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NORM AND LOWER BOUNDS OF OPERATORS ON WEIGHTED SEQUENCE SPACES

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Abstract. This paper is concerned with the problem of finding the upper and lower bounds of matrix operators from weighted sequence spaces $l_p(v, I)$ into $l_p(v, F)$. We consider certain matrix operators such as Cesàro, Copson and Hilbert which were recently considered in [7, 8, 11, 13] on the usual weighted sequence spaces $l_p(v)$.

1. Introduction

We study the norm and lower bounds of certain matrix operators from $l_p(v, I)$ into $l_p(v, F)$ which were considered in [1, 2, 3, 4, 12] on l_p spaces and in [7, 8, 9, 10, 11, 13] on $l_p(v)$ and Lorentz sequence spaces $d(v, p)$, for certain matrix operators such as Cesàro, Copson and Hilbert operators.

If $p \geqslant 1$ and $v = (v_n)$ is a decreasing non-negative sequence such that $\lim_{n\to\infty} v_n = 0$ and $\sum_{n=1}^{\infty} v_n = \infty$, we define the weighted sequence space $l_p(v)$ as follows:

$$l_p(v) := \left\{ x = (x_n) : \sum_{n=1}^{\infty} v_n |x_n|^p \text{ is finite} \right\},$$

with norm $\|\cdot\|_{p,v}$ which is defined in the following way:

$$\|x\|_{p,v} = \left(\sum_{n=1}^{\infty} v_n |x_n|^p \right)^{1/p}.$$

Let F be a partition of positive integers. If $F = (F_n)$, where each F_n is a finite interval of positive integers and also $\max F_n < \min F_{n+1}$ $(n = 1, 2, \dots)$, we define the weighted sequence space $l_p(v, F)$ as follows:

$$l_p(v, F) := \left\{ x = (x_n) : \sum_{n=1}^{\infty} v_n |\langle x, F_n \rangle|^p \text{ is finite} \right\},$$

where $\langle x, F_n \rangle = \sum_{j\in F_n} x_j$. The norm of $l_p(v, F)$ is denoted by $\|\cdot\|_{p,v,F}$ and it is defined as follows:

$$\|x\|_{p,v,F} = \left(\sum_{n=1}^{\infty} v_n |\langle x, F_n \rangle|^p \right)^{1/p}.$$

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For $I_n = \{n\}$, $I = (I_n)$ is a partition of positive integers and $l_p(v, I) = l_p(v)$ and $\|x\|_{p,v,I} = \|x\|_{p,v}$.

We write $\|A\|_{p,v,F}$ for the norm of A as an operator from $l_p(v, I)$ into $l_p(v, F)$, and $\|A\|_{p,v}$ for the norm of A as an operator from $l_p(v)$ into itself.

We consider the lower bounds L of the form

$$\|Ax\|_{p,v,F} \geqslant L\|x\|_{p,v,I},$$

for all decreasing non-negative sequences x. The constant L is not depending on x. We seek the largest possible value of L, and denote the best lower bound by $L_{p,v,F}$ for matrix operators from $l_p(v, I)$ into $l_p(v, F)$; it is denoted by $L_{p,v}(A)$ when $l_p(v, F)$ and $l_p(v, I)$ are substituted by $l_p(v)$.

The following statements give us some conditions adequate for the operators considered below, ensuring that $\|A\|_{p,v,F}$ is determined by decreasing, non-negative sequences.

(1) For all i, j, $a_{i,j} \geqslant 0$.

(2) For all subsets M, N of natural numbers having m, n elements respectively, we have

$$\sum_{i\in M}\sum_{j\in N} a_{i,j} \leqslant \sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j}.$$

(3) $\sum_{i=1}^{\infty} v_i \sum_{j\in F_i} a_{j,1}$ is convergent.

