the alternation properties of the eigenvalues of problems $L(q,\pi,0)$ and $L(q,\pi,\beta)$ (see [1, p. 261], we have the inequalities

$$\begin{aligned} &\mu_0(q,\pi,\beta) < \mu_0(q,\pi,0) < \mu_1(q,\pi,\beta) < \cdots \\ &\qquad \cdots < \mu_n(q,\pi,\beta) < \mu_n(q,\pi,0) < \mu_{n+1}(q,\pi,\beta) < \cdots \end{aligned} \tag{2.2}$$

for arbitrary $\beta \in (0,\pi)$. Then, by the alternation of the eigenvalues of the problems $L(q,\pi,\beta)$ and $L(q,\alpha,\beta)$ (for arbitrary $\alpha \in (0,\pi)$) we have the inequalities

$$\begin{aligned} &\mu_0(q,\alpha,\beta) < \mu_0(q,\pi,\beta) < \mu_1(q,\alpha,\beta) < \cdots \\ &\qquad \cdots < \mu_n(q,\alpha,\beta) < \mu_n(q,\pi,\beta) < \mu_{n+1}(q,\alpha,\beta) < \cdots \, . \end{aligned} \tag{2.3}$$

(2.1), (2.2), (2.3) together give us the correct enumeration of the eigenvalues of the problems $L(q,\alpha,\beta)$ for arbitrary $\alpha \in (0,\pi]$ and $\beta \in [0,\pi)$.

Let us represent an arbitrary positive number γ in the form $\gamma = \alpha + \pi n$, where $\alpha \in (0,\pi]$ and $n = 0,1,2,\dots$; and arbitrary $\delta \in (-\infty,\pi)$ we represent as $\delta = \beta - \pi m$, where $\beta \in [0,\pi)$ and $m = 0,1,2,\dots$.

Definition 1. The function $\mu(q,\gamma,\delta) = \mu(\gamma,\delta)$ in two arguments, defined on $(0,\infty) \times (-\infty,\pi)$ by the formula

$$\mu(\gamma,\delta) = \mu(\alpha+\pi n, \beta - \pi m) := \mu_{n+m}(q,\alpha,\beta), \tag{2.4}$$

where $\mu_k(q,\alpha,\beta)$, $k = 0,1,2,\dots$, are the eigenvalues of $L(q,\alpha,\beta)$, enumerated by (2.1)–(2.3), we shall call the eigenvalues function (EVF) of the family of the problems $\{L(q,\alpha,\beta),\ \alpha \in (0,\pi],\ \beta \in [0,\pi)\}$. In particular, for fixed $\beta \in [0,\pi)$, the function of one argument $\mu^+(\gamma)$, defined on $(0,\infty)$ by the formula

$$\mu^+(\gamma) \equiv \mu(\gamma,\beta) = \mu(\alpha+\pi n,\beta) = \mu_n(q,\alpha,\beta)$$

we shall call the EVF of the family $\{L(q,\alpha,\beta),\ \alpha\in(0,\pi]\}$, and for fixed $\alpha\in(0,\pi]$, the function in one argument $\mu^-(\delta)$, defined on $(-\infty,\pi)$ by the formula

$$\mu^-(\delta)=\mu(\alpha,\delta)=\mu(\alpha,\beta-\pi m)=\mu_m(q,\alpha,\beta),\quad \beta\in[0,\pi),\ m=0,1,2,\dots,$$

we shall call the EVF of the family $\{L(q,\alpha,\beta),\ \beta\in(0,\pi]\}$.

Remark 2. Sometimes we omit some arguments (for example $\mu(q,\alpha,\beta)=\mu(\alpha,\beta)$) to emphasize the principal arguments, according to which we make investigations.

