МАТЕМАТИЧКИ ВЕСНИК
58 (2006), 91–96

UDK 517.28
оригинални научни рад
research paper

PROPERTIES OF FUNCTIONS OF GENERALIZED BOUNDED VARIATION

R. G. Vyas

Abstract. The class of functions of $\Lambda BV^{(p)}$ shares many properties of functions of bounded variation. Here we have shown that $\Lambda BV^{(p)}$ is a Banach space with a suitable norm, the intersection of $\Lambda BV^{(p)}$, over all sequences Λ, is the class of functions of $\mathrm{BV}^{(p)}$ and the union of $\Lambda BV^{(p)}$, over all sequences Λ, is the class of functions having right- and left-hand limits at every point.

INTRODUCTION. Looking to the feature of Jordan's class the concept of bounded variation has been generalized in many ways and many interesting results are obtained in Analysis [1–6]. In most of the case these new notations have been introduced because of their applicability to the study of Fourier series. In 1972 Waterman [1] introduced the class of functions of ΛBV. In 1980 Shiba [4] generalized this class. He introduced the class $\Lambda BV^{(p)}$ $(p \geq 1)$.

DEFINITION. Given an interval I, and a sequence of non-decreasing positive real numbers $\Lambda = \{\lambda_m\}$ $(m = 1, 2, \dots)$ such that $\sum_{m=1}^{n}(1/\lambda_m)$ diverges and $1 \leq p < \infty$, we say that $f \in \Lambda BV^{(p)}(I)$ (that is f is a function of p-Λ-bounded variation over I) if

$$V_\Lambda(f, p, I) = \sup_{\{I_m\}} V_\Lambda(\{I_m\}, f, p, I) < \infty,$$

where $V_\Lambda(\{I_m\}, f, p, I) = \left(\sum_m \dfrac{|f(a_m) - f(b_m)|^p}{\lambda_m}\right)^{1/p}$, and $\{I_m\}$ is a sequence of non-overlapping subintervals $I_m = [a_m, b_m] \subset I = [a, b]$.

Note that, if $p = 1$, one gets the class $\Lambda BV(I)$; if $\lambda_m \equiv 1$ for all m, one gets the class $BV^{(p)}$; if $p = 1$ and $\lambda_m \equiv m$ for all m, one gets the class Harmonic $BV(I)$. If $p = 1$ and $\lambda_m \equiv 1$ for all m, one gets the class $BV(I)$. Moreover, for any $f \in BV^{(P)}(I)$ it follows $f \in \Lambda BV^{(p)}(I)$.

D. Waterman [1, 2] has studied Fourier coefficients properties of this class. He also proved the following result.

AMS Subject Classification: 26A45.
Keywords and phrases: Banach space, right and left-hand limits, p-Λ-bounded variation.

Theorem A. *The class of functions* $\Lambda BV(I)$ *is a Banach space.*

Perlman [6] has also studied some properties of this class. He proved the following results.

Theorem B. *If* $f \in \Lambda BV(I)$ *then* f *has right- and left-hand limits at every point of* I.

Theorem C. *If* $f \in \Lambda BV(I)$ *for every sequence* Λ, *then* $f \in BV(I)$.

Theorem D. *If a function* f *has a right- and left-hand limit at each point of* I, *then* $f \in \Lambda BV(I)$.

Here we have extended these results for the class of functions of $\Lambda BV^{(p)}$. We will show that the class of functions of $\Lambda BV^{(p)}$ lies between the regulated functions and the class of functions of $BV^{(p)}$. That is, we will prove that the the union of $\Lambda BV^{(p)}$-functions over all sequences Λ are the regulated functions and the intersection of $\Lambda BV^{(p)}$-functions over all sequences Λ are the functions of $BV^{(p)}$.

Theorem 1. *The class of functions of* $\Lambda BV^{(p)}(I)$ *is a Banach space.*

Theorem 2. *If* $f \in \Lambda BV^{(p)}(I)$ *then* f *has right- and left-hand limits at every point of* I.

Theorem 1 and Theorem 2 generalize Theorem A and Theorem B respectively because $p = 1$ reduces the class $\Lambda BV^{(p)}(I)$ to the class $\Lambda BV(I)$.

Theorem 3. *If* $f \in \Lambda BV^{(p)}(I)$, *for every sequence* Λ *then* $f \in BV^{(p)}(I)$.

Hence the intersection of $\Lambda BV^{(p)}(I)$, taken over all sequences Λ, is the class of functions of $BV^{(p)}(I)$.