If $A = (a_{i,j})$ is a matrix operator from $l_p(v, I)$ into $l_p(v, F)$ satisfying conditions (1) and (2), then decreasing, non-negative sequences are sufficient to determine the norm of A. Condition (3) ensure that at least finite sequences are mapped into $l_p(v, F)$, see [6].

2. Upper bounds of matrix operators

The purpose of this section is to consider the norm of certain matrix operators from $l_p(v, I)$ into $l_p(v, F)$, the problem analogous to the one considered in [6, 8, 11, 13] on $l_p(v)$ and Lorentz sequence spaces $d(v, p)$ for certain matrix operators such as Cesàro, Copson and Hilbert operators.

Proposition 2.1. *Let $p \geqslant 1$ and $N \geqslant 1$. Also, let $F_i = \{N_i - N + 1, N_i - N + 2, \ldots, N_i - 1, N_i\}$ and $v_n = 1/n^{\alpha}$, where $0 < \alpha \leqslant 1$. If A is a bounded operator from $l_p(v)$ into itself, then A is a bounded operator from $l_p(v, I)$ into $l_p(v, F)$ and also*

$$\|A\|_{p,v,F} \leqslant N^{\alpha/p}\|A\|_{p,v}.$$

Proof. Suppose $x \in l_p(v, I)$ and $y = Ax$. We have

$$\begin{aligned}\|Ax\|_{p,v,F} &= \left(\sum_{i=1}^{\infty} v_i |\langle y, F_i\rangle|^p\right)^{1/p} \\ &= \left(\sum_{i=1}^{\infty} \frac{1}{i^\alpha} |y_{N_i-N+1} + y_{n_i-N+2} + \cdots + y_{N_i}|^p\right)^{1/p} \\ &\leqslant \left(\sum_{i=1}^{\infty} \frac{|y_{N_i-N+1}|^p}{i^\alpha}\right)^{1/p} + \left(\sum_{i=1}^{\infty} \frac{|y_{N_i-N+2}|^p}{i^\alpha}\right)^{1/p} + \cdots + \left(\sum_{i=1}^{\infty} \frac{|y_{N_i}|^p}{i^\alpha}\right)^{1/p} \\ &\leqslant N^{\alpha/p} \left(\sum_{k=1}^{\infty} \frac{|y_k|^p}{k^\alpha}\right)^{1/p} \leqslant N^{\alpha/p} \|Ax\|_{p,v}.\end{aligned}$$

This completes the proof. ■

In the following, we consider the above statement for $p = 1$.

THEOREM 2.1. *Suppose that $A = (a_{i,j})$ is a matrix operator satisfying conditions (1), (2) and (3). If $\sup U_n/V_n < \infty$, where $U_n = u_1 + \cdots + u_n$ and $u_n = \sum_{i=1}^{\infty} v_i \sum_{j\in F_i} a_{j,n}$ and $V_n = v_1 + \cdots + v_n$, then A is a bounded operator from $l_1(v, I)$ into $l_1(v, F)$ and*

$$\|A\|_{1,v,F} = \sup_n \frac{U_n}{V_n}.$$

Proof. Let x be in $l_1(v, I)$ such that $x_1 \geqslant x_2 \geqslant \ldots \geqslant 0$ and $M = \sup U_n/V_n$. Then

$$\begin{aligned}\|Ax\|_{1,v,F} &= \sum_{n=1}^{\infty} v_n \langle Ax, F_n\rangle = \sum_{n=1}^{\infty} v_n \left(\sum_{j\in F_n} \sum_{k=1}^{\infty} a_{j,k} x_k\right) \\ &= \sum_{n=1}^{\infty} u_n x_n = \sum_{n=1}^{\infty} U_n (x_n - x_{n+1}).\end{aligned}$$