Let $\varphi(x,\mu,\gamma)$ and $\psi(x,\mu,\delta)$ denote the solutions of (1.1), satisfying the initial conditions

$$\begin{aligned}\varphi(0,\mu,\gamma)&=\sin\gamma,\quad \varphi'(0,\mu,\gamma)=-\cos\gamma,\quad \gamma\in\mathbf{C},\\ \psi(\pi,\mu,\delta)&=\sin\delta,\quad \psi'(\pi,\mu,\delta)=-\cos\delta,\quad \delta\in\mathbf{C},\end{aligned}\tag{2.5}$$

respectively. The eigenvalues $\mu_n=\mu_n(q,\alpha,\beta),$, $n=0,1,2,\dots$, of $L(q,\alpha,\beta)$ are the solutions of the equation

$$\chi(\mu):=\varphi(\pi,\mu,\alpha)\cos\beta+\varphi'(\pi,\mu,\alpha)\sin\beta=0,\tag{2.6}$$

or the equation

$$\chi_1(\mu):=\psi(0,\mu,\beta)\cos\alpha+\psi'(0,\mu,\beta)\sin\alpha=0.$$

The functions $\varphi_n(x)=\varphi(x,\mu_n,\alpha)$ and $\psi_n(x)=\psi(x,\mu_n,\beta)$, $n=0,1,2,\dots$, are the eigenfunctions, corresponding to the eigenvalue μ_n. The squares of the L^2-norms of these eigenfunctions:

$$a_n=\int_0^\pi|\varphi_n(x)|^2dx,\qquad b_n=\int_0^\pi|\psi_n(x)|^2\,dx,\tag{2.7}$$

are called the norming constants.

Now we prove that EVF $\mu(\gamma,\delta)$ is analytic at the arbitrary point $(\gamma_0,\delta_0)\in(0,\infty)\times(-\infty,\pi)$. Let $\gamma_0=\alpha+\pi n$, $\delta_0=\beta-\pi m$, where $\alpha\in(0,\pi]$, $\beta\in[0,\pi)$ and $n,m=0,1,2,\dots$. And let $\mu_0=\mu(\gamma_0,\delta_0)=\mu(\alpha+\pi n,\beta-\pi m)=\mu_{n+m}(q,\alpha,\beta)$ is the value of EVF at point (γ_0,δ_0). Then $\chi(\mu_0)=0$. Since the eigenvalues are simple, $\frac{\partial\chi(\mu_0,\alpha,\beta)}{\partial\mu}\neq0$ ([1, p. 261]). Then, by the implicit function theorem (see [9, p. 166]), there exists a "complex" neighbourhood V of (γ_0,δ_0), on which one-valued analytic function $\tilde\mu(\gamma,\delta)$ is defined such that $\tilde\mu(\gamma_0,\delta_0)=\mu_0$, $\chi(\tilde\mu(\gamma,\delta),\gamma,\delta)\equiv\chi(\mu_0,\gamma_0,\delta_0)=0$ for all $(\gamma,\delta)\in V$. In particular, for real pair $(\gamma,\delta)\in V$, $\tilde\mu(\gamma,\delta)=\mu(\gamma,\delta)$. Since (γ_0,δ_0) was an arbitrary point from $(0,\infty)\times(-\infty,\pi)$, we have proved the (real) analyticity of $\mu(\gamma,\delta)$ on the whole set $(0,\infty)\times(-\infty,\pi)$. In particular, it follows the "real analyticity" (see [9, p. 167]) of $\mu^+(\cdot)$ on $(0,\infty)$ and $\mu^-(\cdot)$ on $(-\infty,\pi)$.

Remark 3. Thus, according to the definition (2.4), each $\mu_n(q,\alpha,\beta)$ is "a part" of "analytic surface" $\mu(q,\gamma,\delta)$, and therefore it is an analytic function in α and β. In other words, we can say that $\mu_n(q,\alpha,\beta)=\mu_n(\alpha,\beta)$ is analytic in $(0,\pi)\times(0,\pi)$ and at the boundaries of $[0,\pi]\times[0,\pi]$ the function $\mu_n(\alpha,\beta)$ analytically transforms into $\mu_{n+1}(\alpha,\beta)$ or $\mu_{n-1}(\alpha,\beta)$.

Let us also prove that the ranges of values of $\mu^+(\cdot)$ and $\mu^-(\cdot)$ are the whole real axis (hence the range of $\mu(\cdot,\cdot)$ also is $\mathbf{R}$). In the case of μ^- it is sufficient to prove that for every $\mu_0 \in \mathbf{R}$ there exists $\beta_0 \in [0,\pi)$ such that

$$\chi_(\mu_0) = \varphi(\pi,\mu_0,\alpha)\cos\beta_0 + \varphi'(\pi,\mu_0,\alpha)\sin\beta_0 = 0.$$

Really, if $\varphi(\pi,\mu_0,\alpha) = 0$, then we take $\beta_0 = 0$, if $\varphi(\pi,\mu_0,\alpha) \neq 0$, then we take $\beta_0 = \operatorname{arcctg}\left(-\frac{\varphi'(\pi,\mu_0,\alpha)}{\varphi(\pi,\mu_0,\alpha)}\right)$. The case of μ^+ is proved in the same way.

