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GENERALIZED MAXIMUM PRINCIPLES FOR LINEAR ELLIPTIC EQUATIONS

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Abstract. In this paper we extend the classical generalized maximum principle for linear elliptic equations which holds in relatively compact subdomains to a generalized maximum principle which holds in any domain.

1. Introduction

We introduce some new results concerning the maximum principles for second order linear elliptic partial differential equations defined on a noncompact Riemannian manifold,also we prove existence, uniqueness and integral representation for solutions of the nonhomogeneous equation.

Let us first introduce some terminology and results which we need in this paper.

Let P be a linear , second order, elliptic operator defined on a noncompact, connected, C^3-smooth Riemannian manifold Ω of dimension d. Here P is an elliptic operator with real, Hölder continuous coefficients which in any coordinate system $(U; x_1, \dots, x_d)$ has the form

$$P(x, \partial_x) = -\sum_{i,j=1}^{d} a_{ij}(x)\partial_i\partial_j + \sum_{i=1}^{d} b_i(x)\partial_i + c(x), \tag{1.1}$$

where $\partial_i = \partial/\partial_{x_i}$. We assume that for every $x \in \Omega$ the real quadratic form

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \ , \ \ \xi = (\xi_1, \dots, \xi_d) \in \mathbf{R}^d \tag{1.2}$$

is positive definite.

We denote the cone of all positive (classical) solutions of the elliptic equation $Pu = 0$ in Ω by $C_P(\Omega)$. In case that the coefficients of P are smooth enough, we denote by P^* the formal adjoint of P.

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Let $\{\Omega_k\}_{k=1}^{\infty}$ be *an exhaustion* of Ω, i.e. a sequence of smooth, relatively compact domains such that $\Omega_1 \neq \emptyset, \overline{\Omega_k} \subset \Omega_{k+1}$ and $\bigcup_{k=1}^{\infty} \Omega_k = \Omega$. Assume that $C_P(\Omega) \neq \emptyset$. Then for every $k \geq 1$ the Dirichlet Green function $G_P^{\Omega_k}(x,y)$ exists and is positive. By the generalized maximum principle, $\left\{G_P^{\Omega_k}(x,y)\right\}_{k=1}^{\infty}$ is an increasing sequence which, by Harnak inequality, converges uniformly in any compact subdomain of Ω either to $G_P^{\Omega}(x,y)$, the *positive minimal Green function* of P in Ω and P is said to be a *subcritical operator* in Ω, or to infinity and in this case P is *critical* in Ω. The operator P is said to be *supercritical*in Ω if $C_P(\Omega) = \emptyset$.

It follows that P is critical (resp. subcritical) in Ω if and only if P^* is critical (resp. subcritical) in Ω. Furthermore, if P is critical in Ω, then $C_P(\Omega)$ is a one dimensional cone and any positive supersolution of the equation $Pu = 0$ in Ω is a solution. In this case $\phi \in C_P(\Omega)$ is called a *ground state* of P in Ω.

We fix a reference point $x_0 \in \Omega_1$. From time to time, we consider the convex set

$$K_P(\Omega\ ) := \{\, u \in C_P(\Omega) \mid u(x_0) = 1\,\}$$

of all *normalized* positive solutions.

Remark 1.1. We would like to point out that the criticality theory, and in particular the results of this paper, are also valid for the class of weak solutions of elliptic equations in divergence form and also for the class of strong solutions of strongly elliptic equations with locally bounded coefficients. Nevertheless, for the sake of clarity, we prefer to present our results only for the class of classical solutions.

Subcriticality is a stable property in the following sense. If P is subcritical in Ω and $V \in C_0^{\alpha}(\Omega)$ is a real function, then there exist $\epsilon > 0$ such that $P - \mu V$ is subcritical, for all $|\mu| < \epsilon$ [3,4]. On the other hand, if P is critical in Ω and $V \in C^{\alpha}(\ \Omega)$ is a nonzero, nonnegative function, then for every $\epsilon > 0$ the operator $P + \epsilon V$ is subcritical and $P - \epsilon V$ is supercritical in Ω.

