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FORCING SIGNED DOMINATION NUMBERS IN GRAPHS

S. M. Sheikholeslami

Abstract. We initiate the study of forcing signed domination in graphs. A function $f : V(G) \longrightarrow \{-1,+1\}$ is called *signed dominating function* if for each $v \in V(G)$, $\sum_{u\in N[v]} f(u) \geq 1$. For a signed dominating function f of G, the *weight* f is $w(f) = \sum_{v\in V} f(v)$. The *signed domination number* $\gamma_s(G)$ is the minimum weight of a signed dominating function on G. A signed dominating function of weight $\gamma_s(G)$ is called a $\gamma_s(G)$*-function.* A $\gamma_s(G)$-function f can also be represented by a set of ordered pairs $S_f = \{(v, f(v)) : v \in V\}$. A subset T of S_f is called a *forcing subset* of S_f if S_f is the unique extension of T to a $\gamma_s(G)$-function. The *forcing signed domination number* of S_f, $f(S_f, \gamma_s)$, is defined by $f(S_f, \gamma_s) = \min\{|T| : T$ is a forcing subset of $S_f\}$ and the *forcing signed domination number* of G, $f(G, \gamma_s)$, is defined by $f(G, \gamma_s) = \min\{f(S_f, \gamma_s) : S_f$ is a $\gamma_s(G)$-function$\}$. For every graph G, $f(G, \gamma_s) \geq 0$. In this paper we show that for integer a, b with a positive, there exists a simple connected graph G such that $f(G, \gamma_s) = a$ and $\gamma_s(G) = b$. The forcing signed domination number of several classes of graph, including paths, cycles, Dutch-windmills, wheels, ladders and prisms are determined.

1. Introduction

Let G a be graph with vertex set $V(G)$. For every vertex $v \in V(G)$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and its *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. For a function $f : V(G) \longrightarrow \{-1, 1\}$ and a subset S of $V(G)$ we define $f(S) = \sum_{u\in S} f(u)$. If $S = N[v]$ for some $v \in V$, then we denote $f(S)$ by $f[v]$. For a function $f : V(G) \longrightarrow R$, the *weight* f is $w(f) = \sum_{v\in V} f(v)$. A *signed dominating function* of G is a function $f : V(G) \longrightarrow \{+1, -1\}$ such that $f[v] \geq 1$ for all $v \in V$. The *signed domination number* $\gamma_s(G)$ is the minimum weight of a signed dominating function on G. A signed dominating function of weight $\gamma_s(G)$ is defined a $\gamma_s(G)$*-function.* For every graph G, we have $\gamma_s(G) \in Z$. The signed domination number was introduced by Dunbar et al. in [2] and since then many results have also been obtained on the parameter $\gamma_s(G)$ (see for instance [3, 4, 8, 9, 11, 13]. We use [12] for terminology and notation which are not defined here.

A signed dominating function f of G can also be represented by a set of ordered pairs $S_f = \{(v, f(v)) \mid v \in V\}$. Let f be a $\gamma_s(G)$-function. A subset T of S_f is called a *forcing subset* of S_f if S_f is the unique extension of T to a $\gamma_s(G)$-

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function. The *forcing signed domination number of* S_f, $f(S_f, \gamma_s)$, is defined by $f(S_f, \gamma_s) = \min\{|T| : \text{T is a forcing subset of } S_f\}$. The *forcing signed domination number* $f(G, \gamma_s)$ is defined by $f(G, \gamma_s) = \min\{f(S_f, \gamma_s) \mid S_f \text{ is a } \gamma_s(G)\text{-function}\}$. Hence for every graph G, $f(G, \gamma_s) \geq 0$.

The concept of forcing numbers has been studied in different areas of combinatorics and graph theory, including the chromatic numbers [10], domination numbers [1, 6] and semi-H-cordial labeling of a graph [7]. In this paper we initiate the study of forcing signed domination numbers in graphs. The paper is organized as follows: In Section 2, we give some preliminary results for $f(G, \gamma_s)$. We also prove that for every two integer a, b of which a is positive, there exists a simple connected graph G such that $f(G, \gamma_s) = a$ and $\gamma_s(G) = b$. In section 3, we find the forcing signed domination number of paths and cycles. In Section 4, we determine the forcing signed domination number of Dutch-windmill graphs and wheels. Section 5 is devoted to determine the forcing signed domination number of ladders and prisms.