Now we prove, that EVF $\mu(\gamma,\delta)$ increases with γ and decreases with δ. Let $\gamma_0 = \alpha_0 + \pi n$ be fixed and $\delta = \beta - \pi m$. Then the solution $\psi(x,\mu(\gamma_0,\delta),\beta) = \psi(x,\mu_{n+m}(\alpha_0,\beta),\beta) = \psi_{n+m}(x,\alpha_0,\beta)$ is an eigenfunction. For eigenfunctions $\psi(x,\mu(\gamma_0,\delta),\beta) = \psi(x,\delta)$ and $\psi(x,\mu(\gamma_0,\delta_1),\beta_1) = \psi(x,\delta_1)$ ($\delta_1 = \beta_1 - \pi m$) is true the identity

$$\frac{d}{dx}\left(\psi'(x,\delta_1)\psi(x,\delta) - \psi(x,\delta_1)\psi'(x,\delta)\right) \equiv (\mu(\gamma_0,\delta) - \mu(\gamma_0,\delta_1)) \cdot \psi(x,\delta)\psi(x,\delta_1).$$

Integrating the last identity from 0 to π, we obtain

$$\sin(\delta_1 - \delta) = [\mu(\gamma_0,\delta) - \mu(\gamma_0,\delta_1)] \int_0^\pi \psi(x,\delta_1)\cdot\psi(x,\delta)\,dx.$$

It follows that there exists the derivative

$$\frac{\partial\mu(\gamma_0,\delta)}{\partial\delta} = -\frac{1}{\int_0^\pi \psi^2(x,\delta)\,dx} < 0, \tag{2.8}$$

and therefore, $\mu(\gamma,\delta)$ is decreasing function by δ. In the same way we get the identity

$$\frac{\partial\mu(\gamma,\delta_0)}{\partial\gamma} = \frac{1}{\int_0^\pi \varphi^2(x,\mu(\gamma,\delta_0),\alpha)\,dx} > 0, \tag{2.9}$$

from which we conclude that EVF $\mu(\gamma,\delta)$ is increasing by γ.

Let us fix $\alpha \in (0,\pi]$ and consider the function $\mu^-(\delta) = \mu(\alpha,\delta)$. Since $\mu^-(\cdot)$ is strongly decreasing on $(-\infty,\pi)$, analytic (here its continuity is sufficient), and its range of values is the whole real axis, we get that $\lim_{\delta\to\pi}\mu^-(\delta) = -\infty$ and there exists a unique point δ_0 such that $\mu^-(\delta_0) = 0$. Similarly $\lim_{\gamma\to 0}\mu^+(\gamma) = -\infty$ and there exists a unique point γ_0, such that $\mu^+(\gamma_0) = 0$ (for any fixed $\beta \in [0,\pi)$, i.e. $\gamma_0 = \gamma_0(\beta)$).

3. The EVF $\mu(0,\gamma,\delta)$ and the properties of $\delta_n(\alpha,\beta)$

For $q(x) \equiv 0$ the solution $y = \varphi(x,\lambda^2,\alpha)$ of Cauchy problem (1.1), (2.5) ($\mu = \lambda^2$) has the form

$$\varphi(x,\lambda^2,\alpha) = \sin\alpha\cos\lambda x - \cos\alpha\cdot\frac{\sin\lambda x}{\lambda},$$

and the characteristic equation (2.6) has the form

$$\chi(\lambda^2) = \sin(\alpha - \beta)\cos\lambda\pi - \left(\frac{\cos\alpha\cos\beta}{\lambda} + \lambda\sin\alpha\sin\beta\right)\sin\lambda\pi = 0. \tag{3.1}$$