We associate to Ω a fixed exhaustion $\{\Omega n\}_{n=1}^{\infty}$. For every $k \geq 1$, we denote $\Omega_k^* =\ \Omega\backslash\overline{\Omega_k}$ and for every $k > k_0$ we denote by Ω_{k,k_0} the 'annulus' $\Omega_k\backslash\overline{\Omega_{k_0}}$. Let $f, g \in C(\Omega)$ be positive functions. We say that f *is equivalent to* g on Ω and use the notation $f \approx g$, if there exists a positive constant C such that

$$C^{-1}g(x) \leq f(x) \leq Cg(x) \ \text{ for all } x \in \Omega.$$

We denote by f^+ (resp. f^-) the positive (resp. negative) part of a function f. So, $f = f^+ - f^-$. By [1], we denote the constant function on Ω taking at any point $x \in \Omega$ the value 1.

Let B be a Banach space and B^* its dual. If T is a (bounded) operator in B we denote by T^* its adjoint. The range and the kernel of T are denoted by $R(T)$ and $N(T)$. For every $f \in B$ and $g^* \in B^*$ we use the notation $\langle g^*, f\rangle := g^*(f)$. We denote the spectrum of an operator T acting on B by $\sigma(T)$.

Definition 1.2. Let P be an elliptic operator defined on Ω. A function $u \in C(\Omega_n^*)$ is said to be a *positive solution of the operator P of minimal growth in a neighborhood of infinity in* Ω if u satisfies the following two conditions:

(i) The function u is a positive solution of the equation $Pu = 0$ in Ω_n^*;

(ii) If v is a continuous function on $\overline{\Omega_k^*}$ for some $k > n$ which is a positive solution of the equation $Pu = 0$ in Ω_k^*, and $u \le v$ on $\partial\Omega_k$, then $u \le v$ on Ω_k^*.

DEFINITION 1.3. Let P_i, $i = 1, 2$ be two subcritical operators in Ω. We say that the Green functions $G_{P_1}^{\Omega}(x, y)$ and $G_{P_2}^{\Omega}(x, y)$ are *equivalent* (resp. *semi-equivalent*) if $G_{P_1}^{\Omega} \approx G_{P_2}^{\Omega}$ on $\Omega \times \Omega \setminus \{(x, x) \mid x \in \Omega\}$ (resp. $G_{P_1}^{\Omega}(\cdot, y_0) \approx G_{P_2}^{\Omega}(\cdot, y_0)$ on $\Omega \setminus \{y_0\}$ for some fixed $y_0 \in \Omega$).

Our maximum principles are of Phragmĕn-Lindelöf type and state that if $Pu \ge 0$ in ΩP, is subcritical in Ω and u satisfies some growth condition near infinity in Ω, then $u \ge 0$ (see theorems 1.4 and 2.6). As a consequence, we prove existence and uniqueness theorems for suitable solutions of the nonhomogeneous equation (see theorems 1.6 and 2.8).

Let P be a subcritical operator in Ω and let $\phi \in C(\Omega)$ be a positive function such that ϕ is a solution of the equation $Pu = 0$ in Ω_1^* which has a minimal growth in a neighborhood of infinity in Ω.

We denote by B the real ordered Banach space

$$B = \{u \in C(\Omega) \mid |u(x)| \le c\phi(x) \text{ for some } c > 0 \text{ and all } x \in \Omega\},$$

equipped with the norm

$$\|u\|_B = \inf\{c > 0 \mid |u(x)| \le c\phi(x) \ \forall x \in \Omega\}.$$

The ordering on B is the natural pointwise ordering of functions. For the purpose of spectral theory, we consider also the canonical complexification of B without changing our notation.

In [6] the following maximum principle for supersolutions in B is proved.