Since $\|x\|_{1,v,I} = \sum_{n=1}^{\infty} V_n (x_n - x_{n+1})$, we have $\|Ax\|_{1,v,F} \leqslant M \|x\|_{1,v,I}$. Therefore

$$\|A\|_{1,v,F} \leqslant M. \tag{I}$$

Further, we take $x_1 = x_2 = \cdots = x_n = 1$ and $x_k = 0$ for all $k \geqslant n + 1$, then $\|x\|_{1,v,I} = V_n$, $\|Ax\|_{1,v,F} = U_n$. Hence

$$\|A\|_{1,v,F} \geqslant M. \tag{II}$$

Applying (I), (II) completes the proof of the theorem. ■

The Cesàro operator A is defined by $y = Ax$, where

$$y_n = \frac{1}{n}(x_1 + x_2 + \cdots + x_n), \qquad \text{for each } n.$$

It is given by the Cesãro matrix $a_{n,k} = \begin{cases} \frac{1}{n}, & \text{for } n \geqslant k, \\ 0, & \text{for } n < k. \end{cases}$

Theorem 2.2. *Suppose that A is the Cesàro operator and $p \geqslant 1$. If $N \geqslant 1$ and $F_i = \{N_i - N+1, N_i - N+2, \dots, N_i - 1, N_i\}$ and $v_n = 1/n^\alpha$, where $0 < \alpha < 1$, then A is a bounded operator from $l_p(v, I)$ into $l_p(v, F)$. Also, we have*

$$\|A\|_{1,v,F} \leqslant N^\alpha \zeta(1+\alpha),$$

and

$$\|A\|_{p,v,F} \leqslant N^{\alpha/p} p^*, \quad \text{for } p > 1 \text{ and } p^* = p/(p-1).$$

Proof. Applying Proposition 5.1 of [8] and Proposition 2.1 the statement follows. ∎

The Copson operator C is defined by $y = Cx$, where

$$y_n = \sum_{k=n}^{\infty} \frac{x_k}{k}, \qquad \text{for all } n.$$

It is given as the transpose of the Cesàro operator: $c_{n,k} = \begin{cases} \frac{1}{k}, & \text{for } n \leqslant k, \\ 0, & \text{for } n > k. \end{cases}$.

Firstly, we obtain the norm of Copson operator as the one from $l_1(v, I)$ into $l_1(v, F)$.

Lemma 2.1. ([8], Lemma 2.6) *If $\alpha > 0$, then*

$$\frac{1}{n^{1-\alpha}} \sum_{i=1}^{n} \frac{1}{i^\alpha}$$

is decreasing with n and tends to $1/(1-\alpha)$ as $n \to \infty$.

As an immediate consequence, we have

Theorem 2.3. *Let C be the Copson operator. If $F_i = \{2i-1, 2i\}$ and $v_n = 1/n^\alpha$, where $0 < \alpha < 1$, then C is a bounded operator from $l_1(v, I)$ into $l_1(v, F)$, and*

$$\|C\|_{1,v,F} = \frac{2^\alpha}{1-\alpha}.$$

Proof. With the above notation,

$$u_n = \sum_{i=1}^{\infty} v_i (c_{2i-1,n} + c_{2i,n}).$$

Hence, $\dfrac{u_{2n}}{v_{2n}} = \dfrac{2^\alpha V_n}{n^{1-\alpha}}$ and $\dfrac{u_{2n-1}}{v_{2n-1}} = \dfrac{2V_{n-1} - v_n}{(2n-1)v_{2n-1}}$. Applying Lemma 2.1, we have $\sup \dfrac{u_{2n}}{v_{2n}} = \dfrac{2^\alpha}{1-\alpha}$ and also

$$\frac{u_{2n-1}}{v_{2n-1}} \leqslant \frac{u_{2(2n-1)}}{v_{2(2n-1)}} \leqslant \sup_n \frac{u_{2n}}{v_{2n}} = \frac{2^\alpha}{1-\alpha}.$$

Therefore $\|C\|_{1,v,F} = \sup \dfrac{u_n}{v_n} = \dfrac{2^\alpha}{1-\alpha}$. This establishes the proof of the theorem. ∎

In the following theorem a similar result is obtained for the Copson operator for $p \geqslant 1$.