When $\alpha = \pi$ and $\beta = 0$, this equation has the form $\frac{\sin\lambda\pi}{\lambda} = 0$, and therefore the eigenvalues $\mu_n(0,\pi,0) = \lambda_n^2 = (n+1)^2$, $n = 0,1,2,\dots$. In particular, the lowest eigenvalue $\mu_0(0,\pi,0) = 1$. Since EVF $\mu^-(\delta) = \mu(0,\pi,\delta)$ is decreasing in δ and must obtain all the values up to $-\infty$, it follows that when δ changes from 0 to π, $\mu_0(0,\pi,\delta)$ takes all the values from 1 to $-\infty$, i.e. $\lim_{\beta\to\pi}\mu_0(0,\pi,\beta) = -\infty$. Since $\mu_0(0,\alpha,\beta) < \mu_0(0,\pi,\beta)$, when $0 < \alpha < \pi$, it follows that $\lim_{\beta\to\pi}\mu_0(0,\alpha,\beta) = -\infty$ for arbitrary $\alpha \in (0,\pi]$. Similarly (using that increases in α) we obtain that $\lim_{\alpha\to 0}\mu_0(0,\alpha,\beta) = -\infty$ for arbitrary $\beta \in [0,\pi)$. Since "the integral part" in (1.10) is bounded (see §4 below), we have proved the assertion of Theorem 1, relating to $\mu_0(q,\alpha,\beta)$.

The remaining eigenvalues (i.e. $\mu_n(0,\alpha,\beta)$ for $n \geqslant 1$) $\mu_n(0,\alpha,\beta)$ can be defined also for $\alpha = 0$ and $\beta = \pi$ by:

$$\mu_n(0,0,\beta) := \mu_{n-1}(0,\pi,\beta), \qquad (\beta \in [0,\pi)),$$
$$\mu_n(0,\alpha,\pi) := \mu_{n-1}(0,\alpha,0), \qquad (\alpha \in (0,\pi]).$$

These definitions are correct since (the another definition, which follows from analyticity, is:)

$$\begin{aligned}\mu_n(0,0,\beta) &:= \lim_{\alpha\to 0}\mu_n(0,\alpha,\beta) = \lim_{\alpha\to 0}\mu(0,\alpha+\pi n,\beta)\\ &= \mu(0,\pi n,\beta) = \mu(0,\pi+\pi(n-1),\beta) = \mu_{n-1}(0,\pi,\beta)\end{aligned}$$

and

$$\begin{aligned}\mu_n(0,\alpha,\pi) &:= \lim_{\beta\to\pi}\mu_n(0,\alpha,\beta) = \lim_{\beta\to\pi}\mu(0,\alpha,\beta-\pi n)\\ &= \mu(0,\alpha,\pi-\pi n) = \mu(\alpha,-\pi(n-1)) = \mu_{n-1}(0,\alpha,0).\end{aligned}$$

So, we can consider the eigenvalues $\mu_n(0,\alpha,\beta)$ for $n \geqslant 1$ as the functions, defined on $[0,\pi]\times[0,\pi]$. For all $\alpha,\beta \in [0,\pi]$ there are the relations

$$\begin{aligned}(n-1)^2 &= \mu_{n-2}(0,\pi,0) = \mu_{n-1}(0,\pi,\pi) \leqslant \mu_{n-1}(0,\pi,\beta) = \mu_n(0,0,\beta) \leqslant \mu_n(0,\alpha,\beta)\\ &\leqslant \mu_n(0,\pi,\beta) \leqslant \mu_n(0,\pi,0) = (n+1)^2.\end{aligned}$$

Therefore, it is natural to consider the functions $\lambda_n(0,\alpha,\beta) := \sqrt{\mu_n(0,\alpha,\beta)}$ ($n = 1,2,\dots$) in two arguments $(\alpha,\beta) \in [0,\pi]\times[0,\pi]$ in the form $\lambda_n(0,\alpha,\beta) = n + \delta_n(\alpha,\beta)$, where $\delta_n(\alpha,\beta)$ must satisfy the inequalities $-1 \leqslant \delta_n(\alpha,\beta) \leqslant 1$ and, by the properties of EVF, be increasing in α and decreasing in β.

