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# NEW APPROACH TO SHOCKS GENERATION FOR CONSERVATION LAWS. EXAMPLE: GLOBAL SOLUTION TO HOPF EQUATION

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**Abstract.** We present a new method for constructing global solution to Hopf equation. Propagation and formation of nonlinear waves are described.

## 1. Introduction

In general, the use of asymptotic method for solving a differential equation means the construction of function which depends on some small parameter, say $\varepsilon$, and satisfy a differential equation up to $\mathcal{O}_S(\varepsilon^\alpha)$ when $\varepsilon \to 0$ and $\alpha > 0$. Here, $\mathcal{O}_S(\varepsilon^\alpha)$ means that expression $\frac{\mathcal{O}_S(\varepsilon^\alpha)}{\varepsilon^\alpha}$ stays bounded in $S$ sense when $\varepsilon \to 0$, where $S$ is suitable functional space, usually $C^k$. In our new approach, $S = \mathcal{D}'(\mathbf{R})$. That means that for a given differential equation we look for a function which satisfies this equation in $\mathcal{D}'(\mathbf{R})$ or in a *weak sense*. Therefore, the name of this method is *the weak asymptotic method* (see [3]).

In what follows, we use our method on the following problem:

$$Lu = u_t + (u^2)_x = 0,\ x \in \mathbf{R},\ t > 0, \tag{1}$$

$$u\big|_{t=0} = u_0(x), \tag{2}$$

where $u_0$ is decreasing, Lipschitz continuous and takes values in some compact interval. Asymptotic solutions for this problem has already been searched by several authors as J.Witham and A.M.Il'in. These authors were interested exactly in the scalar conservation law. On the other hand, more general approach (analysis of the system of conservation laws) was proceeded by J.Glimm (see [1]) and Bressan (see [8]). (We mentioned only the authors which completed appropriate approaches.)

Witham uses Florin-Hopf-Cole linearization of equation $u_t + (u^2)_x = \varepsilon u_{xx}$. In that manner he obtains its exact solution. Moreover, he analyzes it as $\varepsilon \to 0$ for different periods of time.

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This method cannot be used for more general situation of the scalar conservation law since in general it is not known how to linearize corresponding problem with vanishing viscosity. Successful attempt to generalize Whitham approach was made by Il'in. He introduced "the matching method". In his work he considers Cauchy problem $u_t + (f(u))_x = \varepsilon u_{xx}$, $u(0,x) = \varphi(x)$ where $\varphi$ is bounded, piecewise smooth function. For fixed $\varepsilon$ one knows that the last problem has bounded infinitely differentiable solution everywhere for $t > 0$ with the exception of the discontinuity points of the initial function. On the contrary, when $\varepsilon \to 0$ the solution stops to be even continuous and the discontinuity line appears. So, the solution of the problem he searches in the form of two asymptotical series. He chooses the first one to be the solution of the problem in an arbitrary domain which does not comprise the discontinuity line while the second series is the solution of the problem near the discontinuity line. He shows that those two series coincides in the vicinity of the discontinuity line. The solution obtained in a such way has complicated form.

The drawback of mentioned approaches lies in the fact that they cannot be applied on the systems of hyperbolic conservation laws. More comprehensive approach was offered by Glimm ("random choice method") and Bressan ("front tracking method"). (Front tracking method was first proposed by DiPerna for $2 \times 2$ systems.)

Our idea in this paper is the closest to Glimm's ideas. He breaks $x$-axis as well as $t$-axis creating a mesh in $xt$-plane. He assigns recursively, (beginning from the time $t_0 = 0$) some constant value on each of intervals $(x_{i-1}, x_{i+1}) \times \{t_s\}$, $i \in \mathbf{Z}$, $s \in \mathbf{N}$, $i + s$ even. In that manner, he obtains a sequence of Riemann problems for equation (1) on disjoint intervals $(x_{2i-1}, x_{2i+3})$ (or $(x_{2i}, x_{2i+2})$) with the initial moment $t = t_s$ where $i \in \mathbf{Z}_2$, $s \in \mathbf{N}_2 + 1$ (or $i \in \mathbf{Z}_2$, $s \in \mathbf{N}_2$). He breaks $t$-axis in such a way that no two shocks created from mentioned Riemann problems interact till $t = t_{s+1}$. In $t = t_{s+1}$ he reapproximates the solution with new step function and so he obtains the Cauchy problem as in the beginning (with the difference that initial moment is not $t = 0$ but $t = t_{s+1}$). Finally, he proves that solution constructed in this way tends to the admissible weak solution of the original problem. Shortcoming of this approach lies in the fact that it does not give any explicit formula which would represent the asymptotic solution of problem (1), (2) but it gives the proof of existence of the Cauchy problem for the initial data with small variation. Method can be applied on Hopf type equation with an arbitrary nonlinearity. (Since Bressan approach is quite different from ours we will not mention it. One can find more information about it in [8].)

We will search the weak asymptotic solution of problem (1), (2) with the accuracy $\varepsilon^{1-\mu}$, $\varepsilon \to 0$, $0 < \mu < 1$. That means that we look for a net of smooth functions $u_\varepsilon(x,t)$, $x \in \mathbf{R}$, $t \in \mathbf{R}^+$, $\varepsilon < 1$, which satisfies

$$Lu_\varepsilon = \mathcal{O}_{\mathcal{D}'}(\varepsilon^{1-\mu}) \tag{3}$$

$$u_\varepsilon\big|_{t=0} - u\big|_{t=0} = \mathcal{O}_{L_1^{loc}}(\varepsilon^{1-\mu}), \tag{4}$$

or, more precisely, we search for $u_\varepsilon(x,t)$, $x \in \mathbf{R}$, $t \in \mathbf{R}^+$, $0 < \varepsilon < 1$, smooth for

$\varepsilon > 0$ and such that for every $\phi \in C_0^\infty(\mathbf{R}_x^1)$ we have in the distributional sense,

$$\int L u_\varepsilon \phi dx = \mathcal{O}(\varepsilon^{1-\mu})$$
$$\|u_\varepsilon\big|_{t=0} - u\big|_{t=0}\|_{L_1^{loc}(\mathbf{R})} = \mathcal{O}(\varepsilon^{1-\mu}), \quad \varepsilon \to 0.$$

First, we approximate given initial condition by a polygon, and then, we solve equation (3) with this polygon as initial condition. It is well known that, as time goes, ends of intervals will reach one another (we will call that phenomenon interaction of points). So, sooner or later one or few shock waves will appear and ends of intervals which do not take part in shocks formation will continue to interact with the formed shocks.

In the analysis of this situation we start from the case when we have two weak discontinuities, and then when we have one weak discontinuity and one shock wave. We use such an approach to analyze the situation when we have arbitrary many weak discontinuities. Here, we additionally assume that no three consequent point will interact simultaneously. We can assume that interactions happens in the moments $t_1^* < t_2^* < \cdots < t_n^* < \cdots$. First, we will look for the partial solutions of equation (3) in the intervals $[0, t_1^* + \delta_1)$, $[t_1^*, t_2^* + \delta_2)$, … , and prove that in common parts of every two intervals partial solutions match. Then, we will connect them by the partition of unity of the time interval.

## 2. Some weak asymptotic formulas

We will introduce necessary definitions and formulas. The proofs can be found in [3] or in [6].

Proposition 2.1. *Let $\omega(z) \in \mathbf{S}(\mathbf{R}^1)$, where $\mathbf{S}$ is the Schwartz space. For any function $\eta(x) \in C_0^\infty$ we have*

$$\langle \frac{1}{\varepsilon}\omega\Big(\frac{x-a}{\varepsilon}\Big), \eta(x)\rangle = \sum_{k=0}^{n} \Omega_k \frac{\varepsilon^k}{k!}(-1)^k \langle \delta^{(k)}(x-a), \eta\rangle + \mathcal{O}(\varepsilon^n), \quad \varepsilon > 0, \tag{5}$$

*where*

$$\Omega_k = \int \omega(z) z^k \, dz, \quad k = 1, \dots, n. \tag{6}$$

Expression (5) we call the weak asymptotic of the function (distribution) $\frac{1}{\varepsilon}\omega\Big(\frac{x-a}{\varepsilon}\Big)$.

Definition 2.2. We denote by $O_{\mathcal{D}'}(\varepsilon^\alpha)$ an element of $\mathcal{D}'$ such that for every function $\eta(x) \in C_0^\infty$ we have

$$f(x,\varepsilon) = O_{\mathcal{D}'}(\varepsilon^\alpha) \Leftrightarrow \langle f(x,\varepsilon), \eta(x)\rangle = O(\varepsilon^\alpha).$$

Proposition 2.3. *Let $\omega_1(z), \omega_2(z) \in \mathbf{S}(\mathbf{R})$. We have*

$$\omega_1\Big(\frac{x-a_1}{\varepsilon}\Big)\omega_2\Big(\frac{x-a_2}{\varepsilon}\Big) = \frac{1}{2}[\varepsilon\delta(x-a_1) + \varepsilon\delta(x-a_2)] B\Big(\frac{\Delta a}{\varepsilon}\Big) + O_{\mathcal{D}'}(\varepsilon^2), \tag{7}$$

*where*

$$B\Big(\frac{\Delta a}{\varepsilon}\Big) = \int \omega_1(z)\omega_2\Big(z - \frac{\Delta a}{\varepsilon}\Big)\,dz = \int \omega_1\Big(z + \frac{\Delta a}{\varepsilon}\Big)\omega_2(z)\,dz. \qquad (8)$$

Remark 2.4. Under the assumptions of Proposition 2.1 and the condition that $\int \omega_i(z)\,dz = 1$, the functions $\omega_i((x-a_i)/\varepsilon)$ are weak approximations (weak asymptotic) of the functions $\varepsilon\delta(x-a_i)$,

$$\omega_i((x-a_i)/\varepsilon) = \varepsilon\delta_{\varepsilon,i}(x-a_i)$$

Hence we can rewrite (8) as

$$\varepsilon\delta_{\varepsilon,1}(x-a_1)\varepsilon\delta_{\varepsilon,2}(x-a_2) = \frac{1}{2}[\varepsilon\delta(x-a_1)+\varepsilon\delta(x-a_2)]B(\Delta a/\varepsilon) + O_{\mathcal{D}'}(\varepsilon^2).$$

In a similar way, under the assumptions of Proposition 2.3 $\omega_i((x-a_1)/\varepsilon) = \theta_{\varepsilon,i}(x-a_i)$ are approximations of the Heaviside $\theta$-function. Hence we can rewrite (8) as

$$\theta_{\varepsilon,1}(x-a_1)\theta_{\varepsilon,2}(x-a_2) = \theta(x-a_1)B_1(\Delta a/\varepsilon) + \theta(x-a_2)B_2(\Delta a/\varepsilon) + O_{\mathcal{D}'}(\varepsilon). \quad (9)$$

## 3. Uniform in t ∈ R solution of the Hopf equation with the “simple” functions as an initial condition

We remind that our task is to connect two states of the solution of Cauchy problem (1), (2). The first state is the continuous one (i.e. before blow up of the classical solution), and the other one is discontinuous state. As we have already said, for accomplishing our task, we will use well known idea of replacing (approximating) old initial data with a more simple function. In the case of random choice method or wave front tracking method, the old initial data are replaced by piecewise constant functions, or, in other words, the old initial data are locally replaced by the Heaviside function. In our case, we have to replace the old initial data with some continuous function (since we want to inspect the passage from continuous to discontinuous state we have to start from something which is continuous). Naturally, in the beginning we have to consider Hopf equation with the initial function equal to the function we will locally replace our old initial data with.

### 3.1. Interaction of weak discontinuities. Generation of shock waves

We will consider the Hopf equation

$$Lu = u_t + (u^2)_x = 0 \qquad (10)$$

and pose the following initial condition

$$u\big|_{t=0} = u_0^0 + u_1^0(a_1 - x)_+ - u_1^0(a_2 - x)_+,$$

where $a_1 > a_2$, $z_+ = z\theta(z)$, $u_i^0 = \text{const} > 0$.