Here are some well-known results on $\gamma_s(G)$.

Theorem A. [2] *If f is a signed dominating function for a graph G, then each endvertex and each vertex adjacent with an endvertex of G is assigned the value 1 under f.*

Theorem B. [5] *For $n \geq 2$, $\gamma_s(P_n) = n - 2\lfloor \frac{n-2}{3} \rfloor$.*

Theorem C. [5] *For $n \geq 3$, $\gamma_s(C_n) = n - 2\lfloor \frac{n}{3} \rfloor$.*

Theorem D. [8] *For $n \geq 2$,*

$$\gamma_s(P_2 \times P_n) = \begin{cases} n & \text{if} \quad n \text{ is even;} \\ n-1 & \text{if} \quad n \text{ is odd.} \end{cases}$$

Theorem E. [8] *For $n \geq 3$,*

$$\gamma_s(P_2 \times C_n) = \begin{cases} n & \text{if} \quad n \equiv 0 \pmod 4; \\ n+2 & \text{if} \quad n \equiv 2 \pmod 4; \\ n+1 & \text{if} \quad n \quad \text{is} \quad \text{odd.} \end{cases}$$

2. Realizability

We have already noted that if G is a graph with $f(G, \gamma_s) = a$ and $\gamma_s(G) = b$, then $a \geq 0$ and $b \in Z$. In this section we prove the corresponding realization result.

The following observations will be useful in this note.

Observation 1. Let G be a graph with $\Delta \leq 3$, g be a signed dominating function of G and $u, v \in V(G)$. If $g(u) = g(v) = -1$, then $d(u, v) \geq 3$.

Observation 2. For a graph G, $f(G, \gamma_s) = 0$ if and only if G has a unique $\gamma_s(G)$-function. Moreover, $f(G, \gamma_s) = 1$ if and only if G does not have a unique $\gamma_s(G)$-function but some pair $(v, \pm 1)$ belongs to exactly one $\gamma_s(G)$-function.

The following result is a direct consequence of Observation 2

Corollary 3. *For a graph G, $f(G, \gamma_s) > 1$ if and only if every pair $(v, \pm 1)$ of each $\gamma_s(G)$-function belongs to at least two $\gamma_s(G)$-functions.*

Theorem 4. *For every graph G of order n, if $\gamma_s(G) = n$ then $f(G, \gamma_s) = 0$.*

Proof. Let $\gamma_s(G) = n$. We show that every non isolated vertex is either an endvertex or adjacent to an endvertex. Consider a vertex v that is neither an endvertex nor adjacent to an endvertex. Then we can assign -1 to v and $+1$ to each other vertex, to produce a signed dominating function on G of weight $n-2$, which is a contradiction. This proves our claim and the theorem is true by Theorem A. ■

Corollary 5. *For $n \geq 1$, $f(K_{1,n}, \gamma_s) = 0$.*

Proof. By Theorem A, $\gamma_s(K_{1,n}) = n$ and the result follows by Theorem 4. ■

Next theorem shows that for every pair a, b of integers, with a positive, there exists a simple connected graph G such that $f(G, \gamma_s) = a$ and $\gamma_s(G) = b$.

Theorem 6. *For every two integers a and b, with a positive, there exists a simple connected graph G such that $a = f(G, \gamma_s)$ and $b = \gamma_s(G)$.*

Proof. Let G be obtained from complete graph $K_{8|b|+8}$ whose vertex set is $\{v_1, \ldots, v_{8|b|+8}\}$, by adding $24|b| + 24$ new vertices, say $u_1, u_2, \ldots, u_{8|b|+8}$, $w_1, w_2, \ldots, w_{8|b|+8}$, $z_1, z_2, \ldots, z_{8|b|+8}$ and new edges $u_iv_i, v_1w_i, v_2w_i, v_3z_i, v_4z_i$ for each i. We consider three cases.

Case 1. $b = 0$. Obviously $f(G, \gamma_s) = \gamma_s(G) = 0$. Suppose now that $a > 0$. Let G_1 be obtained from G by adding $2a$ new vertices, say m_i, n_i $(1 \leq i \leq a)$, and new edges $v_{8|b|+8}m_i, v_{8|b|+8}n_i$ and m_in_i for $i = 1, \ldots, a$. It is easy to see that $f(G_1, \gamma_s) = a$ and $\gamma_s(G_1) = 0$.