Substituting $\lambda = \lambda_n(0,\alpha,\beta) = n + \delta_n(\alpha,\beta)$ in (3.1), for $\delta_n = \delta_n(\alpha,\beta)$ we obtain the (transcendental) equation:

$$\sin(\alpha-\beta)\cos\pi\delta_n - \left(\frac{\cos\alpha\cdot\cos\beta}{n+\delta_n} + (n+\delta_n)\sin\alpha\sin\beta\right)\sin\pi\delta_n = 0. \tag{3.2}$$

Solving this equation as trigonometrical equation of the type $a\cos\pi\delta_n + b\sin\pi\delta_n = 0$ and using the property $-1 \leqslant \delta_n(\alpha,\beta) \leqslant 1$, we obtain

$$\delta_n(\alpha,\beta) = \frac{1}{\pi}\left[\arccos\frac{\cos\alpha}{\sqrt{(n+\delta_n(\alpha,\beta))^2\sin^2\alpha+\cos^2\alpha}} - \right.$$
$$\left. - \arccos\frac{\cos\beta}{\sqrt{(n+\delta_n(\alpha,b))^2\sin^2\beta+\cos^2\beta}}\right]. \quad (3.3)$$

Thus, Lemma 1 is proved. ■

Although (3.3) is not a representation for $\delta_n(\alpha,\beta)$, but only an equation, many properties of $\delta_n(\alpha,\beta)$ can be derived. For example, it follows from (3.3) that $-1 \leqslant \delta_n(\alpha,\beta) \leqslant 1$. Besides, it is easy to compute the values

$$\begin{gathered}\delta_n(0,0)=0,\quad \delta_n\left(0,\ \frac{\pi}{2}\right) = -\frac{1}{2},\quad \delta_n(0,\pi) = -1,\\ \delta_n\left(\frac{\pi}{2},\ 0\right) = \frac{1}{2},\quad \delta_n\left(\frac{\pi}{2},\ \frac{\pi}{2}\right) = 0,\quad \delta_n\left(\frac{\pi}{2},\pi\right) = -\frac{1}{2},\\ \delta_n(\pi,0)=1,\quad \delta_n\left(\pi,\ \frac{\pi}{2}\right) = \frac{1}{2},\quad \delta_n(\pi,\pi)=0.\end{gathered} \quad (3.4)$$

Differentiating (3.2) in α and in β, we obtain

$$\frac{\partial\delta_n(\alpha,\beta)}{\partial\alpha} = \frac{\cos(\alpha-\beta)\cos\pi\delta_n(\alpha,\beta) + \left(\frac{\sin\alpha\cos\beta}{n+\delta_n(\alpha,\beta)} - (n+\delta_n(\alpha,\beta))\cos\alpha\sin\beta\right)\sin\pi\delta_n(\alpha,\beta)}{\left(\pi\sin(\alpha-\beta) - \frac{\cos\alpha\cos\beta}{(n+\delta_n(\alpha,\beta))^2} + \sin\alpha\sin\beta\right)\sin\pi\delta_n(\alpha,\beta) + \pi\left(\frac{\cos\alpha\cos\beta}{n+\delta_n(\alpha,\beta)} + (n+\delta_n(\alpha,\beta))\sin\alpha\sin\beta\right)\cos\pi\delta_n(\alpha,\beta)}, \quad (3.5)$$

$$\frac{\partial\delta_n(\alpha,\beta)}{\partial\beta} = \frac{\left(\frac{\cos\alpha\sin\beta}{n+\delta_n} - (n+\delta_n)\sin\alpha\cos\beta\right)\sin\pi\delta_n - cos(\alpha-\beta)\cos\pi\delta_n}{\left(\pi\sin(\alpha-\beta) - \frac{\cos\alpha\cos\beta}{(n+\delta_n)^2} - \sin\alpha\sin\beta\right)\sin\pi\delta_n + \left(\frac{\pi\cos\alpha\cos\beta}{n+\delta_n} + \pi(n+\delta_n)\sin\alpha\sin\beta\right)\cos\pi\delta_n}. \quad (3.6)$$