THEOREM 1.4. *Let P be a subcritical operator in Ω and let $\phi \in C(\Omega)$ be a positive function such that ϕ is a solution of the equation $Pu = 0$ in Ω_1^* which has a minimal growth in a neighborhood of infinity in Ω. Suppose that $v \in B$ satisfies the equation $Pv = f \ge 0$ in Ω, where $f \in C^{\alpha}(\Omega)$. Then $v \ge 0$ in Ω.*

In the critical case we have (see [6])

PROPOSITION 1.5. *Let P be a critical operator in Ω and let ϕ_0 be a ground state of the operator P in Ω. Suppose that $Pu \ge 0$ in Ω and that for some $C > 0$, $u \ge -C\phi_0$ in Ω_1^*. Then $u = C_1\phi_0$, where C_1 is a real constant.*

The maximum principle (Theorem 1.4) implies the following theorem concerning the existence, uniqueness and integral representation for solutions of the nonhomogeneous equation [6].

THEOREM 1.6. *Let P be a subcritical operator in Ω and let ϕ be a positive solution of the equation $Pu = 0$ in Ω_1^* which has a minimal growth in a neighborhood of infinity in Ω.*

(i) *Let $f \in C^{\alpha}(\Omega)$, $0 < \alpha \le 1$, be a real function such that*

$$\int_{\Omega} G_P^{\Omega}(x,y)\,|f(y)|\;dy \le C\phi(x) \tag{1.3}$$

for all $x \in \Omega_2^*$. Then there exists a unique solution $u \in B$ of the equation $Pu = f$ in Ω. Moreover,

$$u(x) = \int_{\Omega} G_P^{\Omega}(x,y) f(y)\,dy.$$

(ii) *Suppose that $f \in C^{\alpha}(\Omega)$, $0 < \alpha \le 1$, and $f \ge 0$. Then f satisfies estimate (1.3) if and only if there exists a solution $u \in B$ of the equation $Pu = f$ in Ω. In this case, u is the minimal nonnegative solution of the equation $Pv = f$ in Ω.*

2. The results and their proofs

We present now a new version of the maximum principle which extends Theorem 1.4 and is valid for solutions which do not grow too fast.

We denote by $\beta = \beta^K$ the real ordered Banach space

$$\begin{aligned}\beta = \beta^K := \{\, f \in C(\Omega) \mid |f(x)| \le cu(x) \text{ for some fixed } c > 0 \\ \text{and } u \in K_p(\Omega), \text{ and for all } x \in \Omega \,\}\end{aligned} \tag{2.1}$$

equipped with the norm

$$\|f\|_{\beta} := \inf_{u \in K_p(\Omega)} \inf\{\, c > 0 \mid |f(x)| \le cu(x)\ \forall x \in \Omega \,\}.$$

The ordering on β is the natural pointwise ordering of functions. For the purpose of spectral theory, we consider also the canonical complexification of β whithout changing our notation. Clearly, $B \subset \beta$ and there exist $C > 0$ such that $\|f\|_{\beta} \le C\,\|f\|_B$, for all $f \in B$.

Recall that $\phi \in C(\Omega)$ is a fixed positive function such that ϕ is a solution of the equation $Pu = 0$ in Ω_1^* which has a minimal growth in a neighborhood of infinity in Ω. We consider also the set $B_{\infty}^{K,\phi} := \{\, f \mid f \in C(\overline{\Omega_N^*})$ for some $N \in \mathbf{N}$, and $\forall \epsilon > 0\ \exists n \ge N$, $C_{\epsilon} \ge 0$, $u_{\epsilon} \in K_p(\Omega)$ such that $|f(x)| \le \epsilon u_{\epsilon}(x) + C_{\epsilon}\phi(x)$ in $\Omega_n^* \,\}$.

Remark 2.1. (i) Note that $B_{\infty}^{K,\phi} \cap C(\Omega)$ is a closed subspace of β. We denote this Banach subspace by $\beta_0 = \beta_0^K$. Clearly, $B \subset \beta_0$.