Case 2. $b > 0$. First let $a = 0$. Let G_2 be obtained from G by adding b pendant edges at $v_{8|b|+8}$, say $v_{8|b|+8}y_1, \ldots, v_{8|b|+8}y_b$. It is easy to see that $\gamma_s(G_2) = b$ and $f(G_2, \gamma_s) = 0$. Suppose now that $a > 0$. Let G_3 be obtained from G_2 by adding $2a$ new vertices m_i, n_i $(1 \leq i \leq a)$ and new edges $v_{8|b|+8}m_i, v_{8|b|+8}n_i$ and m_in_i for $i = 1, \ldots, a$. One can see that $f(G_3, \gamma_s) = a$ and $\gamma_s(G_3) = b$.

Case 3. $b < 0$. If $a = 0$, then let G_4 be obtained from G by adding $|b|$ new vertices, say $y_1, \ldots, y_{|b|}$, and joining them to both v_5, v_6. Obviously $f(G_4, \gamma_s) = 0$ and $\gamma_s(G_4) = b$. If $a > 0$, then let G_5 be obtained from G_4 by adding $2a$ new vertices m_i, n_i $(1 \leq i \leq a)$ and adding new edges $v_{8|b|+8}m_i, v_{8|b|+8}n_i$ and m_in_i for $i = 1, \ldots, a$. It is easy to verify that $f(G_5, \gamma_s) = a$ and $\gamma_s(G_5) = b$. This completes the proof. ■

3. Forcing signed domination number of paths and cycles

In this section we determine the forcing signed domination number of paths and cycles. We begin with the forcing signed domination number of paths. Since for $1 \leq n \leq 4$, $f(P_n, \gamma_s) = 0$ by Theorem A, we consider paths of order at least 5.

Theorem 7. *For $n \geq 5$,*

$$f(P_n,\gamma_s) = \begin{cases} 0 & \text{if} \quad n \equiv 2 \pmod 3; \\ 1 & \text{if} \quad n \equiv 0 \text{ or } 1 \pmod 3. \end{cases}$$

Proof. Let $P_n = v_1, v_2, \ldots, v_n$ and g be a γ_s-function of P_n. By Theorem A, $g(v_1) = g(v_2) = g(v_n) = g(v_{n-1}) = 1$. By Observation 1, $g(v_i) = g(v_j) = -1$ implies that $|i-j| \geq 3$. Therefore, the number of vertices of P_n which g can assign -1 to them is at most $\lfloor \frac{n-2}{3} \rfloor$. On the other hand, g must assign the value -1 to exactly $\lfloor \frac{n-2}{3} \rfloor$ vertices of P_n by Theorem B. If $n = 3k+2$ for some $k \in \mathbf{N}$, then obviously $g(v_{3i}) = -1$ for $i = 1, \ldots, k$ and g assigns the value 1 to each other vertex. Thus $f(P_{3k+2}, \gamma_s) = 0$.

Now let $n \not\equiv 2 \pmod 3$. Define $g, h : V(P_n) \longrightarrow \{-1, +1\}$ by

$$g(v_i) = \begin{cases} -1 & \text{if} \quad i = 3, 6, \ldots, 3(\lfloor \frac{n}{3} \rfloor - 1); \\ 1 & \text{otherwise,} \end{cases}$$

and

$$h(v_i) = \begin{cases} -1 & \text{if} \quad i = 4, 7, \ldots, 3(\lfloor \frac{n}{3} \rfloor - 1) + 1; \\ 1 & \text{otherwise .} \end{cases}$$

It is easy to see that g and h are $\gamma_s(P_n)$-function. It follows that $f(P_n, \gamma_s) \geq 1$ by Observation 2. Consider two cases.

Case 1. $n \equiv 0 \pmod 3$. Let $T = \{(v_{n-2}, 1)\}$. We claim that T is a forcing subset for g. Let f be a $\gamma_s(P_n)$-function such that $f(v_{n-2}) = 1$. This forces $f(v_{3i}) = -1$ for $1 \leq i \leq \frac{n}{3} - 1$, which implies f assigns the value 1 to each other vertex. Therefore $f = g$ and $f(S_g, \gamma_s) \leq 1$. Thus, $f(P_n, \gamma_s) \leq 1$.