Theorem 4. *If* f *is a continuous function over* I, *then* $f \in \Lambda BV^{(p)}(I)$ *for some sequence* Λ.

Theorem 5. *If* φ *is a monotone function from* I *into* $[c, d]$ *and* $f \in \Lambda BV^{(p)}[c, d]$ *then* $f \circ \varphi \in \Lambda BV^{(p)}(I)$.

As a partial converse of Theorem 2 we have the following result.

Theorem 6. *If* f *has right- and left-hand limits at every point of* I, *then* $f \in \Lambda BV^{(p)}(I)$ *for some sequence* Λ.

It follows from Theorem 2 and Theorem 6 that the union of $\Lambda BV^{(p)}$, taken over all sequences Λ, is the class of functions having right- and left-hand limits at every point.

Theorem 7. *If* g *is continuous and* $F \in \Lambda BV^{(p)}(I)$, *then* $g \circ F \in \Lambda' BV^{(p)}(I)$ *for some sequence* Λ'.

To prove these results we need the following Lemmas.

Lemma 1. [6, Lemma 1] *Let* $\{a_n\}$ *be a sequence of positive numbers tending to zero. Then there exists a decreasing sequence* $\{b_n\}$ *of positive numbers tending to zero such that* $\sum b_n = \infty$ *and* $\sum a_n b_n < \infty$.

Lemma 2. [6, Lemma 3] *Let* $\{a_n\}$ *be a decreasing sequence of positive numbers. If* $\{b_n\}$ *is a sequence of positive numbers tending to zero and* $\{B_n\}$ *is the sequence* $\{b_n\}$ *rearranged in decreasing order, then* $\sum a_k b_k \leq \sum a_k B_k$.

Lemma 3. [6, Theorem, p. 207] *A function has right- and left-hand limits at each point if and only if it is the composition of a continuous function with a monotone function.*

Lemma 4. [5, Lemma 1.6] *If* $f \in \Lambda BV^{(p)}(I)$ *then* f *is bounded over* I.

Proof of Theorem 2. It is sufficient to prove the result for left-hand limits only. Suppose that there is a point x in $(a, b]$ at which f does not have a left-hand limit. Then

$$L = \overline{\lim_{t \to x-}} > l = \underline{\lim_{t \to x-}}.$$

For $\delta = \frac{L-l}{3}$, consider increasing sequences $\{P_n\}$ and $\{p_n\}$ converging to x such that $f(P_n) \geq L - \delta$ and $f(p_n) \leq L + \delta$. We choose subsequences $\{Q_n\}$ of $\{P_n\}$ and $\{q_n\}$ of $\{p_n\}$ such that $q_1 < Q_1 < q_2 < Q_2 < \cdots$. Consider intervals $I_n = [q_n, Q_n]$; for all n, we get $|f(I_n)| \geq (L - \delta) - (l + \delta) = \delta$.

Hence $\sum \frac{|f(I_n)|^p}{\lambda_n} \geq \delta^p (\sum \frac{1}{\lambda_n}) = \infty$, which contradicts our hypothesis. Hence the result follows. ■

Proof of Theorem 3. Let f$\in \Lambda BV^{(p)}(I)$ for at lest one choice of $\Lambda = \{\lambda_n\}$. Then, from Lemma 4, f is bounded over I that is $m \leq f \leq M$. To prove the result it is sufficient to prove that $F = \frac{f-m}{M-m}$ belongs to $BV^{(p)}(I)$.

Suppose that F is not in $BV^{(p)}$. Then there is a point x in I such that F is not of $BV^{(p)}$ on any neighborhood of x. Let $\{a_n\}$ be a sequence of positive numbers such that $\sum a_n = \infty$. Then there is a partition P_1 of I such that

$$\sum_{J \in P_1} |F(J)|^p \geq a_1 + 2.$$

The point x is either an interior point of an interval in P_1 or an endpoint of at most two intervals in P_1. Removing this one, or possibly two, intervals from P_1, for the remaining collection of intervals, say Q_1, since $|F(x)| \leq 1$ for all $x \in I$, we get

$$\sum_{J \in Q_1} |F(J)|^p \geq a_1.$$

If Q_1 has q_1 intervals we get $Q_1 = \{I_k^1 \mid k = 1, 2, \ldots, q_1\}$. Define $\lambda_k = 1$ for all $k = 1, 2, \ldots, q_1$, and we have

$$\sum_{1}^{q_1} \frac{|F(I_k^1)|^p}{\lambda_k} \geq a_1.$$

The result is true for the first step.