(ii) Consider the following closed subspace of β

$$A^K := \left\{\, f \in C(\Omega) \mid \lim_{n \to \infty} \inf_{u \in K_p(\Omega)} \sup_{x \in \Omega_n^*} \{|f(x)|\,/u(x) = 0\} \,\right\}, \tag{2.2}$$

which contains functions that grow slower than functions in $K_p(\Omega)$. Clearly, $A^K \subset \beta_0$ and if $B \subset A^K$, then $A^K = \beta_0$. There are examples where $B \not\subseteq A^K$ [2]. Note that A. Ancona proved recently that if P is symmetric with respect to a Riemannian measure σ, then $B \subset A^K$ [2].

We first prove some lemmas which will imply the maximum principle in β_0.

Lemma 2.2. *Let P be a critical or subcritical operator in Ω. Suppose that $v \in C(\overline{\Omega_2^*}) \cap B_\infty^{K,\phi}$ is a solution of the equation $Pu = 0$ in Ω_2^* such that $v = 0$ in $\partial\Omega_2$. Then $v = 0$.*

Proof. Suppose that $v(x_1) > 0$ for some $x_1 \in \Omega_2^*$ and let $\epsilon > 0$. By definition, there exist $n \in \mathbf{N}$, $C_\epsilon \geq 0$ and $u_\epsilon \in K_p(\Omega)$ such that $|v(x)| \leq \epsilon u_\epsilon + C_\epsilon \phi$ in Ω_n^*. By the generalized maximum principle $|v(x)| \leq \epsilon u_\epsilon(x) + C_\epsilon \phi(x)$ in Ω_2^*. Let

$$C_{0,\epsilon} = \inf \left\{ C \geq 0 \mid \epsilon u_\epsilon + C\phi(x) - v(x) \geq 0 \text{ in } \Omega_2^* \right\}.$$

It follows that $\epsilon u_\epsilon + C_{0,\epsilon}\phi(x) - v(x) > 0$ in Ω_2^*. Since ϕ is a positive solution of the equation $Pu = 0$ in Ω_2^* which has a minimal growth in a neighborhood of infinity in Ω, it follows that there exist $\delta = \delta(\epsilon) > 0$ such that $\epsilon u_\epsilon + C_{0,\epsilon}\phi(x) - v(x) > \delta\phi(x)$ in Ω_2^*. By the minimality of $C_{0,\epsilon}$, we infer that for every $\epsilon > 0$, $C_{0,\epsilon} = 0$. Hence, $\epsilon u_\epsilon - v(x) > 0$ in Ω_2^* for every $\epsilon > 0$, contradicting the hypothesis that $v(x_1) > 0$. ■

Lemma 2.3. *Let P be a critical or subcritical operator in Ω. Suppose that $v \in C(\overline{\Omega_2^*}) \cap B_\infty^{K,\phi}$ is a solution of the equation $Pu = 0$ in Ω_2^* such that $v \gneqq 0$ on $\partial\Omega_2$. Then $v \gneqq 0$ and v/ϕ is bounded in Ω_2^*. Moreover, if $v > 0$ on $\partial\Omega_2$ then v is a positive solution of minimal growth in a neighborhood of infinity in Ω.*

Proof. Denote by w_k, $k > 2$ the solutions of the following Dirichlet problems

$$\begin{aligned} Pu &= 0 \quad \text{in} \quad \Omega_{k,2}, \\ u &= v \quad \text{on} \quad \partial\Omega_2, \\ u &= 0 \quad \text{on} \quad \partial\Omega_k. \end{aligned}$$

By the generalized maximum principle, the sequence $\{w_k\}_{k>2}$ is a nondecreasing bounded sequence of nonnegative solutions which converges to a function $w \leq C_{1\phi}$. Recall that $v \neq 0$ on $\partial\Omega_2$. It follows that w is a nonzero, nonnegative solution of the equation $Pu = 0$ in Ω_2^* (in fact, $w > 0$ on at least one component of Ω_2^*). Moreover, if $v > 0$ on $\partial\Omega_2$, then w is a positive solution of minimal growth in a neighborhood of infinity in Ω.