Case 2. $n \equiv 1 \pmod 3$. We show that $T = \{(v_{n-4}, -1)\}$ is a forcing subset of g. Let f be a $\gamma_s(P_n)$-function such that $f(v_{n-4}) = -1$. This forces $f(v_{n-3}) = f(v_{n-2}) = f(v_{n-5}) = f(v_{n-6}) = 1$ by Theorem A and Observation 1. Since f must assign the value -1 exactly to $\lfloor \frac{n-2}{3} \rfloor$ vertices of P_n by Theorem B, we must have $f(v_3) = f(v_6) = \ldots = f(v_{n-4}) = -1$. It follows that f must assign the value 1 to each other vertex. Thus $f = g$ and $f(S_g, \gamma_s) \leq 1$. Therefore $f(P_n, \gamma_s) \leq 1$ and the proof is complete. ∎

Next we determine $f(C_n, \gamma_s)$ for all cycles. Obviously, $f(C_n, \gamma_s) = 1$ when $n = 3, 4, 5$. Therefore, we consider cycles of order at least 6.

Theorem 8. *For $n \geq 6$,*

$$f(C_n,\gamma_s) = \begin{cases} 1 & \text{if} \quad n \equiv 0 \pmod 3; \\ 2 & \text{if} \quad n \equiv 1 \text{ or } 2 \pmod 3. \end{cases}$$

Proof. Let $C_n = v_1, v_2, \ldots, v_n$. By Theorem C, $\gamma_s(C_n) = n - 2\lfloor \frac{n}{3} \rfloor$. Define $g, h : V(C_n) \longrightarrow \{-1, +1\}$ by

$$g(v_i) = \begin{cases} -1 & \text{if} \quad i = 1, 4, \ldots, 3(\lfloor \frac{n}{3} \rfloor - 1) + 1; \\ 1 & \text{otherwise,} \end{cases}$$

and

$$h(v_i) = \begin{cases} -1 & \text{if} \quad i = 2, 5, \ldots, 3(\lfloor \frac{n}{3} \rfloor - 1) + 2; \\ 1 & \text{otherwise} . \end{cases}$$

It is easy to see that g and h are $\gamma_s(C_n)$-function. It follows that $f(C_n, \gamma_s) \geq 1$ by Observation 2. First let $n \equiv 0 \pmod 3$. We claim that $T = \{(v_1, -1)\}$ is a forcing subset for g. Let f be an extension of T to a $\gamma_s(C_n)$-function. This forces $f(v_2) = f(v_3) = f(v_n) = f(v_{n-1}) = 1$ by Observation 1. Since f is a $\gamma_s(C_n)$-function, f must assign the value -1 to exactly $\lfloor \frac{n}{3} \rfloor$ vertices of C_n by Theorem C. This forces $f(v_4) = f(v_7) = \ldots = f(v_{n-2}) = -1$. It follows that f assigns the value 1 to each other vertex of C_n. Therefore $f = g$ and $f(C_n, \gamma_s) \leq 1$.

Now let $n \equiv 1 \pmod 3$. Then $n = 3k + 1$ for some $k \geq 2$. First we show that every set $T = \{(v, \varepsilon) \mid v \in V(C_n) \text{ and } \varepsilon = +1 \text{ or } -1\}$ which has an extension to a $\gamma_s(C_n)$-function, has at least two extension to a $\gamma_s(C_n)$-function which implies $f(C_n, \gamma_s) \geq 2$. Without loss of generality, we can assume $v = v_1$ and $\varepsilon = -1$. Define $g, h : V(C_n) \longrightarrow \{-1, +1\}$ by

$$g(v_i) = \begin{cases} -1 & \text{if} \quad i = 1, 4, \ldots, 3(k-1) + 1; \\ 1 & \text{otherwise,} \end{cases}$$

and

$$h(v_i) = \begin{cases} -1 & \text{if} \quad i = 1, 5, \ldots, 3(k-1) + 2; \\ 1 & \text{otherwise} . \end{cases}$$

It is easy to see that g and h are $\gamma_s(C_n)$-function such that $g(v_1) = h(v_1) = -1$. It follows that $f(C_n, \gamma_s) \geq 2$ by Corollary 3. Now it is easy to verify that $T = \{(v_1, -1), (v_{n-2}, 1)\}$ is a forcing subset of g which implies $f(C_n, \gamma_s) \leq 2$. Thus $f(C_n, \gamma_s) = 2$.