It follows from (3.4)–(3.6) that

$$\begin{gathered}\frac{\partial\delta_n(0,0)}{\partial\alpha} = \frac{n}{\pi},\quad \frac{\partial\delta_n\left(0,\frac{\pi}{2}\right)}{\partial\alpha} = \frac{1}{\pi}\left(n-\frac{1}{2}\right),\quad \frac{\partial\delta_n(0,\pi)}{\partial\alpha} = \frac{n-1}{\pi},\\ \frac{\partial\delta_n\left(\frac{\pi}{2},0\right)}{\partial\alpha} = \frac{1}{\pi\left(n+\frac{1}{2}\right)},\quad \frac{\partial\delta_n\left(\frac{\pi}{2},\frac{\pi}{2}\right)}{\partial\alpha} = \frac{1}{\pi n},\quad \frac{\partial\delta_n\left(\frac{\pi}{2},\pi\right)}{\partial\alpha} = \frac{1}{\pi\left(n-\frac{1}{2}\right)},\\ \frac{\partial\delta_n(\pi,0)}{\partial\alpha} = \frac{n+1}{\pi},\quad \frac{\partial\delta_n\left(\pi,\frac{\pi}{2}\right)}{\partial\alpha} = \frac{1}{\pi}\left(n+\frac{1}{2}\right),\quad \frac{\partial\delta_n(\pi,\pi)}{\partial\alpha} = \frac{n}{\pi}.\end{gathered}$$

and

$$\frac{\partial\delta_n(0,0)}{\partial\beta}=-\frac{n}{\pi},\qquad \frac{\partial\delta_n\left(0,\frac{\pi}{2}\right)}{\partial\beta}=-\frac{1}{\pi\left(n-\frac{1}{2}\right)},\ \frac{\partial\delta_n(0,\pi)}{\partial\beta}=-\frac{n-1}{\pi},$$
$$\frac{\partial\delta_n\left(\frac{\pi}{2},0\right)}{\partial\beta}=-\frac{n+\frac{1}{2}}{\pi},\frac{\partial\delta_n\left(\frac{\pi}{2},\frac{\pi}{2}\right)}{\partial\beta}=-\frac{1}{\pi n},\qquad \frac{\partial\delta_n\left(\frac{\pi}{2},\pi\right)}{\partial\beta}=-\frac{n-\frac{1}{2}}{\pi},$$
$$\frac{\partial\delta_n(\pi,0)}{\partial\beta}=-\frac{n+1}{\pi},\ \frac{\partial\delta_n\left(\pi,\frac{\pi}{2}\right)}{\partial\beta}=-\frac{1}{\pi\left(n+\frac{1}{2}\right)},\ \frac{\partial\delta_n(\pi,\pi)}{\partial\beta}=-\frac{n}{\pi}\,.$$

These formulae are useful for obtaining the asymptotics of the norming constants a_n and b_n, since by (1.8) and (2.7)–(2.9)

$$\frac{1}{a_n(\alpha,\beta)}=\frac{\partial\mu(\alpha+\pi n,\beta)}{\partial\gamma}=2\left[n+\delta_n(\alpha,\beta)\right]\cdot\frac{\partial\delta_n(\alpha,\beta)}{\partial\alpha}+o(1),$$
$$\frac{1}{b_n(\alpha,\beta)}=\frac{\partial\mu(\alpha,\beta+\pi n)}{\partial\delta}=2\left[n+\delta_n(\alpha,\beta)\right]\cdot\frac{\partial\delta_n(\alpha,\beta)}{\partial\beta}+o(1),$$

when $n\to\infty$.

Now we will show that (1.4)–(1.7) are the consequence of (1.8) and (1.9). It is evident from (3.3) that $\delta_n(\pi,0)=1$. So we obtain (1.7) from (1.8) and Lemma 2. Let us write temporarily $x_{n,\alpha}(\beta)=\frac{\cos\beta}{\sqrt{(n+\delta_n(\alpha,\beta))^2\sin^2\beta+\cos^2\beta}}$. Then $\delta_n(\pi,\beta)=1-\frac{1}{\pi}\arccos x_{n,\pi}(\beta)$. If $\sin\beta\neq 0$ ($\beta\in(0,\pi)$), $x_{n,\pi}(\beta)=O\left(\frac{1}{n}\right)$ and

$$x_{n,\pi}(\beta)=\frac{\operatorname{ctg}\,\beta}{\sqrt{\left(n+\delta_n(\pi,\beta)\right)^2+\operatorname{ctg}^2\beta}}=\frac{\operatorname{ctg}\beta}{n}+\operatorname{ctg}\beta\cdot O\left(\frac{1}{n^2}\right).$$