Let v_k, $k > 2$, be the solutions of the Dirichlet problems

$$\begin{aligned} Pu &= 0 \quad \text{in} \quad \Omega_{k,2}, \\ u &= 0 \quad \text{on} \quad \partial\Omega_2, \\ u &= v \quad \text{on} \quad \partial\Omega_k. \end{aligned}$$

Then $v = w_k + v_k$ and therefore, the sequence $\{v_k\}_{k>2}$ converges to a function v_0. On the other hand, since $v \in B_\infty^{K,\phi}$, it follows that for every $\epsilon > 0$ there exist $n \in \mathbf{N}$, $C_\epsilon \geq 0$ and $u_\epsilon \in K_p(\Omega)$ such that $|v(x)| \leq \epsilon u_\epsilon + C_\epsilon\phi$ in Ω_n^*. By the generalized maximum principle, for every $k \geq n$ we have $|v_k(x)| \leq \epsilon u_\epsilon + C_\epsilon\phi$ in $\Omega_{k,2}^*$. Hence, $|v_0(x)| \leq \epsilon u_\epsilon + C_\epsilon\phi$ in Ω_2^* and $v_0 \in B_\infty^{K,\phi}$. Lemma 2.2 implies that $v_0 = 0$. Hence, $v = w \geq 0$ and v is a nonnegative solution such that v/ϕ is bounded in Ω_2^*. ■

Lemma 2.4. *Let P be a critical or subcritical operator in Ω. Let $f \in C^{\alpha}(\overline{\Omega_2^*})$ be a nonnegative function. Suppose that $v \in C(\overline{\Omega_2^*}) \cap B_{\infty}^{K,\phi}$ is a solution of the equation $Pu = f$ in Ω_2^* such that $v \geq 0$ on $\partial\Omega_2$. Then $v \geq 0$. Moreover,*

$$v(x) = h(x) + \int_{\Omega_2^*} G_P^{\Omega_2^*}(x,y) f(y)\, dy, \tag{2.3}$$

where $h \in C(\overline{\Omega_2^*})$ is a nonnegative solution of the equation $Pu = 0$ in Ω_2^* which is bounded by $C\phi$ (for some constant $C > 0$) and satisfies the boundary condition $h = v$ on $\partial\Omega_2$. In particular, $\int_{\Omega_2^*} G_P^{\Omega_2^*}(x,y) f(y)\, dy < \infty$.

Proof. Since $|v| = v^+ + v^-$, it follows that $v^{\pm} \in B_{\infty}^{K,\phi}$. Let $w_{k,\pm}$, $k > 2$ be the nonnegative solutions of the Dirichlet problems

$$Pu = f^{\pm} \ \text{ in } \ \Omega_{k,2}, \qquad u = v^{\pm} \ \text{ on } \ \partial\Omega_2, \qquad u = v^{\pm} \ \text{ on } \ \partial\Omega_k.$$

By the generalized maximum principle and the definition of $B_{\infty}^{K,\phi}$ it follows that for every $\epsilon > 0$ there exist $n \in \mathbf{N}$, $C_\epsilon \geq 0$ and $u_\epsilon \in K_p(\Omega)$ such that for every $k \geq n$ we have $0 \leq w_{k,-}(x) \leq \epsilon u_\epsilon + C_\epsilon \phi$ in $\Omega_{k,2}^*$. By a standard elliptic argument, the sequence $\{w_{k,-}\}$ has a converging subsequence to a nonnegative solution of the equation $Pu = 0$ in Ω_2^* which takes the value zero on $\partial\Omega_2^*$. Since any such solution is in $B_{\infty}^{K,\phi}$, it follows from Lemma 2.2 that it is the zero solution and $\lim_{k\to\infty} w_{k,-} = 0$.