We shall seek the weak asymptotic solution in the form of the (regularized) broken line

$$u_\varepsilon(x,t) = u_0^0 + u_1(t,\varepsilon)(\varphi_1(t,\varepsilon)-x)\theta_{\varepsilon,1}(-x+\varphi_1(t,\varepsilon)) - u_2(t,\varepsilon)(\varphi_2(t,\varepsilon)-x)\theta_{\varepsilon,2}(-x+\varphi_2(t,\varepsilon)).$$

The unknown functions $\varphi_i$ and $u_i$, $i=1,2$, appearing in the last expression belong to $C^1(\mathbf{R}^+)$ for every fixed $\varepsilon$ and they satisfy the following initial conditions:

$$\varphi_i(0,\varepsilon) = a_i, \qquad u_i(0,\varepsilon) = u_1^0, \quad i=1,2.$$

Substituting the approximation of $u_\varepsilon(x,t)$ into equation (10) and taking into account the definitions and calculations given in the previous section we obtain

$$\big(u_1(\varphi_1-x)_+\big)_t - \big(u_2(\varphi_2-x)_+\big)_t + \big(u_1^2(\varphi_1-x)_+^2\big)_x + \big(u_2^2(\varphi_2-x)_+^2\big)_x + 2\big[u_0u_1(\varphi_1-x)_+\big]_x - 2\big[u_0u_2(\varphi_2-x)_+\big]_x - 2\big[u_1u_2(\varphi_1-x)(\varphi_2-x)\theta(\varphi_1-x)\big]_x B_1(\Delta\varphi/\varepsilon) - 2\big[u_1u_2(\varphi_1-x)(\varphi_2-x)\theta(\varphi_2-x)\big]_x B_2(\Delta\varphi/\varepsilon) = O_{\mathcal{D}'}(\varepsilon),\ \Delta\varphi = \varphi_2-\varphi_1.$$

Let us consider the domain $\varphi_2 < x \le \varphi_1$. Notice that in this case $\theta_1 = \theta(\varphi_1 - x) = 1$ and $\theta_2 = \theta(\varphi_2 - x) = 0$. Accordingly, we have

$$u_{1t}(\varphi_1-x) + u_1\varphi_{1t} + 2\big[u_0u_1(\varphi_1-x)\big]_x + \big[u_1^2(\varphi_1-x)^2\big]_x + 2u_1u_2(\varphi_1-x)B_1 + 2u_1u_2(\varphi_2-x)B_1 =$$
$$u_{1t}(\varphi_1-x)+u_1\varphi_{1t}-2u_0u_1+2u_1^2(\varphi_1-x)+2u_1u_2(\varphi_1-x)B_1+2u_1u_2(\varphi_2-x)B_1 = 0$$

Setting $x=\varphi_1$ we obtain,

$$\varphi_{1t} - 2u_0 + 2u_2\Delta\varphi B_1 = 0. \tag{11}$$

Substituting this relation into the last equation, we arrive at the following equation for the function $u_1$:

$$u_{1t} - 2u_1^2 + 4u_1u_2B_1 = 0. \tag{12}$$

In a similar way, considering the domain $-\infty < x \le \varphi_2$, we obtain the other two equations

$$\varphi_{2t} - 2u_0 + 2u_1\Delta\varphi B_2(\Delta\varphi/\varepsilon) = 0, \tag{13}$$
$$u_{2t} + 2u_2^2 - 4u_1u_2B_2(\Delta\varphi/\varepsilon) = 0, \quad \Delta\varphi = \varphi_2 - \varphi_1. \tag{14}$$

Let $\Delta\varphi < 0$, then, up to $O(\varepsilon^N)$, we have $B_1(\Delta\varphi/\varepsilon) = 0$, $B_2(\Delta\varphi/\varepsilon) = 1$ and we obtain the following system of equations describing the evolution of the broken line until it turns over (or, more formally, until $\varphi_1 > \varphi_2$):

$$(\varphi_{10})'_t - 2u_0 = 0, \qquad (\varphi_{20})'_t - 2u_0 + 2u_{10}(\varphi_{20}-\varphi_{10}) = 0, \tag{15}$$
$$(u_{10})'_t - 2(u_{10})^2 = 0, \qquad (u_{20})'_t + 2u_{20}^2 - 4u_{10}u_{20} = 0,$$

Solutions of this system have the form

$$u_{10}(t) = u_{20}(t) = u_1^0/(1-2tu_1^0),$$
$$\varphi_{10} = a_1 + 2u_0 t, \qquad \varphi_{20} = a_2 + 2[u_1^0(a_1-a_2)+u_0]t.$$

We write $\psi_0 = \varphi_{20}(t) - \varphi_{10}(t)$.

At time $t = t^*$ such that $\psi_0(t^*) = 0$ the weak discontinuities merge and a shock wave is generated. To construct formulas that are uniform in $t$ and describe the interaction of weak discontinuities and the generation of a shock wave, we seek the functions $\varphi_i$, $i = 1, 2$, in the form

$$\varphi_k(t,\varepsilon) = \varphi_{k0}(t) + \psi_0\phi_k(\tau), \ \tau = \psi_0/\varepsilon, \ k = 1,2.$$

We also introduce the function $\rho = \rho(\tau)$:

$$\rho(\tau) = \frac{\varphi_2(t,\varepsilon) - \varphi_1(t,\varepsilon)}{\varepsilon}.$$

Substituting this into (11) and (13) and letting $\rho \to -\infty$ we see that $\phi_k$ must satisfy

$$\phi_k(\tau)\big|_{\tau_1\to-\infty} = 0, \qquad \frac{d\phi_k}{d\tau}\big|_{|\tau|\to\infty} = o(\tau^{-1}).$$

We shall seek the functions $u_k(t,\varepsilon)$, $k = 1, 2$, in the form

$$u_k(t,\varepsilon) = \psi_0(0)u_1^0/(\psi_0 + \varepsilon g_k(\tau)), \quad k = 1,2.$$

Here we assume that the functions $g_k(\tau)$ behave in the same way as the functions $\phi_k(\tau)$. From equations (12), (13), (14), (15) we conclude $\tau+g = \tau+g_1 = \tau+g_2 = \rho$ as well as

$$\dot\rho = 1 - 2B_1(\rho), \qquad \rho/\tau\big|_{\tau\to-\infty} \to 1.$$

The stationary solution of this equation is $\rho = \rho_0$, where $\rho_0$ is such that $B_1(\rho_0) = 1/2$. Since $0 < B_1 < 1$ we see that $\dot\rho > 0$. Accordingly, $\rho$ is an increasing function which tends to $\rho_0$.

This allows us to calculate the solution for $\Delta\psi_0 > 0$ (i.e., after the interaction). Notice that in that case $\tau \to \infty$.

We introduce the function $G(\tau) = \tau + g(\tau)$. By the previous, $\dot G = \dot\rho$, $G/\tau\big|_{\tau\to-\infty} \to +1$, and we choose

$$G = -\int_{-\infty}^{\infty} (1 - 2B_1(\rho))\, d\tau' + \rho_0.$$

On the other hand, we can express the functions $u_i$ via the function $G$:

$$u_i = \frac{\psi_0(0)u_1^0}{\varepsilon G} \overset{\tau\to\infty}{\to} \frac{\psi_0(0)u_1^0}{\varepsilon\rho_0}.$$

We calculate the limit $(\varphi_k)^+_t$ as $\tau \to \infty$ of the velocities of the weak discontinuities

$$(\varphi_2)^+_t = 2u_0 - \frac{2\psi_0(0)u_1^0}{\varepsilon\rho_0}\frac{1}{2}\varepsilon\rho_0 = 2u_0 + (a_1 - a_2)u_1^0,$$
$$(\varphi_1)^+_t = 2u_0 - \frac{2\psi_0(0)u_1^0}{\varepsilon\rho_0}\frac{1}{2}\varepsilon\rho_0 = 2u_0 + (a_1 - a_2)u_1^0,$$

which coincides with the velocity of the shock wave

$$U(x,t) := u_0^0 + (a_1 - a_2)u_1^0\theta(-x + \varphi^+(t)),$$

where $\varphi^+ = \varphi_2^+ = \varphi_1^-$.

By using the explicit formula for the solution $u_\varepsilon(x,t)$, we can easily show that

$$w - \lim_{\varepsilon\to 0} u_\varepsilon(x,t) = U(x,t), \qquad t > t^*.$$

To this end, we rewrite the above-constructed solution $u_\varepsilon(x,t)$ in the form

$$u_\varepsilon(x,t) = u_0 + u_1(\varphi_1 - \varphi_2)\theta_{\varepsilon,1}(\varphi_1 - x) + u_1(x - \varphi_2)\big[\theta_{\varepsilon,2}(\varphi_2 - x) - \theta_{\varepsilon,1}(\varphi_1 - x)\big].$$

Consider the second term. We have

$$u_1(\varphi_1 - \varphi_2) = \frac{\psi_0(0)u_1^0\rho}{G} = \psi_0(0)u_1^0 = (a_1 - a_2)u_1^0 \stackrel{\text{def}}{=} U_0.$$

Since $\varphi_1\big|_{t>t^*} \simeq \varphi^+$, the first two terms pass into the shock wave $U(x,t)$ for $t > t^*$. Consider the last term

$$u_1(x - \varphi_2)\big[\theta_{\varepsilon,2}(\varphi_2 - x) - \theta_{\varepsilon,1}(\varphi_1 - x)\big] = u_1(x - \varphi_2)\left[\frac{\theta_{\varepsilon,2}(\varphi_2 - x) - \theta_{\varepsilon,1}(\varphi_1 - x)}{\varphi_1 - \varphi_2}\right]$$

As was already shown, the coefficient of the expression in braces is a constant. The expression in square brackets is an approximation of the $\delta$-function at the point $\varphi_2$. Hence the entire expression in braces is small (in a weak sense) as $\varepsilon \to 0$.

### 3.2. Interaction of a weak discontinuity and a shock wave

In this case mechanism of evolution is as follows: segments of the broken line interact with the step that has already been formed.

In order to study this we consider the Hopf equation again. The initial condition corresponding to this type of interaction has the form

$$u\big|_{t=0} = u_0^0\theta(a_1^0 - x) + u_1^0(a_1 - x)\theta(a_1 - x) - u_1^0(a_2 - x)\theta(a_2 - x), \qquad (16)$$

where $u_0^0$, $u_1^0$ are positive constants and $a_1 > a_2$. Just as before, we construct the weak asymptotic solution in the form

$$u_\varepsilon(x,t) = u_0(t,\varepsilon)\theta_{\varepsilon,1}(\varphi_1(t,\varepsilon) - x) + u_1(t,\varepsilon)(\varphi_1(t,\varepsilon) - x)\theta_{\varepsilon,1}(\varphi_1(t,\varepsilon) - x)$$
$$- u_1(t,\varepsilon)(\varphi_2(t,\varepsilon) - x)\theta_{\varepsilon,2}(\varphi_2(t,\varepsilon) - x), \qquad (17)$$

where $u_i = u_i(t,\varepsilon)$, $\varphi_i = \varphi_i(t,\varepsilon)$ are unknown $C^1(\mathbf{R}^+)$ functions for every fixed $\varepsilon$ which satisfy the following initial conditions:

$$u_i(0,\varepsilon) = u_i^0, \qquad \varphi_i(0,\varepsilon) = a_i, \quad i = 1,2.$$

Denote $\Delta\varphi = \varphi_2 - \varphi_1$. Substituting expression for $u_\varepsilon(x,t)$ into the Hopf equation and repeating the procedure from the previous subsection we obtain the following system of equations determining the unknown functions from (17)

$$\varphi_{1t} - u_0 + 2u_1(\varphi_2 - \varphi_1)B_1(\Delta\varphi/\varepsilon) = 0, \tag{18}$$

$$u_{0t} - u_0 u_1(1 - 2B_1(\Delta\varphi/\varepsilon)) = 0, \tag{19}$$

$$u_{1t} - 2u_1^2 + 4u_1^2 B_1(\Delta\varphi/\varepsilon) = 0, \tag{20}$$

$$\varphi_{2t} - 2u_0 B_2 + 2u_1\psi B_2(\Delta\varphi/\varepsilon) = 0. \tag{21}$$