THEOREM 2.4. *Let C be the Copson operator and $p \geqslant 1$. If $N \geqslant 1$ and $F_i = \{N_i - N + 1, N_i - N + 2, \ldots, N_i - 1, N_i\}$ and $v_n = 1/n^\alpha$, where $0 < \alpha < 1$, then C is a bounded operator from $l_p(v, I)$ into $l_p(v, F)$. Moreover, we have*

$$\|C\|_{p,v,F} = \frac{pN^{\alpha/p}}{1-\alpha}.$$

Proof. Applying Theorem 4.2 of [8] and Proposition 2.1, we deduce that

$$\|C\|_{p,v,F} \leqslant \frac{pN^{\alpha/p}}{1-\alpha}.$$

We now show that the reverse inequality holds. Choose $\varepsilon > 0$ and define r by $\alpha + rp = 1 + \varepsilon$. Let $y = Ax$ and $x_i = 1/i^r$ for all n. Note that (x_i) is decreasing and $x_i \in l_p(v, I)$. Then applying the integral estimate it follows that

$$y_{N_i-N+n} = \sum_{k=N_i-N+n}^{\infty} \frac{1}{k^{1+r}} \geqslant \frac{1}{r(N_i - N + n)^r},$$

for all i and $1 \leqslant n \leqslant N$. Therefore

$$\begin{aligned}
\|Ax\|_{p,v,F} &= \left(\sum_{i=1}^{\infty} v_i (y_{N_i-N+1} + y_{N_i-N+2} + \cdots + y_{N_i-1} + y_{N_i})^p \right)^{1/p} \\
&\geqslant \left(\sum_{i=1}^{\infty} v_i \left(\sum_{n=1}^{N} \frac{1}{r(N_i - N + n)^r} \right)^p \right)^{1/p} \\
&\geqslant \left(\sum_{i=1}^{\infty} v_i \left(\sum_{n=1}^{N} \frac{N}{r(Ni)^r} \right)^p \right)^{1/p} = \frac{N^{1-r}}{r} \|x\|_{p,v,I} \\
&= \frac{pN^{p-1+\alpha/p}}{1-\alpha+\varepsilon} \|x\|_{p,v,I} \geqslant \frac{pN^{\alpha/p}}{1-\alpha+\varepsilon} \|x\|_{p,v,I}.
\end{aligned}$$

The statement of the theorem follows from the above inequality. ■

We recall that the Hilbert operator H is defined by the matrix

$$h_{i,j} = \frac{1}{i+j}, \qquad i, j = 1, 2, \ldots$$

Let $0 < \alpha < 1$. As in the most studies on Hilbert operator, we use the well-known integral

$$\int_0^\infty \frac{1}{t^\alpha (t+c)}\, dt = \frac{\pi}{c^\alpha \sin \alpha\pi}.$$

In the following, we are looking for an upper bound of the Hilbert matrix operator. At first, we consider the case $p = 1$ and give the exact solution for this problem.

THEOREM 2.5. *Let H be the Hilbert matrix operator, and $F_i = \{2i-1, 2i\}$ and $v_n = 1/n^\alpha$, where $0 < \alpha < 1$. Then H is a bounded operator from $l_1(v, I)$ into $l_1(v, F)$, and also*

$$\|H\|_{1,v,F} = \frac{2^\alpha \pi}{\sin \alpha\pi}.$$

Proof. Using the usual notation we have

$$u_n = \sum_{i=1}^{\infty} v_i(h_{2i-1,n} + h_{2i,n}) = \sum_{i=1}^{\infty} \frac{1}{i^\alpha}\left(\frac{1}{2i-1+n} + \frac{1}{2i+n}\right).$$