On the other hand, $w_{k,+} \geq 0$, and since

$$w_{k,+} - w_{k,-} = v, \tag{2.4}$$

it follows that $\lim_{k\to\infty} w_{k,+} = v \geq 0$. Note that

$$w_{k,+}(x) = h_k(x) + g_k(x) + \int_{\Omega_{k,2}} G_P^{\Omega_{k,2}}(x,y) f(y)\, dy,$$

where h_k satisfies

$$Pu = 0 \ \text{ in } \ \Omega_{k,2}, \qquad u = v \ \text{ on } \ \partial\Omega_2, \qquad u = 0 \ \text{ on } \ \partial\Omega_k,$$

and g_k satisfies

$$Pu = 0 \ \text{ in } \ \Omega_{k,2}, \qquad u = 0 \ \text{ on } \ \partial\Omega_2, \qquad u = v \ \text{ on } \ \partial\Omega_k.$$

Clearly, $0 \leq h_k \leq C\phi$, and $\{h_k\}$ converges to a nonnegative solution h of the equation $Pu = 0$ in Ω_2^*, which is bounded by $C\phi$ and satisfies the boundary condition $h = v$ on $\partial\Omega_2$. On the other hand, for every $\epsilon > 0$ there exist $n \in \mathbf{N}$, $C_\epsilon \geq 0$ and $u_\epsilon \in K_p(\Omega)$ such that for every $k \geq n$ we have $0 \leq g_k(x) \leq \epsilon u_\epsilon + C_\epsilon \phi$ in $\Omega_{k,2}^*$. The same argument used for $w_{k,-} \to 0$ demonstrates now that the sequence $\{g_k\}$ converges to zero.

Moreover, the sequence $\left\{ \int_{\Omega_{k,2}} G_P^{\Omega_{k,2}}(x,y) f(y)\, dy \right\}$ is a nondecreasing locally bounded sequence of nonnegative solutions of the equation $Pu = f$. By monotone convergence, this sequence converges to $\int_{\Omega_2^*} G_P^{\Omega_2^*}(x,y) f(y)\, dy$ and the lemma follows. ■

Lemma 2.5. *Let P be a critical or subcritical operator in Ω and let $\Psi \in C(\partial\Omega_2)$ be a real function. Let $f \in C^{\alpha}(\overline{\Omega_2^*})$ be a real function such that*

$$\int_{\Omega_2^*} G_P^{\Omega_2^*}(x,y)\,|f(y)|\;dy \in B_{\infty}^{K,\phi}. \tag{2.5}$$

Then there exists a unique solution $v \in C(\overline{\Omega_2^*}) \cap B_{\infty}^{K,\phi}$ of the equation $Pu = f$ in Ω_2^* which satisfies $v = \Psi$ on $\partial\Omega_2^*$. Moreover,

$$v(x) = h(x) + \int_{\Omega_2^*} G_P^{\Omega_2^*}(x,y) f(y)\,dy,$$

where $h(x)$ is a solution of the homogeneous equation $Pu = 0$ in Ω_2^* which satisfies $h = \Psi$ on $\partial\Omega_2^*$ and $|h(x)| \le C\phi(x)$ in Ω_2^* for some $C > 0$.

Proof. Let $h_{\pm}$ be the unique nonnegative solution of the equation $Pu = 0$ in Ω_2^* which is bounded by $C\phi$ for some constant $C > 0$, and satisfies $h_{\pm} = \Psi^{\pm}$ on $\partial\Omega_2^*$. Consider the function

$$v_{\pm}(x) = h_{\pm}(x) + \int_{\Omega_2^*} G_P^{\Omega_2^*}(x,y) f^{\pm}(y)\,dy.$$

Clearly, $v_{\pm} \in C(\overline{\Omega_2^*}) \cap B_{\infty}^{K,\phi}$ and satisfies the equation $Pv_{\pm} = f^{\pm}$ in Ω_2^* and the boundary condition $v_{\pm} = \Psi^{\pm}$ on $\partial\Omega_2^*$. Therefore, the function $v = v_+ - v_-$ is a desired solution. The uniqueness follows from Lemma 2.2. ∎

Now we establish the maximum principle for solutions which do not grow too fast at infinity.