Before the interaction, we have $\varphi_2 < \varphi_1$, $\Delta\varphi/\varepsilon \sim -\infty$, and $B_1 = 0$, $B_2 = 1$ with arbitrary accuracy in $\varepsilon$. Denoting by $\varphi_{10}$, $u_{10}$, $\varphi_{20}$, $u_{00}$ the solution of system (18)–(21) with $B_1 = 0$, $B_2 = 1$ (i.e. before the interaction), we obtain the following system of equations for these functions:

$$\begin{aligned}
\varphi_{10t} &= u_{00},\\
u_{10t} &= 2u_{10}^2,\\
u_{00t} &= u_{00}u_{10},\\
\varphi_{20t} &= 2(u_{00} - u_{10}\psi_0), \qquad \psi_0 = \varphi_{20} - \varphi_{10}.
\end{aligned}$$

The solutions of this system are:

$$u_{10} = \frac{u_1^0}{1 - 2u_1^0 t}, \qquad u_{00} = \frac{u_0^0}{(1 - 2u_1^0 t)^{1/2}},$$

$$\varphi_{20} = a_2 + 2Ut, \qquad \varphi_{10} = a_1 + \int_0^t u_{00}\,dt, \tag{22}$$

$$\psi_0 = \frac{1}{u_1^0}\Big[(\psi_0^0 u_1^0 - u_0^0)(1 - 2u_0^1 t) - u_0^0\sqrt{1 - 2u_0^1 t}\Big], \tag{23}$$

$$U = u_0^0 + u_1^0(a_1 - a_2).$$

One can easily see that the function $\psi_0(t)$ vanishes at the two points $t_1 = 1/2u_0^1$ and $t^*$ such that

$$\sqrt{1 - 2u_0^1 t^*} = \frac{u_0^0}{U}.$$

Obviously, $t^* < t_1$ and $x = \varphi_{10}$ and $x = \varphi_{20}$ merge at $t = t^*$. In this case we have

$$u_{00}(t^*) = u_{00}^* \equiv U, \qquad u_{10}(t^*) = \frac{U^2}{(u_0^0)^2} u_1^0 < \infty.$$

Thus, in this example the mechanism of formation of a new shock wave consists not in turning over the inclined segment of the broken line, as in the preceding example, but in the disappearance of this inclined segment due to increasing vertical segment.

Subtracting equation (20) from equation (18), we obtain the following equation for the function $\psi$:

$$\psi_t = (u_0 - 2\psi u_1)(1 - 2B_1(\Delta\varphi/\varepsilon))$$

or, denoting $\rho = \Delta\varphi/\varepsilon = (\psi_0 + \psi_0\psi_1(\tau))/\varepsilon$, $\tau = \psi_0/\varepsilon$,

$$\psi_0'\dot\rho = [u_0 - 2\psi u_1](1 - 2B_1(\rho)), \qquad \frac{\rho}{\tau}\Big|_{\tau\to-\infty} \to 1.$$

Note that we can use the formula for $\psi_0$ (and for the functions $u_{00}$ and $u_{10}$) only for $t \in [0, t^* + \delta]$, where $\delta > 0$ is any number such that $\delta < t_1 - t^*$.

To obtain formulas that are global in $t$, we need to choose a number $\delta$ and continue the functions $u_{00}$, $u_{10}$, and $\psi_0$ smoothly to the time $t \geq t^* + \delta$ so that the function $u_\varepsilon$ remains the (weak asymptotic) solution of Cauchy problem (10), (16). Since $u_0 - 2\psi u_1 = \frac{2u_1^0}{\sqrt{1-2u_1^0 t}} + u_0^0 + (a_1 - a_2)u_1^0 > 0$, for $t < t^*$ we see that $\dot\rho > 0$ in that interval. Like in Section 3.1 we conclude that there exists a solution $\rho \to \rho_0$ while $\tau \to +\infty$, where $\rho_0$ is a root of the equation $B_1(\rho) = 1/2$. This implies that $\dot\rho \to 0$, $\tau \to +\infty$ which means that when $t > t^*$ we have $\varphi_1(t) = \varphi_2(t)$. From (20) it is easy to compute $\varphi_i(t)$, $i = 1, 2$ for $t > t^*$. We have, $\varphi_i(t) = U$, $i = 1, 2$, and from initial conditions for $\varphi_i$, $i = 1, 2$, one sees that for $t > t^*$

$$\varphi_1(t,\varepsilon) = \varphi_{10}(t^*) + Ut + \mathcal{O}(\varepsilon), \quad \varphi_2 = \varphi_{20}(t^*) + Ut + \mathcal{O}(\varepsilon). \tag{24}$$

Let us consider the system of equations for the functions $u_0$ and $u_1$. From (19) and (20) we see that

$$u_1(t,\varepsilon) = \frac{u_1^0}{1 - 2u_1^0\int_0^t [1 - 2B_1(\rho(\tau))]\,dt'}.$$

Since $\int_0^{t^*}(1 - 2B_1)\,dt' \leq t^*$, we have $u_1(t^*,\varepsilon) \leq u_{10}(t^*)$. On the other hand, we have $t > t^*$ for $\rho \to \rho_0$. Therefore, $\psi_1(\tau) \to -1$ as $\tau \to \infty$ and hence $(\Delta\varphi) \to 0$ as $\tau \to \infty$ (i.e., for $t > t^*$ and $\varepsilon \to 0$). This implies that for $t > t^*$ we have

$$u_1(t,\varepsilon) = u_{10}(t^*) + o(1), \qquad \varepsilon \to 0.$$

We represent the above-constructed solution in the form

$$\begin{aligned} u_\varepsilon(x,t) &= u_0\theta_{\varepsilon,1}(\varphi_1 - x) + u_1(\varphi_1 - x)\theta_{\varepsilon,1}(\varphi_1 - x) - u_1(\varphi_2 - x)\theta_{\varepsilon,2}(\varphi_2 - x) \\ &= U\theta_{\varepsilon,1}(\varphi_1 - x) + (u_0 - U)\theta_{\varepsilon,1}(\varphi_1 - x) + u_1(\varphi_1 - \varphi_2)\theta_{\varepsilon,1}(\varphi_1 - x) \\ &\quad + [\theta_{\varepsilon,1}(\varphi_1 - x) - \theta_{\varepsilon,2}(\varphi_2 - x)](\varphi_2 - x)u_1. \end{aligned}$$

Since when $t > t^*$ we have $U - u_0 \to 0$ and $\varphi_2 - \varphi_1 \to 0$ the second and the third term in the previous expression tends to zero while the fourth term tends to zero because

$$\begin{aligned} &[\theta_{\varepsilon,1}(\varphi_1 - x) - \theta_{\varepsilon,2}(\varphi_2 - x)](\varphi_2 - x)u_1 \\ &= \frac{\theta_{\varepsilon,1}(\varphi_1 - x) - \theta_{\varepsilon,2}(\varphi_2 - x)}{\varphi_2 - \varphi_1}(\varphi_2 - x)(\varphi_2 - \varphi_1)u_1 \to \delta(\varphi - x)(\varphi_2 - x)(U - u_0) = 0 \end{aligned}$$

Since for $t > t^*$, $\varepsilon \to 0$, from (24) we have $\varphi_1 = a_1 + Ut$, the first term, and therefore constructed function $u_\varepsilon(x,t)$ when $t > t^*$, approximates the shock wave

$$u = U\theta(a_1 + Ut - x).$$

## 4. Standard examples of Cauchy problems for Hopf equation

In the next four sections we will inspect only the development of the initial condition before the second interaction. After the first interaction we will have the analogous problem as in the start of our considerations or the problem will be deduced to some of situations we have already analyzed (more precisely, equation is the same while initial condition is analogous to one of those we have already considered). Using same methods as before one can prove that the functions which describe the solution of appropriate problem from the $k$-th till the $k+2$-nd interaction and from the $k+1$-st till the $k+3$-rd interaction match in the common part of the intervals $(t_k^*, t_{k+1}^* + \delta_k)$, $\delta_k < t_{k+2}^* - t_{k+1}^*$, $k \in \mathbf{N}$, where, as usual, $t_k^*$ is the time of $k$-th interaction. Therefore, we will be able to write the uniform solution using partition of unity of the time axis.

In what follows, functions $\phi$ which we were using for "smoothing" functions describing moving of points $a_i$ will depend not only on the "fast" variable $\tau$ but on the variable $t$, too. Therefore, we will need the following lemma:

Lemma 4.1. *Suppose that for the functions $\varphi(t,\tau)$ and $F(t,\tau)$ where $\tau = \frac{\psi_0(t)}{\varepsilon}$, $\psi_0$ a smooth increasing function which has unique zero in $t = t^*$, we have*

$$(\varphi(t,\tau))_t = F(t,\tau), \tag{25}$$

*as well as $F(\cdot,\tau) - F(\cdot,-\infty) \in \mathcal{S}(\mathbf{R})$ and $F(\cdot,\tau) - F(\cdot,\infty) \in \mathcal{S}(\mathbf{R})$. Then there exist functions $\phi(t,\tau)$, $t \in \mathbf{R}^+$, $\tau \in \mathbf{R}$ such that*

$$\varphi(t,\tau) = \varphi^-(t) + \psi_0(t)\phi(t,\tau) + \mathcal{O}(\varepsilon), \tag{26}$$

*where $\varphi^-(t) = \varphi(t,-\infty)$, $t \in \mathbf{R}^+$. Furthermore, we have*

$$\phi = \frac{1}{\psi_0' \tau} \int_0^\tau \big(F(t,\tau) - \varphi^-(t) - \psi_0(t)\phi_t^+(t)\omega(\tau)\big)\, d\tau', \tag{27}$$

*where $\phi^+$ is given by*

$$\psi_0(t)\phi^+(t) = \int_{t^*}^t \big(F(t,+\infty) - \varphi_t^-(t)\big)\, dt'$$

*while $\omega$ is such that $\lim\limits_{z\to\infty} w(z) = 1$, $\lim_{z\to-\infty} w(z) = 0$, and $\frac{d\omega}{dz} \in \mathcal{S}(\mathbf{R})$.*

*Proof.* Substituting (26) in (25) we obtain

$$\varphi_t^- + \psi_{0t}(\tau\phi)_\tau + \psi_0 \frac{\partial \phi}{\partial t} = F.$$

So, for proving that (27) is the function satisfying the lemma, we have to prove that

$$\varphi_t^- + \big(F - \varphi_t^- - \psi_0\omega\phi_t^+\big) + \psi_0 \frac{\partial \phi}{\partial t} = F + \mathcal{O}(\varepsilon),$$

i.e. that,

$$\varepsilon \frac{\partial}{\partial t}\big(\tau\phi - \tau\omega\phi^+\big) = \mathcal{O}(\varepsilon).$$

Substituting (27) in the above equation we have

$$\varepsilon\frac{\partial}{\partial t}\left(\frac{1}{\psi_0'}\int_0^{\tau}\big(F(t,\tilde\tau)-\varphi_t^-(t)-\psi_0(t)\phi_t^+(t)\omega(\tilde\tau)-\psi_0'(t)\phi^+(t)\omega(\tilde\tau)\big)\,d\tilde\tau\right)=\mathcal{O}(\varepsilon).$$

Using the fact that $\varphi^-$ solves the equation $\varphi_t = F^- = F(:\cdot:,-\infty)$ and that $\varphi_t^- + \psi_0\phi_t^+ + \psi_0'\phi^+ = (\varphi^-\psi_0\phi^+)_t = F^+ = F(:\cdot:,\infty)$ we have,

$$\varepsilon\frac{\partial}{\partial t}\left(\frac{1}{\psi_0'}\int_{-\infty}^{\tau}\big(F(t,\tilde\tau)-F^+(t)\omega(\tilde\tau)-F^-(t)(1-\omega(\tilde\tau))\big)\,d\tilde\tau\right)=\mathcal{O}(\varepsilon). \tag{28}$$