Using $\arccos x=\frac{\pi}{2}-\arcsin x$ and $\arcsin x=x+O\left(x^3\right)$ we obtain

$$\delta_n(\pi,\beta)=1-\frac{1}{2}+\frac{1}{\pi}x_{n,\pi}(\beta)+O\left(x^3_{n,\pi}(\beta)\right)=\frac{1}{2}+\frac{\operatorname{ctg}\beta}{\pi\left(n+\frac{1}{2}\right)}+\operatorname{ctg}\beta\cdot O\left(\frac{1}{n^2}\right).$$

Substituting $\delta_n(\pi,\beta)$ into (1.8) and using Lemma 2 we obtain (1.5). Similarly $\delta_n(\alpha,0)=\frac{1}{\pi}\arccos x_{n,0}(\alpha)=\frac{1}{2}-\frac{1}{\pi}\arcsin x_{n,0}(\alpha)=\frac{1}{2}-\frac{\operatorname{ctg}\alpha}{\pi\left(n+\frac{1}{2}\right)}+\operatorname{ctg}\alpha\cdot O\left(\frac{1}{n^2}\right)$ and we obtain (1.6). When $\sin\alpha\neq 0$, $\sin\beta\neq 0$ ($\alpha,\beta\in(0,\pi)$), similarly we obtain

$$\delta_n(\alpha,\beta)=\frac{1}{\pi n}\left(\operatorname{ctg}\beta-\operatorname{ctg}\alpha\right)+\operatorname{ctg}\alpha\cdot O\left(\frac{1}{n^2}\right)+\operatorname{ctg}\beta\cdot O\left(\frac{1}{n^2}\right)$$

and substituting into (1.8) we obtain (1.4). ■

It is useful to remark that $\{\delta_n(\alpha,\beta)\}_{n=1}^{\infty}$ form a sequence of functions, which are analytic in the domain $(0,\pi)\times(0,\pi)$, continuous on $[0,\pi]\times[0,\pi]$ and the limit function $\delta_\infty(\alpha,\beta)$ is discontinuous:

$$\delta_\infty(\alpha,\beta)=\begin{cases}-1, & \alpha=0,\ \ \beta=\pi,\\ -\frac{1}{2}, & \alpha=0,\ \ \beta\in(0,\pi)\ \text{ and }\ \beta=\pi,\ \ \alpha\in(0,\pi),\\ 0, & (\alpha,\beta)\in(0,\pi)\times(0,\pi),\ \ (\alpha=0,\ \beta=0),\ \ (\alpha=\pi,\ \beta=\pi),\\ \frac{1}{2}, & \alpha=\pi,\ \ \beta\in(0,\pi)\ \text{ and }\ \beta=0,\ \ \alpha\in(0,\pi),\\ 1, & \alpha=\pi,\ \ \beta=0.\end{cases}$$

From this point of view it is interesting to compare the principal part $[n+\delta_n(\alpha,\beta)]^2$ of our formula (1.8) and the remark 1 in article [3, p. 768].

It is also easy to see that for $\alpha\in(0,\pi]$ and $\beta\in[0,\pi)$ $\delta_n(\alpha,\beta)=\delta_n(\pi-\beta,\pi-\alpha)$. This equality is, on the other hand, a consequence of the equality $\mu_n(q,\alpha,\beta)=\mu_n(q^*,\pi-\beta,\pi-\alpha)$ (see [3, p. 763]), where $q^*(x)=q(\pi-x)$.

4. The proof of Lemma 2

Let $y_i(x,\mu)=y_i(x,\mu,q)$, $i=1,2$, be the solutions of (1.1), which satisfy the initial conditions $y_1(0,\mu)=1$, $y_1'(0,\mu)=0$ and $y_2(0,\mu)=0$, $y_2'(0,\mu)=1$. For these solutions the estimates are known (see [10], also [1], [2], [6]), which, in the case of equation $-y''+tq(x)y=\lambda^2y$, can be written in the form ($\sigma_0(x)=\int_0^x|q(s)|\,ds$):

$$\left|y_1(x,\lambda^2,tq)-\cos\lambda x\right|\leqslant\frac{|t|\sigma_0(x)}{|\lambda|}\,e^{|\mathrm{Im}\lambda|x+\frac{t\sigma_0(x)}{|\lambda|}},$$

$$\left|y_2(x,\lambda^2,tq)-\frac{\sin\lambda x}{\lambda}\right|\leqslant\frac{|t|\sigma_0(x)}{|\lambda|^2}\,e^{|\mathrm{Im}\lambda|x+\frac{t\sigma_0(x)}{|\lambda|}}.$$