Theorem 2.6. *Let P be a subcritical operator in Ω. Suppose that $Pv = f \ge 0$ in Ω, where $f \in C^{\alpha}(\Omega)$ and $v \in \beta_0$. Then $v \ge 0$ in Ω.*

Proof. Suppose that there exists $x_1 \in \Omega$ such that $v(x_1) > 0$. Then there exists a ball $B_{\epsilon} = B(\,x_1, \epsilon) \subset \Omega$ such that $v > 0$ in B_{ϵ}. Lemma 2.4 implies that $v \ge 0$ in $B_{\epsilon}^* := \Omega \backslash B_{\epsilon}$ and therefore $v > 0$ in Ω.

So, we may assume that $v \le 0$ in Ω and suppose that there exists $x_1 \in \Omega$ such that $v(x_1) < 0$. Then there exists a ball $B_{\epsilon} = B(x_1, \epsilon) \subset \Omega$ such that $v < 0$ in B_{ϵ}. Without loss of generality, we may assume that $B_{\epsilon} \subset \Omega_1$. Let $u_{\epsilon,k}$ be the solution of the following Dirichlet problem

$$Pu = 0 \ \text{ in } \ \Omega_k \backslash B_{\epsilon}, \qquad u = 0 \ \text{ on } \ \partial B_{\epsilon}, \qquad u = v \ \text{ on } \ \partial\Omega_k.$$

Using again the generalized maximum principle and Lemma 3.6 it follows that $\lim_{k\to\infty} u_{\epsilon,k} = 0$.

Consider also the Dirichlet problem

$$Pu = 0 \ \text{ in } \ \Omega_k \backslash B_{\epsilon}, \qquad u = v \ \text{ on } \ \partial B_{\epsilon}, \qquad u = 0 \ \text{ on } \ \partial\Omega_k,$$

and denote its negative solution by $v_{\epsilon,k}$. Set $v_{\epsilon} := \lim_{k\to\infty} v_{\epsilon,k}$. Clearly, $0 < -v_{\epsilon} \le C_{\epsilon} G_P^{\Omega}(\cdot, x_1)$. Using the well known asymptotic behavior of $G_P^{\Omega}(\cdot, x_1)$ near the pole x_1, it follows that there exists $C > 0$ such that for $\epsilon > 0$ small enough $0 < -\, v_\epsilon \le C\epsilon^{d-2} G_P^\Omega(\cdot, x_1)$ if $d \ge 3$, and $0 < -\, v_\epsilon \le -C(\log \epsilon)^{-1} G_P^\Omega(\cdot, x_1)$ if $d = 2$. Therefore, $\lim_{\epsilon\to 0} v_\epsilon = 0$.

Finally, let $w_{\epsilon,k}$ be the solution of the Dirichlet problem

$$Pu = f \ \text{ in } \ \Omega_k \backslash B_\epsilon, \qquad u = 0 \ \text{ on } \ \partial B_\epsilon, \qquad u = 0 \ \text{ on } \ \partial\Omega_k.$$

Then $w_{\epsilon,k} \ge 0$. On the other hand,

$$v = u_{\epsilon,k} + v_{\epsilon,k} + w_{\epsilon,k}. \tag{2.6}$$

Letting first $k \to \infty$ and then $\epsilon \to 0$ in equation (2.6), we obtain that

$$v = \lim_{\epsilon\to 0}(\lim_{k\to\infty} w_{\epsilon,k}) \ge 0,$$

contradicting the hypothesis that $v \le 0, v \ne 0$. ■

The next proposition deals with the critical case and extends proposition 1.7.

PROPOSITION 2.7. *Let P be a critical operator in Ω and let ϕ_0 be a ground state of the operator P in Ω. Let $W \in C_0^\alpha(\Omega)$ be a nonzero, nonnegative function. Suppose that $Pu \ge 0$ ($Pu \le 0$) in Ω, where $u \in \beta_0^{K_{P+W}}$. Then $u = C_1\phi_0$, where C_1 is a real constant.*

Proof. Suppose that $Pu \gneqq 0$. Then there exists $V \in C_0^\alpha(\Omega)$, a nonzero nonnegative function such that $(P+V)u \ge 0$ in Ω. Since the cones C_{P+W} and $C_{P+\,V}$ are equivalent [4], it follows that $u \in B_0^{K_{P+V,\phi}}$. Applying the maximum principle (Theorem 2.6) it follows that $u \ge 0$. Therefore, u is a nonnegative supersolution of the critical operator P and hence $u = c\phi_0$, where $c \ge 0$.