Since $u_n \geqslant \int_1^\infty \frac{1}{t^\alpha}\left(\frac{1}{2t-1+n} + \frac{1}{2t+n}\right) dt$ and

$$\int_0^\infty \frac{1}{t^\alpha}\left(\frac{1}{2t-1+n} + \frac{1}{2t+n}\right) dt = \frac{\pi}{2^{1-\alpha}\sin\alpha\pi}\left(\frac{1}{(n-1)^\alpha} + \frac{1}{n^\alpha}\right),$$

and also

$$\int_0^1 \frac{1}{t^\alpha}\left(\frac{1}{2t-1+n} + \frac{1}{2t+n}\right) dt \leqslant \frac{2}{n(1-\alpha)},$$

we have

$$n^\alpha u_n \geqslant \frac{\pi}{2^{1-\alpha}\sin\alpha\pi}\left(1 + \frac{n^\alpha}{(n-1)^\alpha}\right) - \frac{2n^\alpha}{n(1-\alpha)} \geqslant \frac{2^\alpha\pi}{\sin\alpha\pi} - \frac{2}{n^{1-\alpha}(1-\alpha)}.$$

Therefore

$$\|H\|_{1,v,F} = \sup_n n^\alpha u_n \geqslant \frac{2^\alpha\pi}{\sin\alpha\pi}.$$

It was shown in [6] that $\|H\|_{1,v} = \frac{\pi}{\sin\alpha\pi}$, and so applying Proposition 2.1. we have

$$\|H\|_{1,v,F} \leqslant \frac{2^\alpha\pi}{\sin\alpha\pi}.$$

This establishes the proof of the theorem. ∎

We now consider the norm of the Hilbert matrix operator for the general case, when $p > 1$.

THEOREM 2.6. *Suppose that H is the Hilbert matrix operator and $p > 1$. If $N \geqslant 1$ and $F_i = \{N_i - N + 1, N_i - N + 2, \ldots, N_i - 1, N_i\}$ and $v_n = 1/n^\alpha$, where $1 - p < \alpha < 1$, then H is a bounded operator from $l_p(v, I)$ into $l_p(v, F)$. Moreover, we have*

$$\|H\|_{p,v,F} = \frac{\pi N^{\alpha/p}}{\sin[(1-\alpha)\pi/p]}.$$

Proof. Applying Theorem 3.2 in [8] and Proposition 2.1, we deduce that

$$\|H\|_{p,v,F} \leqslant \frac{\pi N^{\alpha/p}}{\sin[(1-\alpha)\pi/p]}.$$

To show the reverse inequality, take $r = (1-\alpha)/p$, so that $\alpha + rp = 1$. Fix M, and let

$$x_j = \begin{cases} \frac{1}{j^r}, & \text{for } j \leqslant M, \\ 0, & \text{for } j > M. \end{cases}$$

Then (x_j) is decreasing and $\sum_{j=1}^{\infty} v_j x_j = \sum_{j=1}^{M} \frac{1}{j}$. Also, let $y = Hx$. By routine methods (we omit the details), one finds that

$$\sum_{i=1}^{n} v_i (y_{N_i-N+1} + y_{N_i-N+2} + \cdots + y_{N_i-1} + y_{N_i})^p \geqslant$$

$$\geqslant \frac{\pi N^{1-r}}{\sin r\pi} \sum_{i=1}^{M} \frac{1}{i} - g(r) \geqslant \frac{\pi N^{\alpha/p}}{\sin[(1-\alpha)\pi/p]} \sum_{i=1}^{M} \frac{1}{i} - g(r),$$

where $g(r)$ is independent of M. Clearly, the required statement follows. ■

3. Lower bounds of matrix operators

In this part of the study we are looking for the lower bounds of matrix operators considered in Section 2.

Let $p \geqslant 1$ and A be a matrix operator with non-negative entries. If $y = Ax$ and v is a decreasing sequence, then for each non-negative sequence x, we have

$$\|Ax\|_{p,v,F}^p = \sum_{i=1}^{\infty} v_i \left(\sum_{j\in F_i} y_j \right)^p \geqslant \sum_{i=1}^{\infty} v_i \sum_{j\in F_i} y_j^p$$
$$\geqslant \sum_{i=1}^{\infty} v_i y_i^p = \|Ax\|_{p,v}^p.$$

It follows that $L_{p,v,F}(A) \geqslant L_{p,v}(A)$.