Since we have $F(\cdot,\tau)-F(\cdot,-\infty)\in\mathcal{S}(\mathbf{R})$ and $F(\cdot,\tau)-F(\cdot,\infty)\in\mathcal{S}(\mathbf{R})$, for every $t\in\mathbf{R}^+$, we also have $\frac{\partial}{\partial t}\left(F(\cdot,\tau)-F(\cdot,-\infty)\right)\in\mathcal{S}(\mathbf{R})$ and $\frac{\partial}{\partial t}\left(F(\cdot,\tau)-F(\cdot,\infty)\right)\in\mathcal{S}(\mathbf{R})$. Furthermore, since in (28) we have partial derivative in $t$ (derivative does not affect $\tau$), we conclude that the last integral is bounded and, consequently, that equality given by (28) is correct. ■

In what follows we will need the following functions:

$$\begin{aligned}\tau^{ij} &= \frac{\varphi_{j0}(t)-\varphi_{i0}(t)}{\varepsilon},\\ \rho^{ij} &= \frac{\Delta^{ij}\varphi(t)}{\varepsilon}=\frac{\varphi_j(t)-\varphi_i(t)}{\varepsilon},\\ B_k^{ij} &= B_k(\rho^{ij}),\quad t\in\mathbf{R}^+.\end{aligned}$$

### 4.1. Confluence of weak discontinuities and the shock wave, four point case; the first possibility

In this section we will search asymptotic solution of equation (10) with the initial condition which has three weak discontinuities and one shock wave. We will analyze the transformation of the initial condition till the second interaction. The initial condition is:

$$\begin{aligned}u(x,0) = h &+ u_1^0(a_1-x)-u_1^0(a_2-x)_+ + u_0^0\theta(a_2-x)\\ &+ u_2^0(a_2-x)_+ - u_2^0(a_3-x)_+ + u_3^0(a_3-x)_+ - u_3^0(a_4-x)_+,\end{aligned} \tag{29}$$

where $a_1>a_2>a_3>a_4$ as well as $v_0^0$, $v_1^0$, $v_2^0$, $h>0$, $u_0^0>0$, $u_1^0>0$, $u_2^0>0$ and $u_3^0$ are constants in $\mathbf{R}$ such that $(a_1-a_2)u_1^0=v_0^0$, $(a_2-a_3)u_2^0=v_1^0$ and $(a_3-a_4)u_3^0=v_2^0$. The weak asymptotic solution of problem (10), (29) we have in the form:

$$\begin{aligned}u_\varepsilon(x,t) = h &+ u_1(\varphi_1-x)\theta_\varepsilon(\varphi_1-x)-u_1(\varphi_2-x)\theta_\varepsilon(\varphi_2-x)\\ &+ u_0\theta_\varepsilon(\varphi_2-x)+u_2(\varphi_2-x)\theta_\varepsilon(\varphi_2-x)-u_2(\varphi_3-x)\theta_\varepsilon(\varphi_3-x)\\ &+ u_3(\varphi_3-x)\theta_\varepsilon(\varphi_3-x)-u_3(\varphi_4-x)\theta_\varepsilon(\varphi_4-x),\end{aligned} \tag{30}$$

where $\varphi_i(0)=a_i$, $i=1,2,3,4$, and $u_i(0)=u_i^0$, $i=0,1,2,3$. Like in the previous section we will just write down the equations for the unknown functions $\varphi_i$ and $u_{i-1}$, $i=1,2,3,4$, in the interval $[0,t_1^*+\delta)$. Here, we will assume that the first interaction will happen between the points $a_2$ and $a_3$.

Accordingly, system of equations determining unknown functions in (30) in the interval $[0, t_1^*)$, $\delta < t_2^* - t_1^*$, where $t_2^*$ is time of the second interaction, is:

$$\begin{aligned}
&\varphi_{1t} = 2h\\
&\varphi_{2t} = u_0 + 2h + 2v_0^0 - 2u_2\Delta^{23}\varphi B_1^{23} + 2u_3\Delta^{23}\varphi B_1^{23} = 0\\
&\varphi_{3t} = 2h + 2v_0^0 + 2u_1\Delta^{23}\varphi + 2u_0 B_2^{23} - 2u_1\Delta^{23}\varphi B_2^{23} - 2u_2\Delta^{23}\varphi B_2^{23} = 0\\
&\varphi_{4t} = 2h + 2v_0^0 + 2v_1^0 + v_2^0 + 2u_0\\
&u_{0t} - u_0(u_2 + u_1) + 2u_0 u_2 B_1^{23} - 2u_0 u_3 B_1^{23} = 0\\
&u_{1t} - u_1^2 = 0\\
&u_{2t} - 2u_2^2 - 4u_1u_2B_1^{23} + 4u_1u_3B_1^{23} - 4u_2u_3B_1^{23} - 4u_2^2B_1^{23} = 0\\
&u_{3t} - 2u_3^2 = 0.
\end{aligned} \tag{31}$$

The solutions of the system are:

$$\begin{aligned}
&\varphi_1(t) = 2ht + a_1, \quad \varphi_4(t) = 2(h + v_0^0 + v_1^0 + v_2^0)t + 2\int_0^t u_0(t')dt' + a_4\\
&u_1(t) = \frac{u_1^0}{1 - 2u_1^0 t}, \quad u_3(t) = \frac{u_3^0}{1 - 2u_3^0 t}\\
&u_0(t) = \exp\Big(\int_0^t \big(u_2 + u_1 - 2u_2B_1^{23} + 2u_3B_1^{23}\big)\, dt' + u_0^0\Big),\\
&\varphi_2(t) = \varphi_{20}(t) + \psi_0^{23}(t)\phi_2(t, \tau^{23})\\
&\varphi_3(t) = \varphi_{30}(t) + \psi_0^{23}(t)\phi_3(t, \tau^{23})\\
&u_2(t) = u_{20}(t) + \psi_0^{23}(t)\kappa(t, \tau),
\end{aligned} \tag{32}$$

where $\psi_0^{23} = \varphi_{30} - \varphi_{20}$, $\tau^{23} = \frac{\psi_0^{23}}{\varepsilon}$ and

$$\begin{aligned}
\varphi_{20}(t) &= (2h + 2v_0^0)t + \int_0^t 2u_0(t)dt + a_2,\\
\varphi_{30}(t) &= 2(h + v_0^0 + v_1^0)t + \int_0^t 2u_0(t)dt + a_3,\\
u_{20}(t) &= \frac{u_2^0}{1 - 2u_2^0 t},\\
\dot\rho^{23}\psi_{0t}^{23} &= (u_0 - 2u_2\Delta^{23})(1 - 2B_1^{23}) + 2u_1\Delta^{23}\varphi B_1^{23} - 2u_3\Delta^{23}\varphi B_1^{23}.
\end{aligned}$$

Functions $\rho^{23}(\tau^{23})$, $\phi_2(t, \tau^{23})$, $\phi_3(t, \tau^{23})$ and $\kappa(t, \tau)$ are such that

$$\begin{aligned}
&\frac{\rho^{23}(\tau^{23})}{\tau^{23}}\Big|_{\tau^{23}\to-\infty} = 1,\\
&B_i(\rho^{23}(t, \tau^{23})\big|_{\tau^{23}\to+\infty}) = 1/2, \ i = 1, 2,\\
&\phi_i(t, \tau^{23})\big|_{\tau^{23}\to-\infty} = 0,
\end{aligned}$$

$$\frac{d\phi_i(t,\tau^{23}}{d\tau^{23}}\Big|_{\tau^{23}\to-\infty} = o((\tau^{23})^{-1}),\ i=2,3,$$
$$\kappa(t,\tau^{23})\big|_{\tau^{23}\to-\infty} = 0,$$
$$\frac{d\kappa(t,\tau^{23})}{\tau^{23}} = o((\tau^{23})^{-1}).$$

Expressions for functions $\phi_2(t,\tau^{23})$, $\phi_3(t,\tau^{23})$ and $\kappa(t,\tau)$ we obtain from system (31) and Lemma 4.1.

Remark 4.2. While we obtain expressions for $\phi_2(t,\tau^{23})$ and $\phi_3(t,\tau^{23})$ explicitly from Lemma 4.1, for $\kappa(t,\tau)$ we have $u_{2t} = F(t,\tau,u_2)$ wherefrom we obtain the relation

$$a(t,\tau)\kappa^2 + b(t,\tau)\kappa + c(t,\tau) = 0,$$

for appropriate $C^1(\mathbf{R}^+ \times \mathbf{R})$ functions $a$, $b$ and $c$. From here, using boundary condition for the function $\kappa$ we find the unknown function $\kappa$.

### 4.2. Confluence of weak discontinuities and the shock wave, four point case; the second possibility

The disposure of discontinuities in the initial data is here different than in the previous section. Here, we want to explore interaction of weak discontinuity and the shock wave in the case when the shock wave is above the weak discontinuity. We will analyze the evolution of the initial condition till the second interaction. The initial condition is:

$$\begin{aligned} u(x,0) = h &+ u_1^0(a_1-x) - u_1^0(a_2-x)_+ + u_2^0(a_2-x)_+ \\ &+ u_0^0(a_3-x)_+ - u_2^0(a_3-x)_+ + u_3^0(a_3-x)_+ - u_3^0(a_4-x)_+, \end{aligned} \tag{33}$$

where $a_1 > a_2 > a_3 > a_4$ are constants in $\mathbf{R}$, $u_0^0$, $u_1^0$, $u_2^0$ and $u_3^0$ are positive constants such that $(a_1-a_2)u_1^0 = v_0^0$, $(a_2-a_3)u_2^0 = v_1^0$ and $(a_3-a_4)u_3^0 = v_2^0$. The weak asymptotic solution of problem (10), (33) we will search in the form:

$$\begin{aligned} u_\varepsilon(x,t) = h &+ u_1(\varphi_1-x)\theta_\varepsilon(\varphi_1-x) - u_1(\varphi_2-x)\theta_\varepsilon(\varphi_2-x) + u_2(\varphi_2-x)\theta_\varepsilon(\varphi_2-x) \\ &- u_2(\varphi_3-x)\theta_\varepsilon(\varphi_3-x) + u_0\theta_\varepsilon(\varphi_3-x) + u_3(\varphi_3-x)\theta_\varepsilon(\varphi_3-x) \\ &- u_3(\varphi_4-x)\theta_\varepsilon(\varphi_4-x), \end{aligned} \tag{34}$$

where $\varphi_i(0) = a_i$, $i=1,2,3,4$, and $u_i(0) = u_i^0$, $i=0,1,2,3$. Here, we will assume that the first interaction will happen between the points $a_2$ and $a_3$.