Assume that $Pu = 0$. If $u \ge 0$, then $u = c\phi_0$. Suppose that $u(x_1) < 0$. We may assume that $u < 0$ in Ω_1. Let $V \in C_0^\alpha(\Omega_1)$ be a nonzero, nonnegative function. Using the equivalence of the cones C_{P+W} and $C_{P+\,V}$, it follows that $u \in B_0^{K_{P+V,\phi}}$. Therefore, Lemma 2.3 implies that $u < 0$ in Ω_1^*. Hence $-u$ is a global nonnegative solution of the critical operator P. Thus $u = c\phi_0$, where c is a real constant. ■

In the following theorem we prove existence, uniqueness and integral representation for solutions in β_0 of the nonhomogeneous equation.

THEOREM 2.8. *Let P be a subcritical operator in Ω.*

(i) *Let $f \in C^\alpha(\Omega)$, $0 < \alpha \le 1$, be a real function such that*

$$\int_\Omega G_P^\Omega(x,y)\, |f(y)|\, dy \in \beta_0. \tag{2.7}$$

Then there exists a unique solution $u \in \beta_0$ of the equation $Pu = f$ in Ω. Moreover,

$$u(x) = \int_\Omega G_P^\Omega(x,y) f(y)\, dy.$$

(ii) *Suppose that* $f \in C^\alpha(\Omega)$, $0 < \alpha \leq 1$, *and* $f \geq 0$. *Then*

$$\int_\Omega G_P^\Omega(x,y) f(y)\, dy \in \beta_0$$

if and only if there exists a solution $u \in \beta_0$ *of the equation* $Pu = f$ *in* Ω. *In this case,* u *is the minimal nonnegative solution of the equation* $Pv = f$ *in* Ω.

Proof. (i) Let

$$u_n(x) = \int_{\Omega_n} G_P^{\Omega_n}(x,y) f(y)\, dy.$$

By the Lebesgue dominated convergence theorem and a standard elliptic argument,

$$u(x) = \lim_{n\to\infty} u_n(x) = \int_\Omega G_P^\Omega(x,y) f(y)\, dy$$

is a solution of $Pu = f$ in Ω. It follows that $u \in \beta_0$. The uniqueness follows from Theorem 2.6.

(ii) Suppose that $u \in \beta_0$ is a solution of the equation $Pu = f \geq 0$ in Ω. Theorem 2.6 implies that $u \geq 0$. Consider again the sequence

$$u_n(x) = \int_{\Omega_n} G_P^{\Omega_n}(x,y) f(y)\, dy.$$

Clearly, $0 \leq u_n \leq u$ in Ω_n and therefore,

$$0 \leq w(x) := \int_\Omega G_P^\Omega(x,y) f(y) dy = \lim_{n\to\infty} \int_{\Omega_n} G_P^{\Omega_n}(x,y) f(y)\, dy \leq u(x) \in \beta_0.$$

Part (i) implies now that $w = u \in \beta_0$.

Let $\breve{u}$ $\geq$0 be a solution of the equation $Pv = f$ in Ω and consider again the sequence $\{u_n\}$. By the generalized maximum principle $0 \leq u_n \leq \breve{u}$. Hence, $u(x) = \lim_{n\to\infty} u_n(x) \leq \breve{u}(x)$ and u is the minimal nonnegative solution of the equation $Pv = f$ in Ω. ■

Remark 2.9. Condition (2.7) holds true if f/u is a small perturbation of P in Ω for some $u \in C_P(\Omega)$. In particular, it holds for all $f \in \beta$ if (1)is a small perturbation.

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