Corollary 3.1. *Suppose that A is the Cesàro operator and $p \geqslant 1$. If $v_n = 1/n$, then $L_{p,v,F}(A) \geqslant 1$.*

Proof. If we apply Theorem 4 in [7], we deduce that $L_{p,v}(A) = 1$ and so we have the statement. ■

The Copson matrix is an upper triangular matrix. We will solve the lower bound problem through the next statement. In fact, we characterize a class of operators for which the lower bound constant is equal to one.

Theorem 3.1. *Suppose that A is an upper triangular matrix, i.e. $a_{n,k} = 0$ for $n > k$, and $\sum_{n=1}^{k} a_{n,k} = 1$ for all k (in other words, A is a quasi-summable matrix). Let $p \geqslant 1$ and $v = (v_n)$ be a non-negative decreasing sequence. Then $L_{p,v,F}(A) = 1$.*

Proof. If we apply Proposition 2 in [7], we have $L_{p,v}(A) = 1$. Hence $L_{p,v,F}(A) \geqslant L_{p,v}(A) = 1$. Since $1 \in F_1$ and $Ae_1 = e_1$, we deduce that

$$\|Ae_1\|_{p,v,F} = \|e_1\|_{p,v,I} = v_1.$$

This completes the proof of the theorem. ■

We now generalize Theorem 1 of [7] for certain matrix operators from $l_p(v,I)$ into $l_p(v,F)$ and deduce the lower bound for the Hilbert matrix operator.

Lemma 3.1. [7] *Let $p \geqslant 1$. Suppose that (a_j), (x_j) are non-negative sequences and (x_j) is decreasing and tends to 0. Write $A_n = \sum_{j=1}^{n} a_j$ (with $A_0 = 0$), and $B_n = \sum_{j=1}^{n} a_j x_j$. Then:*

(i) *$B_n^p - B_{n-1}^p \geqslant (A_n^p - A_{n-1}^p)x_n^p$ for all n;*

(ii) *if $\sum_{j=1}^{\infty} a_j x_j$ is convergent, then*

$$\left(\sum_{j=1}^{\infty} a_j x_j\right)^p \geqslant \sum_{n=1}^{\infty} A_n^p (x_n^p - x_{n+1}^p).$$

Corollary 3.2. *If (x_j) is a non-negative decreasing sequence and $X_n = x_1 + \cdots + x_n$, then for each n, $X_n^p - X_{n-1}^p \geqslant [n^p - (n-1)^p]x_n^p$.*

Theorem 3.2. *Suppose that $p \geqslant 1$ and $A = (a_{i,j})$ is a matrix operator from $l_p(v,I)$ into $l_p(v,F)$ with non-negative entries. Write $r_{j,i} = \sum_{k=1}^{i} a_{j,k}$ and*

$$S_i = \sum_{n=1}^{\infty} v_n \left(\sum_{j \in F_n} r_{j,i}\right)^p.$$

Then $L_{p,v,F}^p(A) = \inf_n \dfrac{S_n}{V_n}$.

Proof. Let x be in $l_p(v,I)$ such that $x_1 \geqslant x_2 \ldots \geqslant 0$ and $m = \inf S_n/V_n$. Applying Lemma 3.1, we have $y_i^p \geqslant \sum_{n=1}^{\infty} r_{i,n}^p (x_n^p - x_{n+1}^p)$. Hence

$$\begin{aligned}\|Ax\|_{p,v,F}^p &= \sum_{n=1}^{\infty} v_n \left(\sum_{j \in F_n} y_j\right)^p \geqslant \sum_{n=1}^{\infty} v_n \sum_{i=1}^{\infty} \left(\sum_{j \in F_n} r_{j,i}\right)^p (x_i^p - x_{i+1}^p) \\ &= \sum_{i=1}^{\infty} S_i (x_i^p - x_{i+1}^p).\end{aligned}$$

Since $\|x\|_{p,v,I}^p = \sum_{n=1}^{\infty} V_n (x_n^p - x_{n+1}^p)$, we deduce that

$$\|Ax\|_{p,v,F}^p \geqslant m \|x\|_{p,v,I}^p.$$

Therefore $L_{p,v,F}(A) \geqslant m$.