Accordingly, system of equations determining unknown functions in (34) in the interval $[0, t_1^* + \delta)$, $\delta < t_2^* - t_1^*$, where $t_2^*$ is time of the second interaction, is:

$$\begin{aligned} &\varphi_{1t} = 2h \\ &\varphi_{2t} = 2h + 2v_0^0 + 2u_0B_1^{23} - 2u_2\Delta^{23}\varphi B_1^{23} + 2u_3\Delta^{23}\varphi B_1^{23} \\ &\varphi_{3t} = 2h + u_0 + 2v_0^0 - 2u_1\Delta^{23}\varphi + 2u_1\Delta^{23}\varphi B_2^{23} - 2u_2\Delta^{23}\varphi B_2^{23} = 0 \\ &\varphi_{4t} = 2h + 2v_0^0 + 2v_1^0 + v_2^0 + 2u_0 \end{aligned}$$

$$u_{0t} - u_0(u_2 + u_1) + 2u_0u_2B_1^{23} - 2u_0u_3B_1^{23} = 0$$
$$u_{1t} - u_1^2 = 0$$
$$u_{2t} - 2u_2^2 - 4u_1u_2B_1^{23} + 4u_1u_3B_1^{23} - 4u_2u_3B_1^{23} = 4u_2^2B_1^{23} = 0$$
$$u_{3t} - 2u_3^2 = 0.$$

The solutions of the system are:

$$\varphi_1(t) = 2ht + a_1, \quad \varphi_4(t) = 2(h + v_1^0 + v_2^0 + v_3^0)t + 2\int_0^t u_0(t')dt' + a_4$$
$$u_1(t) = \frac{u_1^0}{1 - 2u_1^0t}, \quad u_3(t) = \frac{u_3^0}{1 - 2u_3^0t}$$
$$\varphi_2(t) = \varphi_{20}(t) + \psi_0^{23}(t)\phi_2(t, \tau^{23}) \tag{35}$$
$$u_0(t) = \exp\Bigg(\int_0^t \left(u_2 + u_1 - 2u_2B_1^{23} + 2u_3B_1^{23}\right)\, dt' + u_0^0\Bigg),$$
$$\varphi_3(t) = \varphi_{30}(t) + \psi_0^{23}(t)\phi_3(t, \tau^{23})$$
$$u_2(t) = u_{20}(t) + \psi_0^{23}(t)\kappa(t, \tau),$$

where $\psi_0^{23} = \varphi_{30} - \varphi_{20}$, $\tau^{23} = \frac{\psi_0^{23}}{\varepsilon}$ and

$$\varphi_{20}(t) = 2(h + v_0^0)t + \int_0^t 2u_0(t)dt + a_2$$
$$\varphi_{20}(t) = 2(h + v_1^0 + v_0^0)t + \int_0^t 2u_0(t)dt + a_3$$
$$u_{20}(t) = \frac{u_2^0}{1 - 2u_2^0t},$$
$$\dot\rho^{23}\psi_{0t}^{23} = (u_0 - 2u_2\Delta^{23})(1 - 2B_1^{23}) - 2u_1\Delta^{23}\varphi B_1^{23} - 2u_3\Delta^{23}\varphi B_1^{23}.$$

Functions $\rho^{23}(\tau^{23})$, $\phi_2(t, \tau^{23})$, $\phi_3(t, \tau^{23})$ and $\kappa(t, \tau^{23})$ have the same properties as in the previous section and we obtain them in the completely same manner.

### 4.3. Confluence of weak discontinuities, four point case

In this section we will search asymptotic solution of equation (10) with the initial condition which has four weak discontinuities. We will analyze the transformation of the initial condition till the second interaction. The initial condition is:

$$u(x, 0) = h + u_1^0(a_1 - x) - u_1^0(a_2 - x)_+ + u_2^0(a_2 - x)_+ \\ - u_2^0(a_3 - x)_+ + u_3^0(a_3 - x)_+ - u_3^0(a_4 - x)_+, \tag{36}$$

where $a_1 > a_2 > a_3 > a_4$ are constants in $\mathbf{R}$, $u_1^0$, $u_2^0$ and $u_3^0$ are positive constants such that $(a_1 - a_2)u_1^0 = v_0^0$, $(a_2 - a_3)u_2^0 = v_1^0$ and $(a_3 - a_4)u_3^0 = v_2^0$. The weak asymptotic solution of problem (10), (36) we will search in the form:

$$u_\varepsilon(x, t) = h + u_1(\varphi_1 - x)\theta_\varepsilon(\varphi_1 - x) - u_1(\varphi_2 - x)\theta_\varepsilon(\varphi_2 - x) + u_2(\varphi_2 - x)\theta_\varepsilon(\varphi_2 - x) \\ - u_2(\varphi_3 - x)\theta_\varepsilon(\varphi_3 - x) + u_3(\varphi_3 - x)\theta_\varepsilon(\varphi_3 - x) - u_3(\varphi_4 - x)\theta_\varepsilon(\varphi_4 - x), \tag{37}$$

where $\varphi_i(0) = a_i$, $i = 1,2,3,4$, and $u_i(0) = u_i^0$, $i = 1,2,3$. Like in the previous section we will just write down the equations for the unknown functions $\varphi_i$ and $u_{i-1}$, $i = 1,2,3,4$, in the interval $[0, t_1^* + \delta)$. We will assume that the first interaction will happen between the points $a_2$ and $a_3$.

Accordingly, system of equations determining unknown functions in (37) in the interval $[0, t_1^* + \delta)$, $\delta < t_2^* - t_1^*$, where $t_2^*$ is time of the second interaction, is:

$$\begin{aligned}
&\varphi_{1t} = 2h\\
&\varphi_{2t} = 2(h + v_0^0) - 2u_2\Delta^{23}\varphi B_1^{23} + 2u_3\Delta^{23}\varphi B_1^{23}\\
&\varphi_{3t} = 2(h + v_0^0) - 2u_1\Delta^{23}\varphi + 2u_1\Delta^{23}\varphi B_2^{23} - 2u_2\Delta^{23}\varphi B_2^{23} = 0\\
&\varphi_{4t} = 2h + 2v_0^0 + 2v_1^0 + v_2^0\\
&u_{1t} - u_1^2 = 0\\
&u_{2t} - 2u_2^2 - 4u_1u_2B_1^{23} + 4u_1u_3B_1^{23} - 4u_2u_3B_1^{23} = 4u_2^2B_1^{23} = 0\\
&u_{3t} - 2u_3^2 = 0.
\end{aligned}$$

The solutions of the system are:

$$\begin{aligned}
\varphi_1(t) = 2ht + a_1, \quad \varphi_4(t) = 2(h + v_1^0 + v_2^0 + v_3^0)t + 2\int_0^t u_0(t')\,dt' + a_4\\
u_1(t) = \frac{u_1^0}{1 - 2u_1^0 t}, \quad u_3(t) = \frac{u_3^0}{1 - 2u_3^0 t}\\
\varphi_2(t) = \varphi_{20}(t) + \psi_0^{23}(t)\phi_2(t, \tau^{23})\\
\varphi_3(t) = \varphi_{30}(t) + \psi_0^{23}(t)\phi_3(t, \tau^{23})\\
u_2(t) = u_{20}(t) + \psi_0^{23}(t)\kappa(t, \tau),
\end{aligned} \tag{38}$$

where $\psi_0^{23} = \varphi_{30} - \varphi_{20}$, $\tau^{23} = \frac{\psi_0^{23}}{\varepsilon}$ and

$$\begin{aligned}
\varphi_{20}(t) = 2(h + v_0^0)t + a_2\\
\varphi_{20}(t) = 2(h + v_1^0 + v_0^0)t + a_3\\
u_{20}(t) = \frac{u_2^0}{1 - 2u_2^0 t},\\
\dot{\rho}^{23} = 1 - 2B_1^{23}
\end{aligned}$$

Functions $\rho^{23}(\tau^{23})$, $\phi_2(t, \tau^{23})$, $\phi_3(t, \tau^{23})$ and $\kappa(t, \tau^{23})$ have the same properties as in the previous section and we obtain them in the completely same manner.

### 4.4. Confluence of two shock waves connected with the straight line

We will analyze the transformation of the initial condition till the second interaction. The initial condition is:

$$\begin{aligned}
u(x,0) = h + u_1^0(a_1 - x)_+ - u_1^0(a_2 - x)_+ u_1^0(a_2 - x)_+ + u_2^0(a_2 - x)_+\\
+ u_2^0(a_3 - x)_+ - u_2^0(a_3 - x)_+ + u_3^0(a_3 - x)_+ - u_3^0(a_4 - x)_+,
\end{aligned} \tag{39}$$

where $a_1 > a_2 > a_3 > a_4$ are constants in $\mathbf{R}$, $u_0^0$, $u_1^0$, $u_2^0$ and $u_3^0$ are positive constants such that $(a_1 - a_2)u_1^0 = v_0^0$, $(a_2 - a_3)u_2^0 = v_1^0$ and $(a_3 - a_4)u_3^0 = v_2^0$. The weak asymptotic solution of problem (10), (39) we have in the form:

$$\begin{aligned} u_\varepsilon(x,t) = h + u_1(\varphi_1 - x)\theta_\varepsilon(\varphi_1 - x) - u_1(\varphi_2 - x)\theta_\varepsilon(\varphi_2 - x) + u_{01}\theta_\varepsilon(\varphi_2 - x) \\ + u_2(\varphi_2 - x)\theta_\varepsilon(\varphi_2 - x) + -u_2(\varphi_3 - x)\theta_\varepsilon(\varphi_3 - x) + u_{02}\theta_\varepsilon(\varphi_3 - x) \\ + u_3(\varphi_3 - x)\theta_\varepsilon(\varphi_3 - x) - u_3(\varphi_4 - x)\theta_\varepsilon(\varphi_4 - x), \quad (40) \end{aligned}$$

where $\varphi_i(0) = a_i$, $i = 1,2,3,4$, and $u_i(0) = u_i^0$, $i = 0,1,2,3$. Like in the previous section we will just write down the equations for the unknown functions $\varphi_i$ and $u_{i-1}$, $i = 1,2,3,4$, in (40) in the interval $[0, t_1^* + \delta)$. Here, we will assume that the first interaction will happen between the points $a_2$ and $a_3$.

System of equations determining unknown functions in (40) in the interval $[0, t_1^* + \delta)$, $\delta < t_2^* - t_1^*$, where $t_2^*$ is time of the second interaction, is:

$$\begin{aligned} &\varphi_{1t} = 2h \\ &\varphi_{2t} = 2h + 2v_0^0 + u_{01} + 2u_{02}B_1^{23} - 2u_2\Delta^{23}\varphi B_1^{23} + 2u_3\Delta^{23}\varphi B_1^{23} \\ &\varphi_{3t} = 2h + u_{02} - 2u_{01}B_2^{23} + 2v_0^0 - 2u_1\Delta^{23}\varphi B_1^{23} - 2u_2\Delta^{23}\varphi B_2^{23} \\ &\varphi_{4t} = 2h + 2v_0^0 + 2v_1^0 + v_2^0 + 2u_{01} + 2u_{02} \\ &u_{01t} - u_{01}(u_2 + u_1) + 2u_{01}u_2B_1^{23} - 2u_{01}u_3B_1^{23} = 0 \\ &u_{02t} - u_{02}(u_3 + u_2) + 2u_{02}u_1B_1^{23} - 2u_{02}u_2B_1^{23} = 0 \\ &u_{1t} - u_1^2 = 0 \\ &u_{2t} - 2u_2^2 - 4u_1u_2B_1^{23} + 4u_1u_3B_1^{23} - 4u_2u_3B_1^{23} = 4u_2^2B_1^{23} = 0 \\ &u_{3t} - 2u_3^2 = 0. \end{aligned}$$

The solutions of the system are:

$$\varphi_1(t) = 2ht + a_1,\ \varphi_4(t) = 2(h + v_1^0 + v_2^0 + v_3^0)t + 2\int_0^t (u_{01}(t') + u_{02}(t'))dt' + a_4$$

$$u_1(t) = \frac{u_1^0}{1 - 2u_1^0 t}, \quad u_3(t) = \frac{u_3^0}{1 - 2u_3^0 t}$$

$$u_0(t) = \exp\left(\int_0^t \left(u_2 + u_1 - 2u_2B_1^{23} + 2u_3B_1^{23}\right) dt' + u_0^0\right),$$

$$\varphi_2(t) = \varphi_{20}(t) + \psi_0^{23}(t)\phi_2(t, \tau^{23})$$

$$\varphi_3(t) = \varphi_{30}(t) + \psi_0^{23}(t)\phi_3(t, \tau^{23})$$

$$u_2(t) = u_{20}(t) + \psi_0^{23}(t)\kappa(t, \tau),$$

where $\psi_0^{23} = \varphi_{30} - \varphi_{20}$, $\tau^{23} = \frac{\psi_0^{23}}{\varepsilon}$ and

$$\varphi_{20}(t) = 2(h + v_0^0)t + \int_0^t 2u_0(t)dt + a_2$$

$$\varphi_{20}(t) = 2(h + v_1^0 + v_0^0)t + \int_0^t 2u_0(t)dt + a_3$$
$$u_{20}(t) = \frac{u_2^0}{1 - 2u_2^0 t},$$
$$\dot\rho^{23}\psi_{0t}^{23} = (u_0 - 2u_2\Delta^{23})(1 - 2B_1^{23}) - 2u_1\Delta^{23}\varphi B_1^{23} - 2u_3\Delta^{23}\varphi B_1^{23}.$$

Functions $\rho^{23}(\tau^{23})$, $\phi_2(t,\tau^{23})$, $\phi_3(t,\tau^{23})$ and $\kappa(t,\tau)$ have the same properties and we obtain them in the same way as before.