Assume the result is true for n steps and we have to prove the result for the next step. The one, or possible two, intervals that were removed from P_n to form Q_n, form a neighborhood U_n of x. Since F is not of $BV^{(p)}$ on U_n there is a finite partition P_{n+1} of U_n so that

$$\sum_{J\in P_{n+1}} |F(J)|^p \geq (n+1)a_{n+1} + 2.$$

The point x is either an interior point of one interval or an endpoint of at most two intervals in P_{n+1}. If we remove this one, or possible two, intervals from P_{n+1} and call the remaining collection of intervals Q_{n+1}, then

$$\sum_{J\in Q_{n+1}} |F(J)|^p \geq (n+1)a_{n+1}.$$

If Q_{n+1} has q_{n+1} intervals we write $Q_{n+1} = \{I_k^{n+1} \mid k = 1, 2, \ldots, q_{n+1}\}$ and define $\lambda_{r_n+i} = n+1$, for all i=1 to q_{n+1}, where $r_n = \sum_0^n q_k$ and $q_0 = 0$. We have

$$\sum_1^{q_{n+1}} \frac{|F(I_k^{n+1})|^p}{\lambda_{r_n+k}} \geq a_{n+1}.$$

Observe that the intervals of Q_{n+1} are within U_n and all the intervals of $Q_1 \cup Q_2 \cup \cdots \cup Q_{n+1}$ are pairwise non-overlapping. Then

$$\sum_{i=1}^{n+1}\sum_{k=1}^{q_i} \frac{|F(I_k^i)|^p}{\lambda_{r_{i-1}+k}} \geq \sum_{i=1}^{n+1} a_i.$$

Thus, we construct a sequence of non-decreasing positive numbers $\{\lambda_k\}$ and a sequence $\{I_k^n \mid k = 1, 2, \ldots, q_n;\ n = 1, 2, \ldots\}$ of non-overlapping subintervals of I such that $\frac{1}{\lambda_k}$ decreases to zero, $\sum \frac{1}{\lambda_k} = \infty$ and

$$\sum_{i=1}^{\infty}\sum_{k=1}^{q_i} \frac{|F(I_k^i)|^p}{\lambda_{r_{i-1}+k}} = \infty.$$

Thus F is not in $\Lambda BV^{(P)}$ for this particular sequence of λ's which contradict our hypothesis. Hence the result follows. ■

Proof of Theorem 4. For $\delta > 0$ and $p \geq 1$ the p-modulus of continuity of f over I (that is $\omega(p,\delta)$ over I) is defined as

$$\omega(p,\delta) = \| \, |T_h f - f|^p \|_{\infty, I},$$

where $(T_h f)(x) = f(x+h)$, $\forall x$. Clearly, $\omega(p,\delta)$ is increasing and converges to zero as $\delta \to 0$ because of the uniform continuity of f on I.

Let $I_n = [a_n, b_n]$ be a sequence of non-overlapping subintervals of I. Define

$$E_m = \{I_k \mid \omega(p, \frac{b-a}{m}) \geq |f(I_k)|^p > \omega(p, \frac{b-a}{m+1})\}, \qquad m = 1, 2, \ldots.$$

If $|f(I_k)|^p \le \frac{b-a}{m+1}$, then

$$|f(I_k)|^p = |f(b_k) - f(a_k)|^p \le \omega(p, |b_k - a_k|) \le \omega(p, \frac{b-a}{m+1}).$$

Thus $I_k \in E_m$ only if $|I_k| > \frac{b-a}{m+1}$. Thus E_m contains at most m intervals. Also if $I_p \in E_r$ and $I_q \in E_{r+s}$ then

$$|f(I_q)|^p \le \omega(p, \frac{b-a}{r+s}) \le \omega(p, \frac{b-a}{r+1}) < |f(I_p)|^p.$$

Thus by considering those intervals in E_1, then those in E_2, etc., and rearranging the intervals we get J_k such that

$$|f(J_1)|^p \ge |f(J_2)|^p \ge \cdots \ge |f(J_n)|^p \ge \cdots \to 0, \tag{1}$$

where

$$|f(J_m)|^p \le \omega(p, \frac{b-a}{m}). \tag{2}$$

Namely, if m is an integer for which $|f(J_m)|^p > \omega(p, \frac{b-a}{m})$, then

$$|f(J_1)|^p \ge |f(J_2)|^p \ge \cdots \ge |f(J_m)|^p > \omega(p, \frac{b-a}{m})$$

implies $|J_k| > \frac{b-a}{m}$ $(k = 1, 2, \ldots, m)$, which is impossible since the intervals J_k $(k = 1, 2, \ldots, m)$ are non-overlapping and contained in $[a, b]$. Thus (2) holds for all m.