Further, we take $x_1 = x_2 = \cdots = x_n = 1$ and $x_k = 0$ for all $k \geqslant n+1$; then $\|x\|_{p,v,I}^p = V_n$ and $\|Ax\|_{p,v,F}^p = S_n$. Hence

$$L_{p,v,F}(A) \leqslant m.$$

This establishes the proof of the theorem. ∎

Note 3.1. For $p > 1$, the last part of Theorem 3.2 shows that $\|A\|_{p,v,F}^p \geqslant \sup_n S_n/V_n$, but $l_p(v,F) = l_p(v)$ when $F_i = \{i\}$ and equality does not hold (see [8]). Write

$$t_n = \sum_{i=1}^{\infty} v_i \left(\sum_{j \in F_i} a_{j,n}\right)^p,$$

and

$$s_n = S_n - S_{n-1} = \sum_{i=1}^{\infty} v_i \left[\left(\sum_{j \in F_i} r_{j,n} \right)^p - \left(\sum_{j \in F_i} r_{j,n-1} \right)^p \right],$$

where $S_n = s_1 + \cdots + s_n$. For $p = 1$, we have $t_n = s_n$. It is elementary that $\inf_n(S_n/V_n) \geqslant \inf_n(s_n/v_n)$. We now apply Lemma 3.1 to deduce the following result.

PROPOSITION 3.1. *If A satisfies all conditions mentioned in Theorem 3.1 and $(a_{i,j})$ decreases with j for each i, then*

$$L_{p,v,F}(A)^p \geqslant \inf_n [n^p - (n-1)^p] \frac{t_n}{v_n}.$$

Proof. It follows from Corollary 3.2 that

$$\left(\sum_{j \in F_i} r_{j,n} \right)^p - \left(\sum_{j \in F_i} r_{j,n-1} \right)^p \geqslant [n^p - (n-1)^p] \left(\sum_{j \in F_i} a_{j,n} \right)^p.$$

Thus

$$s_n \geqslant [n^p - (n-1)^p] \sum_{i=1}^{\infty} \left(\sum_{j \in F_i} a_{j,n} \right)^p = [n^p - (n-1)^p]\, t_n$$

and so we have the statement. ■

In the following statement we consider the lower bound constant for the Hilbert operator H.

THEOREM 3.3. *Suppose that H is the Hilbert operator, and $p \geqslant 1$. Let $F_i = \{2i-1, 2i\}$ and $v_n = 1/n^\alpha$, where $0 < \alpha < 1$. Then*

$$L_{p,v,F}(H)^p \geqslant \sum_{k=1}^{\infty} \frac{1}{k^\alpha (k+1/2)^p}.$$

Proof. Let $E_k = \{i \in Z : (k-1)n < i \leqslant kn\}$, where $k \geqslant 1$. If $i \in E_k$, then $(\frac{i}{n})^\alpha (2i+n)^p \leqslant k^\alpha (2kn+n)^p$. Since E_k has n members,

$$n^{p+\alpha-1} \sum_{i \in E_k} \frac{1}{i^\alpha (2i+n)^p} \geqslant \frac{n^p}{k^\alpha (2kn+n)^p} = \frac{1}{k^\alpha (2k+1)^p}.$$