## 5. Interaction of weak discontinuities and generation of shock waves; n-point case.

In this section we will solve equation (10) with the polygon with $n$ knots as an initial condition. First, we will write down the solution of the problem in the interval $[0, t_1^* + \delta_1)$, where, as usual, $t_i^*$ is the time of the $i$-th interaction and $\delta_i < t_{i+1}^* - t_i^*$. Using results from Section 4 we will then recursively write down the solutions in the intervals $[t_i^*, t_{i+1}^* + \delta_{i+1})$, $i = 1, \dots, n-2$, and $[t_{n-1}^*, \infty)$.

In the beginning we will introduce some auxiliary notations.

$$u_{l,0}(t) = \frac{u_l^0}{1 - 2u_l^0 t}, \quad l = 1, 2, \dots, n,$$
$$v_0^0 = 0, \quad v_k^0 = (a_{k-1} - a_k)u_{k-1}^0, \quad k = 2, \dots, n+1,$$
$$\varphi_{1,0}(t) = a_1, \quad \varphi_{l-1,0}(t) = 2\sum_{k=1}^{l-1} v_k^0 t + a_l, \quad l = 2, \dots, n+1,$$
$$\psi_{l-1,0} = \varphi_{l,0} - \varphi_{l-1,0}, \quad \tau^{l-1,l} = \frac{\psi_{l,0}}{\varepsilon}, \quad l = 2, \dots, n+1,$$
$$\rho_l^{[i]}(\tau^{l-1,l}) = \frac{\varphi_l^{[i]} - \varphi_{l-1}^{[i]}}{\varepsilon}, \quad l = 2, \dots, n+1$$
$$B_{[i]k}^{l-1,l} = B_k(\rho_l^{[i]}), \quad k = 1, 2, \quad l = 2, \dots, n+1, \quad i = 1, \dots, n,$$
$$u_s^{[i]} \equiv 0, \quad s \in \mathbf{Z}_0{}^-, \quad i = 1, \dots, n+1.$$

Let the initial condition for Hopf equation be:

$$u|_{t=0} = u_1^0(a_1 - x)_+ + (u_2^0 - u_1^0)(a_2 - x)_+ + (u_3^0 - u_2^0)(a_3 - x)_+ + \cdots + (u_{n-1}^0 - u_{n-2}^0)(a_{n-1} - x)_+ + (u_n^0 - u_{n-1}^0)(a_n - x)_+ - u_n^0(a_{n+1} - x)_+ \quad (41)$$

We know that after certain time point $a_i$ will reach point $a_j$ or oppositely. Let us suppose that the moments of interactions are $t_1^* < t_2^* < \cdots < t_n^*$ and that in the moment $t = t_i^*$ points $a_{l(i-1)}$ and $a_{l(i)}$ will interact. By an induction argument, we can write the uniform formula which describes propagation of initial wave. Like before, we will analyze behavior of solution in the intervals $[0, t_1^* + \delta_1)$, $[t_1^*, t_2^* + \delta_2)$, $\dots$, $[t_{n-2}^*, \infty)$ where $t_0^* = 0$ and $\delta_i < t_{i+1}^* - t_i^*$, $i = 1, \dots, n-2$. The weak asymptotic solution in the interval $[t_i^*, t_{i+1}^* + \delta_i)$ we will denote with $u_\varepsilon^{[i]}$.

As earlier, we assume that the approximate solution in the interval $[0, t_1^* + \delta_1)$ is in the form

$$
\begin{aligned}
u_\varepsilon^{[1]}(x,t) &= u_1^{[1]}(\varphi_1^{[1]} - x)\theta_{1,\varepsilon}(\varphi_1^{[1]} - x) - u_1^{[1]}(\varphi_2^{[1]} - x)\theta_{2\varepsilon}(\varphi_2^{[1]} - x) \\
&+ \cdots + u_{l(1)-1}^{[1]}(\varphi_{l(1)-1}^{[1]} - x)\theta_{l(1)-1,\varepsilon}(\varphi_{l(1)-1}^{[1]} - x) - u_{l(1)-1}^{[1]}(\varphi_{l(1)}^{[1]} - x)\theta_{l(1),\varepsilon}(\varphi_{l(1)}^{[1]} - x) \\
&+ u_{l(1)}^{[1]}(\varphi_{l(1)-1}^{[1]} - x)\theta_{l(1),\varepsilon}(\varphi_{l(1)}^{[1]} - x) - \cdots + u_n^{[1]}(\varphi_n^{[1]} - x)\theta_{n,\varepsilon}(\varphi_n^{[1]} - x) \\
&- u_n^{[1]}(\varphi_{n+1} - x)\theta_{n+1,\varepsilon}(\varphi_{n+1} - x). \qquad (42)
\end{aligned}
$$

Since in the first step only the interaction of weak discontinuities is possible, by the analogy with Section 4.3 we have:

$$
\begin{aligned}
\varphi_s^{[1]}(t) &= \varphi_{s,0}(t) \quad s \neq l(1) \text{ and } s \neq l(0), \quad s = 1, \dots, n+1, \\
\varphi_{l(0)}(t,\varepsilon) &= \varphi_{l(0),0}(t) + \psi_{l(0),0}(t)\phi_1^{[1]}(\tau^{l(0),l(1)}, t) \\
\varphi_{l(1)}(t,\varepsilon) &= \varphi_{l(1),0}(t) + \psi_{l(0),0}(t)\phi_2^{[1]}(\tau^{l(0),l(1)}, t) \\
u_s^{[1]}(t) &= u_{s,0}(t), \quad s \neq l(1) - 1, \quad s = 1, \dots, n, \\
u_{l(0)}(t,\varepsilon) &= u_{l(0),0}(t) + \kappa^{[1]}(\tau^{l(0),l(1)}, t),
\end{aligned}
$$

where functions $\phi_k^{[1]}(\tau^{l(0),l(1)}, t)$, $k = 1, 2$, and $\kappa^{[1]}(\tau^{l(0),l(1)}, t)$ for every fixed $t$ satisfy ($\tau^{23}$ is independent variable here),

$$
\begin{aligned}
\left(\varphi_{l(0)}\right)_t &= \sum_{k=0}^{l(0)} v_k^0 + 2u_0^{[1]} B_{[1]1}^{l(0),l(1)} \\
\left(\varphi_{l(1)}\right)_t &= \sum_{k=0}^{l(0)} v_k^0 + 2u_0^{[1]} B_{[1]1}^{l(0),l(1)} + 2u_0^{[1]} B_{[1]2}^{l(0),l(1)}, \\
u_0^{[1]} &= u_{l(0)}^{[1]} \left(\varphi_{l(0)}^{[1]} - \varphi_{l(1)}^{[1]}\right) \\
u_{0t}^{[1]} &= -2u_{l(0)-1}^{[1]} u_0^{[1]} B_{[1]1}^{23} + 2u_{l(1)}^{[1]} u_0^{[1]} B_{[1]1}^{23}.
\end{aligned}
$$

with the initial conditions $\kappa^{[1]}(\tau, t) = \phi_j^{[1]}(\tau) = \mathcal{O}(\tau^{-N})$, $j = l(0)$ and $j = l(1)$, $\tau \to -\infty$, $N \in \mathbf{N}$ arbitrary.

Now, we will analyze the Hopf equation in the intervals $[t_i^*, t_{i+1}^* + \delta_{i+1})$, $i = 2, \dots, n-2$, and $[t_{n-1}^*, +\infty)$. By an induction argument, from previous sections we conclude that after $i$-th interaction we have:

$$
\begin{aligned}
\varphi_1 &= \cdots = \varphi_{j(1)}, \\
\varphi_{j(1)+1} &= \cdots = \varphi_{j(2)} \\
&\cdots \\
\varphi_{j(s-1)+1} &= \cdots = \varphi_{j(s)},
\end{aligned}
$$

where $j(s) = n + 1$. Let us assume that in the moment $t = t_{i+1}^*$ the points $a_{j(k)}$ and $a_{j(k+1)}$ interact. Then we have the following possibilities:

**a)** $\varphi_{j(k)}(t) \neq \varphi_m(t)$ for every $m \in \{1,2,\dots,n+1\}\backslash\{j(k)\}$ and $\varphi_{j(k+1)}(t) \neq \varphi_m(t)$ for every $m \in \{1,2,\dots,n+1\}\backslash\{j(k+1)\}$ in the interval $[t_i^*, t_i^* + \delta_i)$. In this case we have the interaction of weak discontinuities (notice that therefore $j(k+1) = j(k)+1$). The solution will have the form:

1. $x \leq \varphi_{j(k+2)}$ and $x \geq \varphi_{j(k-1)}$. In this case we take $u_\varepsilon^{[i+1]}(x,t) = u_\varepsilon^{[i]}(x,t)$.
2. $\varphi_{j(k+2)} \geq x \geq \varphi_{j(k-1)}$. In this interval the solution looks like:

$$\begin{aligned}
&u_\varepsilon^{[i+1]}(x,t) = \\
&\sum_{p=1}^{j(k-1)} v_p^0 + u_{j(k-1)}^{[i+1]}(\varphi_{j(k-1)}^{[i+1]} - x)\theta_\varepsilon(\varphi_{j(k-1)}^{[i+1]} - x) \\
&- u_{j(k-1)}^{[i+1]}(\varphi_{j(k)}^{[i+1]} - x)\theta_\varepsilon(\varphi_{j(k)}^{[i+1]} - x) + u_{j(k)}^{[i+1]}(\varphi_{j(k)}^{[i+1]} - x)\theta_\varepsilon(\varphi_{j(k)}^{[i+1]} - x) \\
&- u_{j(k)}^{[i+1]}(\varphi_{j(k+1)}^{[i+1]} - x)\theta_\varepsilon(\varphi_{j(k+1)}^{[i+1]} - x) + u_{j(k+1)}^{[i+1]}(\varphi_{j(k+1)}^{[i+1]} - x)\theta_\varepsilon(\varphi_{j(k+1)}^{[i+1]} - x) \\
&- u_{j(k+1)}^{[i+1]}(\varphi_{j(k+2)}^{[i+1]} - x)\theta_\varepsilon(\varphi_{j(k+2)}^{[i+1]} - x).
\end{aligned}$$

This is the situation as in Section 4.3. The difference is in the fact that here we do not begin from the moment $t = 0$ but from $t = t_i^*$. Therefore, for initial condition here we will have:

$$\begin{aligned}
&\varphi_{j(k-1)}^{[i+1]}(t_i^*) = \varphi_{j(k-1)}^{[i]}(t_i^*) \qquad \varphi_{j(k)}^{[i+1]}(t_i^*) = \varphi_{j(k)}^{[i]}(t_i^*) \\
&\varphi_{j(k+1)}^{[i+1]}(t_i^*) = \varphi_{j(k+1)}^{[i]}(t_i^*) \qquad \varphi_{j(k+2)}^{[i+1]}(t_i^*) = \varphi_{j(k+2)}^{[i]}(t_i^*) \\
&u_{j(k-1)}^{[i+1]}(t_i^*) = u_{j(k-1)}^{[i]}(t_i^*) \qquad u_{j(k)}^{[i+1]}(t_i^*) = u_{j(k)}^{[i]}(t_i^*) \\
&u_{j(k+1)}^{[i+1]}(t_i^*) = u_{j(k+1)}^{[i]}(t_i^*).
\end{aligned}$$

The unknown functions are given by set of formulas (38) with an obvious difference in indexing.