If $n \equiv 2 \pmod 3$, then an argument similar to that described in case $n \equiv 1 \pmod 3$ shows that $f(C_n, \gamma_s) = 2$. This completes the proof. ∎

4. The Dutch-windmill graphs and wheels

The Dutch-windmill graph, $K_3^{(m)}$, is a graph which consists of m copies of K_3 with a vertex in common. The wheel, W_n, is a graph with $n+1$ vertices $\{v_0, v_1, \ldots, v_n\}$ and edges $\{v_0 v_i \mid 1 \leq i \leq n\} \cup \{v_1 v_2, v_2 v_3, \ldots, v_{n-1} v_n, v_n v_1\}$. In this section we find the forcing signed domination number of $K_3^{(m)}$ and W_n.

Lemma 9. *For every positive integer m, $\gamma_s(K_3^{(m)}) = 1$.*

Proof. By Theorem C, we may assume $m \geq 2$. Let v, u_i, w_i are the vertices of the i-th copy of K_3 in $K_3^{(m)}$ (v is the common vertex). Define $g : V(K_3^{(m)}) \longrightarrow \{-1, +1\}$ by

$$g(w) = \begin{cases} 1 & \text{if} \quad w = v, w_i \text{ and } 1 \leq i \leq m; \\ -1 & \text{if} \quad w = u_i \text{ and } 1 \leq i \leq m. \end{cases}$$

Obviously g is a signed dominating function for $K_3^{(m)}$. Thus, $\gamma_s(K_3^{(m)}) \leq 1$. Now let h be a γ_s-function of $K_3^{(m)}$. Then $h(v) = 1$, for otherwise h must assign the value $+1$ to each other vertex which leads to $\gamma_s(h) = 2m-1 > 1$, a contradiction. Now h can assign the value -1 to exactly one of the vertices w_i or v_i for each i. Thus, $w(h) \geq 1$ and $\gamma_s(K_3^{(m)}) = 1$. ■

Theorem 10. *For every positive integer m, $f(K_3^{(m)}, \gamma_s) = m$.*

Proof. Let g be the γ_s-function of $K_3^{(m)}$ defined in Lemma 9. It is easy to see that $T = \{(u_i, -1) \mid 1 \leq i \leq m\}$ is a forcing subset of g. Therefore, $f(K_3^{(m)}, \gamma_s) \leq m$. Now we show that $f(K_3^{(m)}, \gamma_s) \geq m$. Let $S = \{(w, \varepsilon_w) \mid w \in V(K_3^{(m)})$ and $\varepsilon_w = +1$ or -1$\}$ where $|S| < m$ and S has at least an extension to a γ_s-function of $K_3^{(m)}$. Without loss of generality we may assume S does not intersect the first copy of K_3. Define $S_1 = S \cup \{(w_1, 1), (u_1, -1)\}$ and $S_2 = S \cup \{(w_1, -1), (u_1, 1)\}$. Now we can extend S_1 and S_2, to a γ_s-function of $K_3^{(m)}$. It follows that S is not a forcing subset for any γ_s-function of $K_3^{(m)}$. Thus, $f(K_3^{(m)}, \gamma_s) \geq m$ and the proof is complete. ■

Since $\gamma_s(W_3) = 2$ and $\gamma_s(W_4) = 3$, we consider W_n with $n \geq 5$.

Lemma 11. *For $n \geq 5$, $\gamma_s(W_n) = n + 1 - 2\lfloor \frac{n}{3} \rfloor$.*

Proof. Define $g : W_n \longrightarrow \{-1, +1\}$ by

$$g(w) = \begin{cases} -1 & \text{if} \quad w = v_{3i+1} \ \text{ and } \ 0 \leq i \leq \lfloor \frac{n}{3} \rfloor - 1; \\ 1 & \text{otherwise} . \end{cases}$$