Hence

$$n^{p+\alpha-1} \sum_{k=1}^{\infty} \frac{1}{k^\alpha (2k+n)^p} \geqslant \sum_{k=1}^{\infty} \frac{1}{k^\alpha (2k+1)^p},$$

and also

$$\inf_n n^{p+\alpha-1} \sum_{k=1}^{\infty} \frac{1}{k^\alpha (2k+n)^p} = \sum_{k=1}^{\infty} \frac{1}{k^\alpha (2k+1)^p}.$$

We now apply Proposition 3.1 and with the above notation,

$$t_n = \sum_{i=1}^{\infty} \frac{1}{i^\alpha} \left(\frac{1}{2i-1+n} + \frac{1}{2i+n} \right)^p,$$

and $L_{p,v,F}(H)^p \geqslant \inf_n [n^p - (n-1)^p] n^\alpha t_n$.

Since $n^p-(n-1)^p \geqslant n^{p-1}$, we have

$$L_{p,v,F}(H)^p \geqslant \inf_n n^{p+\alpha-1}\sum_{i=1}^{\infty}\frac{1}{i^\alpha}\left(\frac{1}{2i-1+n}+\frac{1}{2i+n}\right)^p$$
$$\geqslant 2^p \inf_n n^{p+\alpha-1}\sum_{i=1}^{\infty}\frac{1}{k^\alpha(2k+n)^p} = 2^p\sum_{k=1}^{\infty}\frac{1}{k^\alpha(2k+1)^p}.$$

This completes the proof of the theorem. ■

As mentioned in Theorem 3 in [7], we have

$$L_{p,v}(H)^p = \sum_{k=1}^{\infty}\frac{1}{k^\alpha(k+1)^p}.$$

Therefore we have shown that $L_{p,v,F}(H) > L_{p,v}(H)$.

REFERENCES

[**1**] G. Bennett, *Factorizing the classical inequalities*, Mem. Amer. Math. Soc. **576** (1996), 1–130.

[**2**] G. Bennett, *Inequalities complimentary to Hardy*, Quart. J. Math. Oxford (2), **49** (1998), 395–432.

[**3**] G. Bennett, *Lower bounds for matrices*, Linear Algebra Appl. **82** (1986), 81–98.

[**4**] G. Bennett, *Lower bounds for matrices II*, Canadian J. Math. **44** (1992), 54–74.

[**5**] G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities*, 2nd ed., Cambridge University Press, Cambridge 2001.

[**6**] G. J. O. Jameson, *Norms and lower bounds of operators on the Lorentz sequence spaces* $d(w,1)$, Illinois J. Math. **43** (1999), 79–99.

[**7**] G. J. O. Jameson and R. Lashkaripour, *Lower bounds of operators on weighted* l_p *spaces and Lorentz sequence spaces*, Glasgow Mat. J. **42** (2000), 211–223.

[**8**] G. J. O. Jameson and R. Lashkaripour, *Norms of certain operators on weighted* l_p *spaces and Lorentz sequence spaces*, J. Inequalities in Pure Appl. Math., Vol. 3, Issue 1, Article 6 (2002).

[**9**] R. Lashkaripour, *Weighted mean matrix on weighted sequence spaces*, WSEAS Transaction on Mathematics, Issue 4, Vol. 3 (2004), 789–793.

[**10**] R. Lashkaripour, *Transpose of the weighted mean operators on weighted sequence spaces*, WSEAS Transaction on Mathematics, Issue 4, Vol. 4 (2005), 380–385.

[**11**] R. Lashkaripour and D. Foroutannia, *Inequalities involving upper bounds for certain matrix operators*, Proc. Indian Acad. Sci. (Math. Sci.), **116** (August 2006), 325–336.

[**12**] R. Lyons, *A lower bound for the Cesaro operator*, Proc. Amer. Math. Soc. **86** (1982), 694.

[**13**] J. Pečarić, I. Perić, R. Roki, *On bounds for weighted norms for matrices and integral operators*, Linear Algebra Appl. **326** (2001), 121–135.

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