**b)** $\varphi_{j(k)}(t) \neq \varphi_m(t)$ for every $m \in \{1,2,\dots,n+1\}\backslash\{j(k)\}$ and $\varphi_{j(k+1)}(t) = \varphi_m(t)$ for some $m \in \{1,2,\dots,n+1\}\backslash\{j(k+1)\}$ in the interval $(t_i^*, t_i^* + \delta)_i)$.

In this case we have interaction of weak discontinuity and the shock wave whose disposure is analogous to the one from Section 4.1. The solution we have in the form:

1. $x \leq \varphi_{j(k+2)}$ and $x \geq \varphi_{j(k-1)}$. In this case we take $u_\varepsilon^{[i+1]}(x,t) = u_\varepsilon^{[i]}(x,t)$.
2. $\varphi_{j(k+2)} \geq x \geq \varphi_{j(k-1)}$. In this interval the solution looks like:

$$\begin{aligned}
&u_\varepsilon^{[i+1]}(x,t) = \\
&\sum_{p=1}^{j(k-1)-1} v_p^0 + u_{j(k-1)}^{[i+1]}(\varphi_{j(k-1)}^{[i+1]} - x)\theta_\varepsilon(\varphi_{j(k-1)}^{[i+1]} - x) \\
&- u_{j(k-1)}^{[i+1]}(\varphi_{j(k)}^{[i+1]} - x)\theta_\varepsilon(\varphi_{j(k)}^{[i+1]} - x) + u_{0,j(k)}^{[i+1]}\theta_\varepsilon(\varphi_{j(k)}^{[i+1]} - x) \\
&+ u_{j(k)}^{[i+1]}(\varphi_{j(k)}^{[i+1]} - x)\theta_\varepsilon(\varphi_{j(k)}^{[i+1]} - x) - u_{j(k)}^{[i+1]}(\varphi_{j(k+1)}^{[i+1]} - x)\theta_\varepsilon(\varphi_{j(k+1)}^{[i+1]} - x) \\
&+ u_{j(k+1)}^{[i+1]}(\varphi_{j(k+1)}^{[i+1]} - x)\theta_\varepsilon(\varphi_{j(k+1)}^{[i+1]} - x) - u_{j(k+1)}^{[i+1]}(\varphi_{j(k+2)}^{[i+1]} - x)\theta_\varepsilon(\varphi_{j(k+2)}^{[i+1]} - x).
\end{aligned}$$

The difference from the situation from Section 4.1 is in the fact that for initial condition here we will have:

$$\begin{array}{ll}\varphi^{[i+1]}_{j(k-1)}(t^*_i)=\varphi^{[i]}_{j(k-1)}(t^*_i) & \varphi^{[i+1]}_{j(k)}(t^*_i)=\varphi^{[i]}_{j(k)}(t^*_i)\\ \varphi^{[i+1]}_{j(k+1)}(t^*_i)=\varphi^{[i]}_{j(k+1)}(t^*_i) & \varphi^{[i+1]}_{j(k+2)}(t^*_i)=\varphi^{[i]}_{j(k+2)}(t^*_i)\\ u^{[i+1]}_{j(k-1)}(t^*_i)=u^{[i]}_{j(k-1)}(t^*_i) & u^{[i+1]}_{j(k)}(t^*_i)=u^{[i]}_{j(k)}(t^*_i)\\ u^{[i+1]}_{j(k+1)}(t^*_i)=u^{[i]}_{j(k+1)}(t^*_i) & u^{[i+1]}_{0,j(k)}(t^*_i)=\sum\limits_{p=j(k-1)+1}^{j(k)-1}v^0_p.\end{array}$$

The unknown functions are given by (35) with an obvious difference in indexing.

**c)** $\varphi_{j(k)}(t)=\varphi_m(t)$ for some $m\in\{1,2,\dots,n+1\}\backslash\{j(k)\}$ and $\varphi_{j(k+1)}(t)\neq\varphi_m(t)$ for every $m\in\{1,2,\dots,n+1\}\backslash\{j(k+1)\}$ in the interval $(t^*_i,t^*_i+\delta_i)$. In this case we have interaction of weak discontinuity and the shock wave whose disposure is analogous to the one from Section 4.2. The solution we have in the form:

1. $x\leq\varphi_{j(k+2)}$ and $x\geq\varphi_{j(k-1)}$. In this case we take $u^{[i+1]}_\varepsilon(x,t)=u^{[i]}_\varepsilon(x,t)$.
2. $\varphi_{j(k+2)}\geq x\geq\varphi_{j(k-1)}$. In this interval the solution will look like:

$$\begin{aligned}u^{[i+1]}_\varepsilon(x,t)=&\sum_{p=1}^{j(k-1)-1}v^0_p\\ &+u^{[i+1]}_{j(k-1)}(\varphi^{[i+1]}_{j(k-1)}-x)\theta_\varepsilon(\varphi^{[i+1]}_{j(k-1)}-x)-u^{[i+1]}_{j(k-1)}(\varphi^{[i+1]}_{j(k)}-x)\theta_\varepsilon(\varphi^{[i+1]}_{j(k)}-x)\\ &+u^{[i+1]}_{j(k)}(\varphi^{[i+1]}_{j(k)}-x)\theta_\varepsilon(\varphi^{[i+1]}_{j(k)}-x)-u^{[i+1]}_{j(k)}(\varphi^{[i+1]}_{j(k+1)}-x)\theta_\varepsilon(\varphi^{[i+1]}_{j(k+1)}-x)\\ &+u^{[i+1]}_{0,j(k+1)}\theta_\varepsilon(\varphi^{[i+1]}_{j(k+1)}-x)+u^{[i+1]}_{j(k+1)}(\varphi^{[i+1]}_{j(k+1)}-x)\theta_\varepsilon(\varphi^{[i+1]}_{j(k+1)}-x)\\ &-u^{[i+1]}_{j(k+1)}(\varphi^{[i+1]}_{j(k+2)}-x)\theta_\varepsilon(\varphi^{[i+1]}_{j(k+2)}-x).\end{aligned}$$

The difference from the situation from Section 4.2 is in the fact that for initial condition here we will have:

$$\begin{array}{ll}\varphi^{[i+1]}_{j(k-1)}(t^*_i)=\varphi^{[i]}_{j(k-1)}(t^*_i) & \varphi^{[i+1]}_{j(k)}(t^*_i)=\varphi^{[i]}_{j(k)}(t^*_i)\\ \varphi^{[i+1]}_{j(k+1)}(t^*_i)=\varphi^{[i]}_{j(k+1)}(t^*_i) & \varphi^{[i+1]}_{j(k+2)}(t^*_i)=\varphi^{[i]}_{j(k+2)}(t^*_i)\\ u^{[i+1]}_{j(k-1)}(t^*_i)=u^{[i]}_{j(k-1)}(t^*_i) & u^{[i+1]}_{j(k)}(t^*_i)=u^{[i]}_{j(k)}(t^*_i)\\ u^{[i+1]}_{j(k+1)}(t^*_i)=u^{[i]}_{j(k+1)}(t^*_i) & u^{[i+1]}_{0,j(k)}(t^*_i)=\sum\limits_{p=j(k)+1}^{j(k+1)-1}v^0_p.\end{array}$$

The unknown functions are given by (35) with an obvious difference in indexing.

**d)** $\varphi_{j(k)}(t)=\varphi_m(t)$ for some $m\in\{1,2,\dots,n+1\}\backslash\{j(k)\}$ and $\varphi_{j(k+1)}(t)=\varphi_m(t)$ for some $m\in\{1,2,\dots,n+1\}\backslash\{j(k+1)\}$ in the interval $[t^*_i,t^*_i+\delta)i-1)$. In this case we have the situation analogous to one from Section 4.4. The solution we have in the form:

1. $x\leq\varphi_{j(k+2)}$ and $x\geq\varphi_{j(k-1)}$. In this case we take $u^{[i+1]}_\varepsilon(x,t)=u^{[i]}_\varepsilon(x,t)$.

2. $\varphi_{j(k+2)} \geq x \geq \varphi_{j(k-1)}$. In this interval the solution will look like:

$$
\begin{aligned}
u_\varepsilon^{[i+1]}(x,t) = &\sum_{p=1}^{j(k-1)-1} v_p^0 \\
&+ u_{j(k-1)}^{[i+1]}(\varphi_{j(k-1)}^{[i+1]} - x)\theta_\varepsilon(\varphi_{j(k-1)}^{[i+1]} - x) - u_{j(k-1)}^{[i+1]}(\varphi_{j(k)}^{[i+1]} - x)\theta_\varepsilon(\varphi_{j(k)}^{[i+1]} - x) \\
&+ u_{0,j(k)}^{[i+1]}\theta_\varepsilon(\varphi_{j(k)}^{[i+1]} - x) + u_{j(k)}^{[i+1]}(\varphi_{j(k)}^{[i+1]} - x)\theta_\varepsilon(\varphi_{j(k)}^{[i+1]} - x) \\
&- u_{j(k)}^{[i+1]}(\varphi_{j(k+1)}^{[i+1]} - x)\theta_\varepsilon(\varphi_{j(k+1)}^{[i+1]} - x) + u_{0,j(k+1)}^{[i+1]}\theta_\varepsilon(\varphi_{j(k+1)}^{[i+1]} - x) \\
&+ u_{j(k+1)}^{[i+1]}(\varphi_{j(k+1)}^{[i+1]} - x)\theta_\varepsilon(\varphi_{j(k+1)}^{[i+1]} - x) \\
&- u_{j(k+1)}^{[i+1]}(\varphi_{j(k+2)}^{[i+1]} - x)\theta_\varepsilon(\varphi_{j(k+2)}^{[i+1]} - x).
\end{aligned}
$$

The difference from the situation from Section 4.4 is in the fact that for initial condition here we will have:

$$
\begin{aligned}
&\varphi_{j(k-1)}^{[i+1]}(t_i^*) = \varphi_{j(k-1)}^{[i]}(t_i^*) \qquad && \varphi_{j(k)}^{[i+1]}(t_i^*) = \varphi_{j(k)}^{[i]}(t_i^*) \\
&\varphi_{j(k+1)}^{[i+1]}(t_i^*) = \varphi_{j(k+1)}^{[i]}(t_i^*) \qquad && \varphi_{j(k+2)}^{[i+1]}(t_i^*) = \varphi_{j(k+2)}^{[i]}(t_i^*) \\
&u_{j(k-1)}^{[i+1]}(t_i^*) = u_{j(k-1)}^{[i]}(t_i^*) \qquad && u_{j(k)}^{[i+1]}(t_i^*) = u_{j(k)}^{[i]}(t_i^*) \\
&u_{j(k+1)}^{[i+1]}(t_i^*) = u_{j(k+1)}^{[i]}(t_i^*) \qquad && u_{0,j(k)}^{[i+1]}(t_i^*) = \sum_{p=j(k)+1}^{j(k+1)-1} v_p^0 \\
&u_{0,j(k+1)}^{[i+1]}(t_i^*) = \sum_{p=j(k+1)+1}^{j(k+2)-1} v_p^0
\end{aligned}
$$

The unknown functions are given by (40) with an obvious difference in indexing.

Since in the intervals $(t_i^*, t_i + \delta_i)$ we have $u_\varepsilon^{[i]} = u_\varepsilon^{[i+1]}$, the uniform solution of equation (10) with the initial condition (41) we have in the form:

$$
u_\varepsilon = \sum_{i=1}^{n} \eta_i u_\varepsilon^{[i]} \tag{43}
$$

where $\{\eta_1, \dots, \eta_n\}$ is the partition of unity of the nonnegative part of real line such that

$$
\begin{aligned}
\eta_1(t) &= 1, \quad \text{for } t \in [0, t_1^*), \\
\eta_1(t) &= 0, \quad \text{for } t \in [t_1^* + \delta_1, +\infty), \\
\eta_i(t) &= 1, \quad \text{for } t \in [t_{i-1}^* + \delta_{i-1}, t_i^*), \\
\eta_i(t) &= 0, \quad \text{for } t \notin [t_{i-1}^*, t_i^* + \delta_i), \quad i = 2, \dots, n-1, \\
\eta_n(t) &= 1, \quad \text{for } t \in [t_{n-1}^* + \delta_{n-1}, \infty), \\
\eta_n(t) &= 0, \quad \text{for } t \in [0, t_{n-1}^*).
\end{aligned}
$$

## 6. Arbitrary decreasing initial condition

In this section we will solve Hopf equation (10) with initial condition (2) where $u_0(x)$, $x \in \mathbf{R}$, is an arbitrary decreasing Lipschitz continuous function which takes values in some compact subset of real line. The process of solving will be as follows.