Since sequence $\omega(p, \frac{b-a}{n})$ decreases to zero, from Lemma 1 we get a non-decreasing sequence of positive numbers $\{\lambda_n\}$ such that

$$\frac{1}{\lambda_n} \to 0, \qquad \sum \frac{1}{\lambda_n} = \infty \quad \text{and} \quad \sum \frac{\omega(p, \frac{b-a}{n})}{\lambda_n} < \infty.$$

Applying Lemma 2 to the sequences $\{\lambda_n\}$ and $\{|f(I_n)|^p\}$ we get

$$\sum \frac{|f(I_n)|^p}{\lambda_n} \le \sum \frac{|f(J_n)|^p}{\lambda_n} \le \sum \frac{\omega(p, \frac{b-a}{n})}{\lambda_n} < \infty.$$

Hence the result follows. ∎

Since the convergence in the norm is the uniform convergence, the class of all continuous functions over I becomes a closed subspace of the class functions of $\Lambda BV^{(p)}(I)$.

Proof of Theorem 5. Let $I_n = [s_n, t_n]$ be a sequence of non-overlapping subintervals of I. Let J_n be the interval determined by the points $\varphi(s_n)$ and $\varphi(t_n)$. Then $\varphi(I_n) \subseteq J_n \subseteq [c, d]$ and the intervals J_n are non-overlapping, which implies

$$\begin{aligned}\sum \frac{|f\circ\varphi(I_n)|^p}{\lambda_n} &= \sum \frac{|f\circ\varphi(s_n) - f\circ\varphi(t_n)|^p}{\lambda_n} \\ &= \sum \frac{|f[\varphi(s_n)] - f[\varphi(t_n)]|^p}{\lambda_n} = \sum \frac{|f(J_n|^p}{\lambda_n} < \infty,\end{aligned}$$

as $f \in \Lambda BV^{(p)}$. Hence the result follows. ∎

Proof of Theorem 6. Since f has left- and right-hand limit at every point of I, from Lemma 3 we get $f = F\circ\varphi$ where φ is a monotone function defined on I and F is a continuous function defined on the smallest closed interval, say $[c,d]$, containing the range of φ. Let ψ be a linear one to one mapping from $[c,d]$ onto I. Then $g = \psi\circ\varphi$ is a monotone function from I into I and $h = f\circ\psi^{-1}$ is a continuous function defined on I. Hence the result follows from Theorem 4 and Theorem 5. ■

Proof of Theorem 7. Since $F \in \Lambda BV^{(p)}(I)$, from Theorem 2 F has right- and left-hand limit at every point of I. From Lemma 3 we get $F = h\circ\phi$ where h is a continuous function defined on I and ϕ is a monotone function from I into I. Then, we get $g\circ F = (g\circ h)\circ\phi$ where $g\circ h$ is continuous and ϕ is monotone function. From Lemma 3, $g\circ F$ has a right- and left-hand limit at every point of I. Thus the result follows from Theorem 6. ■

REFERENCES

[**1**] D. Waterman, *On the convergence of Fourier series of functions of bounded variation*, Studia Math. **44** (1972), 107–117.

[**2**] D. Waterman, *On Λ-bounded variation*, Studia Math. **57** (1976), 33–45.

[**3**] J. R. Patadia and R. G. Vyas, *Fourier series with small gaps and functions of generalized variations*, J. Math. Anal. Appl. **182**, 1 (1994),113–126.

[**4**] M. Shiba, *On the absolute convergence of Fourier series of functions class $\Lambda BV^{(p)}$*, Sci. Rep. Fukushima Univ. **30** (1980), 7–10.

[**5**] R. G. Vyas, *On the absolute convergence of small gaps Fourier series of functions of $\Lambda BV^{(p)}$*, J. Inequal. Pure Appl. Math. **6** (1),23 (2005), 1–6.

[**6**] S. Perlman, *Functions of generalized variation*, Fund. Math. **105** (1980), 199–211.

(received 10.09.2004, in revised form 29.10.2006)

Department of Mathematics, Faculty of Science, The Maharaja Sayajirao University of Baroda, Vadodara-390002, Gujarat, India.

E-mail: drrgvyas@yahoo.com