МАТЕМАТИЧКИ ВЕСНИК
56 (2004), 77–80

UDK 515.122.2
оригинални научни рад
research paper

SEMIREGULAR PROPERTIES AND GENERALIZED LINDELÖF SPACES

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Abstract. Let (X,τ) be a topological space and (X,τ^*) its semiregularization. Then a topological property $\mathcal{P}$ is semiregular provided that τ has property $\mathcal{P}$ if and only if τ^* has the same property. In this work we study semiregular property of almost Lindelöf, weakly Lindelöf, nearly regular-Lindelöf, almost regular-Lindelöf and weakly regular-Lindelöf spaces. We prove that all these topological properties, on the contrary of Lindelöf property, are semiregular properties.

1. Introduction

Among various covering properties of topological spaces a lot of attention has been made to those covers which involve regularly open sets and regularly closed sets. In 1959 Frolik [5] introduced the notion of weakly Lindelöf space that was afterwards studied by several authors. In 1982 Balasubramanian [1] introduced and studied the notion of nearly Lindelöf spaces as a generalization of the nearly compact spaces defined by Singal and Mathur [8]. Then in 1986 Mršević et al. [7] gave some characterizations of the classes of nearly Lindelöf spaces. In 1984 Willard and Dissanayake [9] gave the notion of almost Lindelöf spaces and in 1996 Cammaroto and Santoro [4] introduced the notion of nearly regular-Lindelöf, almost regular-Lindelöf and weakly regular-Lindelöf spaces by using the regular covers which were introduced by Cammaroto and Lo Faro [3] in 1981. The purpose of this paper is to prove that all these generalizations of Lindelöf spaces, which are topological properties, are semiregular properties.

Throughout this paper, a space X means a topological space (X,τ) on which no separation axioms are assumed unless explicitly stated otherwise. The interior and the closure of any subset A of (X,τ) will be denoted by $\mathrm{Int}(A)$ or $\mathrm{Int}_\tau(A)$ and $\mathrm{Cl}(A)$ or $\mathrm{Cl}_\tau(A)$ respectively. Recall that a subset $A \subseteq X$ is called regularly open (regularly closed) if $A = \mathrm{Int}(\mathrm{Cl}(A))$ $(A = \mathrm{Cl}(\mathrm{Int}(A))$. The class of all regular open sets in a space (X,τ) is a base for a coarser topology than τ on X, called the

AMS Subject Classification: 54 A 05, 54 A 20, 54 F 45

Keywords and phrases: Semiregular property, semiregularization topology, nearly Lindelöf, almost Lindelöf, weakly Lindelöf, nearly regular-Lindelöf, almost regular-Lindelöf and weakly regular-Lindelöf spaces.

semiregularization topology of X which is denoted by (X,τ^*). Semiregularization topologies were studied by Mršević et al. [6] in 1985. If $\tau=\tau^*$ then X is said to be semiregular. The following results, to be used in the sequel, are known and easily can be proved.

LEMMA 1.1. *Let (X,τ) be a topological space and (X,τ^*) its semiregularization. Then*
(a) *regularly open sets of (X,τ) are the same as regularly open sets of (X,τ^*).*
(b) *regularly closed sets of (X,τ) are the same as regularly closed sets of (X,τ^*).*
(c) $\mathrm{Int}_\tau(C)=\mathrm{Int}_{\tau^*}(C)$ *for any regularly closed set C.*
(d) $\mathrm{Cl}_\tau(A)=\mathrm{Cl}_{\tau^*}(A)$ *for every $A\in\tau$.*

DEFINITION 1.1. (see [2]) Let (X,τ) be a topological space and (X,τ^*) its semiregularization. A topological property $\mathcal{P}$ is called semiregular provided that τ has the property $\mathcal{P}$ if and only if τ^* has the property $\mathcal{P}$.

2. Semiregularization of Lindelöf spaces

DEFINITION 2.1. (see [1], [9] and [5]). A topological space X is said to be nearly Lindelöf, almost Lindelöf and weakly Lindelöf if, for every open cover $\{U_\alpha:\alpha\in\Delta\}$ of X, there exists a countable subset $\{\alpha_n:n\in\mathbf{N}\}\subseteq\Delta$ such that
$$X=\bigcup_{n\in\mathbf{N}}\mathrm{Int}(\mathrm{Cl}(U_{\alpha_n})),\quad X=\bigcup_{n\in\mathbf{N}}\mathrm{Cl}(U_{\alpha_n})\quad\text{and}\quad X=\mathrm{Cl}\Big(\bigcup_{n\in\mathbf{N}}U_{\alpha_n}\Big),$$
respectively.