Obviously g is a signed dominating function for W_n which implies $\gamma_s(W_n) \leq n + 1 - 2\lfloor \frac{n}{3} \rfloor$. Now let h be a γ_s-function of W_n. We claim that $h(v_0) = 1$. Let, to the contrary, $h(v_0) = -1$. Since $\deg(v_i) = 3$ for each i, h must assign the value $+1$ to each other vertex which implies $\gamma_s(h) = n - 1 > n + 1 - 2\lfloor \frac{n}{3} \rfloor$, a contradiction. Therefore $h(v_0) = 1$. Since $deg(v_i) \leq 3$ for each $i \geq 1$, $h(v_i) = h(v_j) = -1$ implies that $|i-j| \geq 3$ by Observation 1. It follows that $|\{w \in V(W_n) : h(w) = -1\}| \leq \lfloor \frac{n}{3} \rfloor$. Thus $\gamma_s(W_n) = w(h) \geq n + 1 - 2\lfloor \frac{n}{3} \rfloor$ and the proof is complete. ■

It is easy to see that $f(W_n, \gamma_s) = 1$ when $n = 3, 4, 5$. Therefore, we assume $n \geq 6$.

Theorem 12. *For $n \geq 6$,*

$$f(W_n, \gamma_s) = \begin{cases} 1 & \text{if} \quad n \equiv 0 \pmod 3; \\ 2 & \text{otherwise} . \end{cases}$$

Proof. First let $n \equiv 0 \pmod 3$. Define $g, h : V(W_n) \longrightarrow \{-1, +1\}$ by

$$g(v_i) = \begin{cases} -1 & \text{if} \quad i = 1, 4, \ldots, 3(\lfloor \frac{n}{3} \rfloor - 1) + 1; \\ 1 & \text{otherwise,} \end{cases}$$

and

$$h(v_i) = \begin{cases} -1 & \text{if} \quad i = 2, 5, \ldots, 3(\lfloor \frac{n}{3} \rfloor - 1) + 2; \\ 1 & \text{otherwise .} \end{cases}$$

Obviously g and h are signed dominating function for W_n. By Lemma 11, g and h are $\gamma_s(W_n)$-functions. It follows that $f(W_n, \gamma_s) \geq 1$ by Observation 2. It is easy to verify that $T = \{(v_1, -1)\}$ is a forcing subset for g which implies$f(W_n, \gamma_s) = 1$.

Now let $n \not\equiv 0 \pmod 3$. Let $T = \{(v, \varepsilon) \mid v \in V(W_n) \text{ and } \varepsilon = +1 \text{ or } -1\}$ and T has an extension to a $\gamma_s(W_n)$-function. If $i = 0$ or $\varepsilon = +1$, then obviously T has at least two extension to a $\gamma_s(W_n)$-function. Suppose now that $i \neq 0$ and $\varepsilon = -1$. Without loss of generality we may assume $v = v_1$. Define $g^*, h^* : V(W_n) \longrightarrow \{-1, +1\}$ by

$$g^*(v_i) = \begin{cases} -1 & \text{if} \quad i = 1, 4, \ldots, 3(\lfloor \frac{n}{3} \rfloor - 1) + 1; \\ 1 & \text{otherwise,} \end{cases}$$

and

$$h^*(v_i) = \begin{cases} -1 & \text{if} \quad i = 1, 5, \ldots, 3(\lfloor \frac{n}{3} \rfloor - 1) + 2; \\ 1 & \text{otherwise .} \end{cases}$$

Obviously g^* and h^* are signed dominating functions for which $g(v_1) = h(v_1) = -1$ and by Lemma 11, g^* and h^* are $\gamma_s(W_n)$-functions. It follows that $f(W_n, \gamma_s) \geq 2$ by Corollary 3. It is straightforward to see that $T_1 = \{(v_1, -1), (v_{n-2}, 1)\}$ if $n \equiv 1 \pmod 3$ and $T_2 = \{(v_1, -1), (v_{n-4}, -1)\}$ when $n \equiv 2 \pmod 3$, is a forcing subset of g^* which implies $f(W_n, \gamma_s) \leq 2$. Thus $f(W_n, \gamma_s) = 2$ and the proof is complete. ■

5. Ladders and Prisms

In this section we find the forcing signed domination number of ladders and prisms. Throughout this section we assume the vertices of the i-th copy of P_2 in ladders $P_2 \times P_n$ (prisms $P_2 \times C_n$) are u_i, v_i for $i = 1, 2, \ldots, n$. (see Figure 1).