The solution $\varphi(x,\mu,tq,\alpha)=y_1(x,\mu)\sin\alpha-y_2(x,\mu)\cos\alpha$ can be written in the form

$$\varphi(x,\lambda^2,tq,\alpha)=(\cos\lambda x+r_1(x,tq,\lambda))\sin\alpha-\left(\frac{\sin\lambda x}{\lambda}+r_2(x,tq,\lambda)\right)\cos\alpha,$$

where for real $\mu=\lambda^2>0$ ($Im\,\mu=Im\,\lambda=0$) $r_1(x,tq,\lambda)=O\left(\frac{1}{\lambda}\right)$ and $r_2(x,tq,\lambda)=O\left(\frac{1}{\lambda^2}\right)$ uniformly in $t\in[0,1]$, $x\in[0,\pi]$ and $q\in BL^1_{\mathbf{R}}[0,\pi]$. Then for eigenfunctions $\varphi_n(x,tq)=\varphi(x,\mu_n(tq,\alpha,\beta),tq,\alpha)$ in the case $\sin\alpha\neq0$ we have $\varphi_n^2(x,tq)=\cos^2\lambda_nx\cdot\sin^2\alpha+O\left(\frac{1}{n}\right)$, and in the case $\sin\alpha=0$ ($\alpha=\pi$), $\varphi_n^2(x,tq,\pi)=\frac{\sin^2\lambda_nx}{\lambda_n^2}+O\left(\frac{1}{n^3}\right)$. It follows that, if $\sin\alpha\neq0$, then

$$\|\varphi_n\|^2=\sin^2\alpha\int_0^\pi\frac{1+\cos2\lambda_nx}{2}\,dx+O\left(\frac{1}{n^3}\right)=\frac{\pi}{2}\sin^2\alpha+O\left(\frac{1}{n}\right),$$

and if $\sin\alpha=0$, we have

$$\|\varphi_n\|^2=\frac{1}{\lambda_n^2}\int_0^\pi\frac{1-\cos2\lambda_nx}{2}\,dx+O\left(\frac{1}{n}\right)=\frac{\pi}{2\lambda_n^2}\left(1+O\left(\frac{1}{n}\right)\right).$$

Thus, if we define the normalized eigenfunctions as $h_n(x,tq)=\frac{\varphi_n(x,tq)}{\|\varphi_n\|}$, then in both cases we have

$$h_n^2(x,tq)=\frac{\varphi_n^2(x,tq)}{\|\varphi_n\|^2}=\frac{1\pm\cos2\lambda_nx}{\pi}+O\left(\frac{1}{n}\right)$$

and the estimate of the rest is uniform in $x\in[0,\pi]$, $t\in[0,1]$, $\alpha,\beta\in[0,\pi]$ and $q\in BL^1_{\mathbf{R}}[0,\pi]$. And so

$$\begin{aligned}&\int_0^1\left[\int_0^\pi q(x)h_n^2(x,tq,\alpha,\beta)\,dx\right]dt\\&\quad=\frac{1}{\pi}\int_0^\pi q(x)\,dx\pm\frac{1}{\pi}\int_0^1\left[\int_0^\pi q(x)\cos2\lambda_n(tq,\alpha,\beta)\,dx\right]dt+O\left(\frac{1}{n}\right)\\&\quad=[q]+r_n(q,\alpha,\beta),\end{aligned}$$

and since $\lambda_n(tq, \alpha, \beta) \to \infty$ when $n \to \infty$, then $r_n(q, \alpha, \beta) = o(1)$ (uniformly in $\alpha, \beta \in [0, \pi]$ and $q \in BL^1_{\mathbf{R}}[0, \pi]$, by Riemann-Lebesgue lemma. Lemma 2 is proved. ∎

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