Note first that a space (X,τ) is nearly Lindelöf if and only if (X,τ^*) is Lindelöf (see [7], Theorem 1). Moreover, the Lindelöf property is not a semiregular property (see [7], Example 1). Note also that, in distinction to the Lindelöf property, the nearly Lindelöf property is a semiregular property (see [7], Corollary 4). Next we prove that almost Lindelöf and weakly Lindelöf properties are semiregular properties as we conclude from the following theorems.

THEOREM 2.1. *Let (X,τ) be a topological space. Then (X,τ) is almost Lindelöf if and only if (X,τ^*) is almost Lindelöf.*

Proof. Suppose that (X,τ) is almost Lindelöf and let $\{U_\alpha:\alpha\in\Delta\}$ be an open cover of (X,τ^*), thus an open cover of the almost Lindelöf space (X,τ) since $\tau^*\subseteq\tau$. So there exists a countable subset $\{\alpha_n:n\in\mathbf{N}\}\subseteq\Delta$ such that $X=\bigcup_{n\in\mathbf{N}}\mathrm{Cl}_\tau(U_{\alpha_n})$. By Lemma 1.1(d), $X=\bigcup_{n\in\mathbf{N}}\mathrm{Cl}_{\tau^*}(U_{\alpha_n})$. This implies that (X,τ^*) is almost Lindelöf.

Conversely, suppose that (X,τ^*) is almost Lindelöf and let $\{V_\alpha:\alpha\in\Delta\}$ be an open cover of (X,τ). Then $\{\mathrm{Int}_\tau(\mathrm{Cl}_\tau(V_\alpha)):\alpha\in\Delta\}$ is an open cover of the almost Lindelöf space (X,τ^*). Thus it has a countable subfamily $\{\mathrm{Int}_\tau(\mathrm{Cl}_\tau(V_{\alpha_n})):n\in\mathbf{N}\}$ such that
$$X=\bigcup_{n\in\mathbf{N}}\mathrm{Cl}_{\tau^*}(\mathrm{Int}_\tau(\mathrm{Cl}_\tau(V_{\alpha_n})))=\bigcup_{n\in\mathbf{N}}\mathrm{Cl}_\tau(\mathrm{Int}_\tau(\mathrm{Cl}_\tau(V_{\alpha_n})))=\bigcup_{n\in\mathbf{N}}\mathrm{Cl}_\tau(V_{\alpha_n}).$$
This shows that (X,τ) is almost Lindelöf and completes the proof. ■

Corollary 2.1. *Almost Lindelöf property is a semiregular property.*

Theorem 2.2. *Let (X, τ) be a topological space. Then (X, τ) is weakly Lindelöf if and only if (X, τ^*) is weakly Lindelöf.*

Proof. The proof is quite similar to the proof of Theorem 2.1 above on using the fact that

$$\mathrm{Cl}_{\tau^*}\Big(\bigcup_{n\in\mathbf{N}} \mathrm{Int}_\tau(Cl_\tau(V_{\alpha_n}))\Big) \subseteq \mathrm{Cl}_{\tau^*}\Big(\bigcup_{n\in\mathbf{N}} V_{\alpha_n}\Big) = \mathrm{Cl}_\tau\Big(\bigcup_{n\in\mathbf{N}} V_{\alpha_n}\Big).$$

Thus we choose to omit the details. ∎

Corollary 2.2. *Weakly Lindelöf property is a semiregular property.*

3. Semiregularization of regular-Lindelöf Spaces

Recall that an open cover $\{U_\alpha : \alpha \in \Delta\}$ of a topological space X is called regular [3] if, for every $\alpha \in \Delta$, there exists a nonempty regularly closed subset C_α of X such that $C_\alpha \subseteq U_\alpha$ and $X = \bigcup_{\alpha\in\Delta} \mathrm{Int}(C_\alpha)$. The following definition was given in [4] and is using regular covers to give some generalizations of Lindelöf spaces.

Definition 3.1. A topological space X is called weakly regular-Lindelöf (resp. almost regular-Lindelöf, nearly regular-Lindelöf) if every regular cover $\{U_\alpha : \alpha \in \Delta\}$ of X admits a countable subfamily $\{U_{\alpha_n} : n \in \mathbf{N}\}$ such that

$$X = \mathrm{Cl}\Big(\bigcup_{n\in\mathbf{N}} U_{\alpha_n}\Big) \quad \Big(\text{ resp. } X = \bigcup_{n\in\mathbf{N}} \mathrm{Cl}(U_{\alpha_n}), \quad X = \bigcup_{n\in\mathbf{N}} \mathrm{Int}(\mathrm{Cl}(U_{\alpha_n}))\Big).$$

The following implications are obvious from the definitions.

$$\begin{array}{ccccccc}
\text{Lindelöf} & \Rightarrow & \text{nearly Lindelöf} & \Rightarrow & \text{almost Lindelöf} & \Rightarrow & \text{weakly Lindelöf} \\
 & & \Downarrow & & \Downarrow & & \Downarrow \\
 & & \text{nearly} & & \text{almost} & & \text{weakly} \\
 & & \text{regular-Lindelöf} & \Rightarrow & \text{regular-Lindelöf} & \Rightarrow & \text{regular-Lindelöf}
\end{array}$$

Now we prove that the nearly regular-Lindelöf property is a semiregular property depending on Lemma 1.1 above.