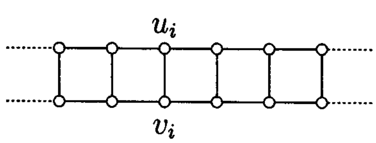

Fig. 1. Ladders $P_2 \times P_n$

Since $P_2 \times P_2 = C_4$, $f(P_2 \times P_2, \gamma_s) = 1$. We assume $n \geq 3$.

THEOREM 13. *For $n \geq 3$, $f(P_2 \times P_n, \gamma_s) = 1$.*

Proof. Define $g, h : V(P_2 \times P_n) \longrightarrow \{-1, +1\}$ by

$$g(w) = \begin{cases} -1 & \text{if} \quad w = u_{4i+1} \text{ and } 0 \leq i \leq \lceil \frac{n}{4} \rceil - 1; \\ -1 & \text{if} \quad w = v_{4i+3} \text{ and } 0 \leq i \leq \lfloor \frac{n+1}{4} \rfloor - 1; \\ 1 & \text{otherwise,} \end{cases}$$

and $h(u_i)=g(v_i)$ and $h(v_i)=g(u_i)$ for each i. It is easy to see that g and h are γ_s-functions for $P_2\times P_n$ by Theorem D. It follows that $f(P_2\times P_n,\gamma_s)\geq 1$ by Observation 2. Let g be the γ_s-function defined above. Let $M=\{w\in V(P_2\times P_n)\mid g(w)=-1\}$. Consider two cases.

Case 1. n is odd. By Theorem D, $|M|=\frac{n+1}{2}$. Now we show that $T=\{(u_1,-1)\}$ is a forcing subset for g. Let f be an extension of T to a $\gamma_s(P_2\times P_n)$-function. By Observation 1, $f(v_1)=f(u_2)=f(v_2)=f(u_3)=1$ and $|M\cap\{u_i,v_i,u_{i-1},v_{i-1},u_{i+1},v_{i+1}\}|\leq 1$ for each i. Since $|M|=\frac{n+1}{2}$, $f(v_3)=-1$. An inductive argument shows that $f(u_{4i+1})=-1$ for $0\leq i\leq\lceil\frac{n}{4}\rceil-1$, $f(v_{4i+3})=-1$ for $0\leq i\leq\lfloor\frac{n+1}{4}\rfloor-1$ and f assigns the value $+1$ to each other vertex. Thus, $f=g$ and $f(P_2\times P_n,\gamma_s)\leq f(S_g,\gamma_s)\leq 1$.

Case 2. n is even. An argument similar to that described in case 1, proves that $T_1=\{(v_{n-1},-1)\}$ and $T_2=\{(u_{n-1},-1)\}$ are forcing subsets of g when $4|n$ and $4\nmid n$, respectively. It follows that $f(P_2\times P_n,\gamma_s)\leq f(S_g,\gamma_s)\leq 1$ and the proof is complete. ■

Finally, we determine $f(P_2\times C_n,\gamma_s)$ for $n\geq 3$. Since $f(P_2\times C_n,\gamma_s)=1$ when n is 3 and 4, we assume $n\geq 5$.

Theorem 14. *For $n\geq 5$,*

$$f(P_2\times C_n,\gamma_s)=\begin{cases}1 & \text{if}\quad n\equiv 0 \pmod 4;\\ 2 & \text{otherwise}\ .\end{cases}$$

Proof. First let $n\equiv 0\pmod 4$. Define $g,h:V(P_2\times C_n)\longrightarrow\{-1,+1\}$ by

$$g(w)=\begin{cases}-1 & \text{if}\quad w=u_{4i+1},v_{4i+3}\ \text{and}\ 0\leq i\leq\lfloor\frac{n}{4}\rfloor-1;\\ 1 & \text{otherwise,}\end{cases}$$

and $h(u_i)=g(v_i)$ and $h(v_i)=g(u_i)$ for each i. Obviously g and h are signed dominating functions for $P_2\times C_n$. Therefore, g and h are $\gamma_s(P_2\times C_n)$-functions by Theorem E. It follows that $f(P_2\times C_n,\gamma_s)\geq 1$ by Observation 2. An argument similar to that described in the Theorem 13, shows that $T=\{(v_{n-1},-1)\}$ is a forcing subset for g. Thus $f(P_2\times C_n,\gamma_s)=1$.

Now let $n\not\equiv 0\pmod 4$. First we show that $f(P_2\times C_n,\gamma_s)\geq 2$. In order to do this, it is sufficient to show that every set $T=\{(w,\varepsilon_w)\mid w\in V(P_2\times C_n),\varepsilon_w=+1\text{ or }-1\}$ which has at least one extension to a γ_s-function, has two extensions to a γ_s-function for $P_2\times C_n$. Without loss of generality, we may assume $w=u_1$ and $\varepsilon=-1$. Define $g,h:V(P_2\times C_n)\longrightarrow\{-1,+1\}$ by

$$g(w)=\begin{cases}-1 & \text{if}\quad w=u_{4i+1}\ \text{and}\ 0\leq i\leq\lfloor\frac{n+1}{4}\rfloor-1;\\ -1 & \text{if}\quad w=v_{4i+3}\ \text{and}\ 0\leq i\leq\lfloor\frac{n}{4}\rfloor-1;\\ 1 & \text{otherwise,}\end{cases}$$

and

$$h(w)=\begin{cases}-1 & \text{if}\quad w=u_1,u_{4i+4}\ \text{and}\ 0\leq i\leq\lfloor\frac{n}{4}\rfloor-1;\\ -1 & \text{if}\quad w=v_{4i+2}\ \text{and}\ 1\leq i\leq\lfloor\frac{n+1}{4}\rfloor-1;\\ 1 & \text{otherwise,}\end{cases}$$

if $n \equiv 3 \pmod 4$, and

$$h(w) = \begin{cases} -1 & \text{if} \quad w = u_1, u_{4i+2}, \quad 1 \le i \le \lfloor \frac{n+1}{4} \rfloor - 1 \text{ and } \lfloor \frac{n+1}{4} \rfloor \ge 2; \\ -1 & \text{if} \quad w = v_{4i+4}, \quad 0 \le i \le \lfloor \frac{n}{4} \rfloor - 1; \\ 1 & \text{otherwise,} \end{cases}$$

when $n \equiv 1$ or $2 \pmod 4$. Obviously g, h are signed dominating functions for $P_2 \times C_n$ in each case and by Theorem E, g and h are $\gamma_s(P_2 \times C_n)$-functions. It follows that $f(P_2 \times C_n, \gamma_s) \ge 2$ by Corollary 3. Now we show that $T_1 = \{(u_1, -1), (u_{n-2}, -1)\}$ if $n \equiv 3 \pmod 4$, $T_2 = \{(u_1, -1), (v_{n-3}, -1)\}$ if $n \equiv 1 \pmod 4$ and $T_3 = \{(u_1, -1), (v_{n-2}, -1)\}$ when $n \equiv 2 \pmod 4$, is a forcing subset for g.

Let f be an extension of T_1 to a γ_s-function for $P_2 \times C_n$. By Observation 1, if $f(u) = f(v) = -1$, then $d(u, v) \ge 3$ which implies $f(v_1) = f(u_2) = f(v_2) = f(u_3) = f(u_n) = f(v_n) = f(u_{n-1}) = f(v_{n-1}) = f(v_{n-2}) = f(u_{n-3}) = f(v_{n-3}) = f(u_{n-4}) = 1$. Since $|\{w \in V(P_2 \times C_n) \mid f(w) = -1\}| = \frac{n-1}{2}$, an inductive argument shows that $f(u_{4i+1}) = -1$ for $0 \le i \le \lfloor \frac{n+1}{4} \rfloor - 1$, $f(v_{4i+3}) = -1$ for $0 \le i \le \lfloor \frac{n}{4} \rfloor - 1$ and f assigns $+1$ to each other vertex. Thus $f = g$ and $f(P_2 \times C_n, \gamma_s) \le 2$. Now an argument similar to that described for T_1 may be applied to show that T_2 and T_3 are forcing subset for g. Thus $f(P_2 \times C_n, \gamma_s) = 2$ and the proof is complete. ∎

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Department of Mathematics, Azarbaijan University of Tarbiat Moallem, Tabriz, I.R. Iran
E-mail: s.m.sheikholeslami@azaruniv.edu