Theorem 3.1. *Let (X, τ) be a topological space. Then (X, τ) is nearly regular-Lindelöf if and only if (X, τ^*) is nearly regular-Lindelöf.*

Proof. Suppose that (X, τ) is nearly regular-Lindelöf and let $\{U_\alpha : \alpha \in \Delta\}$ be a regular cover of (X, τ^*). Since $\tau^* \subseteq \tau$ and by Lemma 1.1 (a, b and d) we have, $\{U_\alpha : \alpha \in \Delta\}$ is a regular cover of the nearly regular-Lindelöf space (X, τ). So it has a countable subset $\{U_{\alpha_n} : n \in \mathbf{N}\}$ such that $X = \bigcup_{n\in\mathbf{N}} \mathrm{Int}_\tau(\mathrm{Cl}_\tau(U_{\alpha_n})) = \bigcup_{n\in\mathbf{N}} \mathrm{Int}_{\tau^*}(\mathrm{Cl}_{\tau^*}(U_{\alpha_n}))$ (by Lemma 1.1(c,d)). This implies that (X, τ^*) is nearly regular-Lindelöf.

Conversely, suppose that (X, τ^*) is nearly regular-Lindelöf and let $\{U_\alpha : \alpha \in \Delta\}$ be a regular cover of (X, τ). Then since $U_\alpha \subseteq \mathrm{Int}_\tau(\mathrm{Cl}_\tau(U_\alpha))$, Lemma 1.1 above implies that $\{\mathrm{Int}_\tau(\mathrm{Cl}_\tau(U_\alpha)) : \alpha \in \Delta\}$ is a regular cover of the nearly regular-Lindelöf space (X, τ^*). Thus there exists a countable subfamily $\{\alpha_n : n \in \mathbf{N}\}$, such that $X = \bigcup_{n\in\mathbf{N}} \mathrm{Int}_{\tau^*}(\mathrm{Cl}_{\tau^*}(\mathrm{Int}_\tau(\mathrm{Cl}_\tau(U_{\alpha_n})))) = \bigcup_{n\in\mathbf{N}} \mathrm{Int}_{\tau^*}(\mathrm{Cl}_{\tau^*}(U_{\alpha_n})) = \bigcup_{n\in\mathbf{N}} \mathrm{Int}_\tau(\mathrm{Cl}_\tau(U_{\alpha_n}))$ (by Lemma 1.1(a,c,d)). This implies that (X, τ) is a nearly regular-Lindelöf space and completes the proof. ■

COROLLARY 3.1. *Nearly regular-Lindelöf property is a semiregular property.*

Next we prove that the almost regular-Lindelöf and weakly regular-Lindelöf properties are semiregular properties.

THEOREM 3.2. *Let (X, τ) be a topological space. Then (X, τ) is almost regular-Lindelöf if and only if (X, τ^*) is almost regular-Lindelöf.*

Proof. The proof is similar to the proof of Theorem 3.1 above, thus we choose to omit the details. ■

COROLLARY 3.2. *Almost regular-Lindelöf property is a semiregular property.*

THEOREM 3.3. *Let (X, τ) be a topological space. Then (X, τ) is weakly regular-Lindelöf if and only if (X, τ^*) is weakly regular-Lindelöf.*

Proof. The proof is similar to the proof of Theorem 3.1 above on using the fact that $\mathrm{Cl}_\tau\Big(\bigcup_{n\in\mathbf{N}}(\mathrm{Int}_\tau(\mathrm{Cl}_\tau(U_{\alpha_n}))\Big) \subseteq \mathrm{Cl}_\tau\Big(\bigcup_{n\in\mathbf{N}} U_{\alpha_n}\Big) = \mathrm{Cl}_{\tau^*}\Big(\bigcup_{n\in\mathbf{N}} U_{\alpha_n}\Big)$. Thus the details are omitted. ■

COROLLARY 3.3. *Weakly regular-Lindelöf property is a semiregular property.*

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(received 01.08.2001, in revised form 15.10.2004)

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