First, we approximate given initial data with a sequence of polygons and then for each of these polygons as initial data we write down the solution of the Hopf equation (we use the results from the previous sections). In choosing the snags of the polygon we have to take care that any three consecutive points must not interact simultaneously (the proof that such a choice is possible one can find in [6] in the case of more general equation $u_t + (f(u))_x = 0$, $f'' > 0$). We will obtain a sequence of solutions and we will prove that it converges in the weak sense to the solution of Hopf equation with the original initial condition. More precisely, we will prove the following theorem:

THEOREM 6.1. *For problem (10), (2) there exists a function* $\hat{u}_\varepsilon = \hat{u}(x,t,\varepsilon) \in C^\infty(\mathbf{R} \times \mathbf{R}^+ \times (0,1))$ *given by (45) such that we have*

$$\int L\hat{u}_\varepsilon \cdot \phi dx = \mathcal{O}(\varepsilon^{1/3}), \quad \phi \in \mathcal{D}(\mathbf{R}),$$

$$\|\hat{u}_\varepsilon(x,0) - u_0(x)\|_{L_1(X)} = \mathcal{O}(\varepsilon^{1-\mu}), \quad x \in \mathbf{R},$$

*for every compact* $X \Subset \mathbf{R}$*, some* $\mu \in (0,1)$ *and every* $t > 0$.

*Furthermore, the weak entropy admissible solution* $\hat{u}(x,t)$[1] *of problem (10), (2) satisfies*

$$\|\hat{u}_\varepsilon(\cdot,t) - \hat{u}(\cdot,t)\|_{L^1(|x|<R)} = \mathcal{O}(\varepsilon^{1-\mu}).$$

*Proof.* Denote by $\hat{u}(x,t)$ weak entropy admissible solution of the given problem. Divide the interval $(-\frac{1}{\varepsilon^{\mu_1}}, \frac{1}{\varepsilon^{\mu_1}})$ by points $a_i$, $i \in \mathbf{N}$, to subintervals of length $\Delta a_i = a_{i-1} - a_i = M\varepsilon^{1-\mu}$, $0 < \mu < 1$, $M \in (1,2)$. Then we approximate initial condition (2) by the polygon $u_{0,\varepsilon}(x)$ with the edges $u_0(a_i)$, $i = 1, 2, \dots, [\frac{2}{\varepsilon^{1-\mu-\mu_1}}] + 1$, $a_i < a_j$ for $i < j$. For $x > \frac{1}{\varepsilon^{\mu_1}}$ we define $u_{0,\varepsilon}(x) = u_{0,\varepsilon}(\frac{1}{\varepsilon^{\mu_1}})$ and for $x < -\frac{1}{\varepsilon^{\mu_1}}$ we define $u_{0,\varepsilon}(x) = u_{0,\varepsilon}(-\frac{1}{\varepsilon^{\mu_1}})$. We select $\mu_1$ and $\mu$ from the interval $(0,1)$ such that $1+\mu_1-\mu < 1/3$. We select points $a_i$, $i \in \mathbf{N}$, in a such way that no three consecutive points interact simultaneously. In other words, we do not have $\varphi_{i-1}(t) = \varphi_i(t) = \varphi_{i+1}(t)$ for any $t < t_i^*$, $i = 2, \dots, n-1$, where $\varphi_i$ describes moving of the point $a_i$ while $t_i^*$ denotes the time of interaction of the points $a_i$ and $a_{i+1}$ (we repeat that the proof that this can be done is given in [6]). According to the previous, approximate solution of the Hopf equation with the initial condition

$$u|_{t=0} = u_{0\varepsilon}(x), \tag{44}$$

will be of the form

$$\hat{u}_\varepsilon(x,t) = \sum_{-\infty}^{+\infty} \eta_i(t) u_{i\varepsilon}(x,t), \tag{45}$$

[1] For the proof that there exists a weak entropy admissible solution of the considered problem see [1].

where $\{\eta_i : i \in \mathbf{N}\}$ is partition of unity of real line such that $\text{supp}\, \eta_i \cap \text{supp}\, \eta_j = \emptyset$, $|i-j| > 1$. We will prove that $\hat{u}_\varepsilon(x,t)$ is the weak asymptotic solution of problem (10), (2) with the accuracy $\varepsilon^{1-\mu}$.

Accordingly, we have to check:

a) $\int_{-\infty}^{+\infty} \left[(\hat{u}_\varepsilon)_t + (\hat{u}_\varepsilon^2)_x\right] \phi(x) dx = \mathcal{O}(\varepsilon^{1/3})$ for every $\phi \in C_0^\infty(\mathbf{R}$,

b) $\|\hat{u}_\varepsilon(\cdot, 0) - u_0(\cdot)\|_{L^1(\mathbf{R})} = \mathcal{O}(\varepsilon^{1/3})$,

To prove a) we substitute solution (45) in equation (2). We obtain $\mathcal{O}(\frac{1}{\varepsilon^{2(1+\mu_1-\mu)}})$ products of the form $\theta_\varepsilon(x-\varphi_i)\theta_\varepsilon(x-\varphi_j)$, $i \neq j$. Using the results from Section 2 we see that each of these products generates remainder equal to $\mathcal{O}(\varepsilon)$. Accordingly, summing all remainders we get that the total remainder is $\mathcal{O}(\varepsilon^{1-2(1+\mu_1-\mu)}) = \mathcal{O}(\varepsilon^{1/3})$. So, expression given in a) is of order $\varepsilon^{1/3}$ what we wanted to prove.

Concerning initial condition (2) on a bounded subset $X$ of $\mathbf{R}$ we have,

$$\int_X (u_{0\varepsilon}(\cdot) - u_0(\cdot)) dx = \sup_{x \in X} |u_{0\varepsilon}(x) - u_0(x)| \cdot \text{diam}\, X \cdot \varepsilon^{1-\mu} = \mathcal{O}(\varepsilon^{1-\mu}). \qquad (46)$$

This proves b) and finishes the proof that $\hat{u}_\varepsilon$ is a weak asymptotic solution of (10), (2).

At the end, we will prove that our weak asymptotic solution is in the sense of $L^1(\mathbf{R})$ convergence "close" to the admissible weak solution of (10), (2). For that reason we will prove that the weak asymptotic solution constructed in the first part of the proof weakly converges to a weak solution of (10), (2). We have for every $\psi \in C_0^\infty([0,T) \times \mathbf{R})$

$$\int_{\mathbf{R}} [u_t + (u^2)_x] \psi(x,t) dx = \mathcal{O}(\varepsilon), \quad t \text{ is fixed.}$$

If we apply $\int_0^T dt$ to the last relation we see that

$$\int_0^T \int_{\mathbf{R}} [u_t + (u^2)_x] \psi(x,t) dx dt = \mathcal{O}(\varepsilon).$$

If we apply the same procedure to the initial condition we conclude that $\text{w-}\lim_{\varepsilon \to 0} u_\varepsilon(x,t) = u(x,t)$, $x \in \mathbf{R}$, $t \in \mathbf{R}^+$ is the weak solution of problem (10), (2). If we prove that the function $u$ is piecewise continuous, from its construction we see that it satisfies Oleinik admissibility condition (see [7]) since the initial condition is decreasing. From [1] one can see that the Oleinik admissibility condition and entropy admissibility condition:

$$\int_0^T \int_{\mathbf{R}} [\partial_t \psi \eta(u) + \partial_x \psi q(u)]\, dx dt + \int_{\mathbf{R}} \psi(x,0) \eta(u_0(x))\, dx \geq 0, \qquad (47)$$

where $q(u) = \int \eta'(u)$ and $\eta \in C^1(\mathbf{R})$ is an arbitrary convex function. Since $\text{w-}\lim_{\varepsilon \to 0} u_\varepsilon(x,t) = u(x,t)$ it is not difficult to conclude that we have

$$\int_0^T \int_{\mathbf{R}} [\partial_t \psi \eta(u_\varepsilon) + \partial_x \psi q(u_\varepsilon)]\, dx\, dt + \int_{\mathbf{R}} \psi(x,0) \eta(u_{\varepsilon 0}(x))\, dx \geq \varepsilon^{1-\mu} \mathcal{O}(1), \quad (48)$$

where $q(u) = \int 2u\eta'(u)du$, are equivalent in the case of piecewise continuous functions.

If we revisit Theorem 6.2.2 in [1] using (48) instead of Definition 6.2.1 of the same book, we obtain

$$\|\check{u}_\varepsilon(\cdot,t)-\hat{u}_\varepsilon(\cdot,t)\|_{L^1(|x|<R)} \le \|\check{u}_\varepsilon(\cdot,0)-\hat{u}_\varepsilon(\cdot,0)\|_{L^1(|x|<R+st)} + \mathcal{O}(\varepsilon^{1-\mu}) = \mathcal{O}(\varepsilon^{1-\mu}),$$

since in this case $\check{u}_\varepsilon(x,0) = \hat{u}_\varepsilon(x,0)$. This implies

$$\|\hat{u}_\varepsilon(\cdot,t) - \hat{u}(\cdot,t)\|_{L^1(|x|<R)} \le \|\check{u}_\varepsilon(\cdot,t) - \hat{u}(\cdot,t)\|_{L^1(|x|<R)} + \|\check{u}_\varepsilon(\cdot,t) - \hat{u}_\varepsilon(\cdot,t)\|_{L^1(|x|<R)} = \mathcal{O}(\varepsilon^{1-\mu}),$$

which implies what we wanted.

Remark 6.2. For additional explanation see [1], Chapter 6.

So, it remains to prove that the function $u$ is piecewise continuous. According to definition we know that two points interacted if they were distanced for the quantity $\mathcal{O}(\varepsilon)$. In every finite interval for fixed $\varepsilon \in (0,1)$ and $t \in \mathbf{R}^+$ we have finite number of points, in the interactions of which we are interested (the points of the partition of the $x$-axis). Therefore, number of formed "shocks" is finite for every fixed $t \in \mathbf{R}^+$ (we put here the quotation marks since in the case of the functions $u_\varepsilon$, $\varepsilon \in (0,1)$ we cannot have proper shocks due to smoothness of appropriate functions; here we have only very steep parts of graphs of the functions $u_\varepsilon$, $\varepsilon \in (0,1)$ from which, when we let $\varepsilon \to 0$, become shocks). When we let $\varepsilon \to 0$ the parts between the shocks will become continuous since those points originates from the initial condition. More precisely, since they do not form any shock, they reached the appropriate state by moving along the characteristics which makes them the points of continuity. So, it is obvious that $u$ must be piecewise continuous. ■

REFERENCES

[**1**] C. M. Dafernos *Hyperbolic Conservation Laws in Continuum Physics*, Springer, 2000.

[**2**] V. G. Danilov, *A new definition of weak solutions of semilinear equations with a small parameter*, Uspekhi Mat. Nauk, **51**:5 (1997), 184 (in Russian). English transl. in Russian Math. Surveys.

[**3**] V. G. Danilov, *Generalized solution describing singularity interaction*, Int. Journal of Mathematics and Math. Sciences, **29**, 22 (2002), 481–494.

[**4**] G. B. Whitham, *Linear and Nonlinear Waves*, John Wiley, New York, 1974.

[**5**] A. M. Il'in, *Compound expansions*, Matem. Sb. (N.S.) **99 (141)**, 4 (1976), 514–537.

[**6**] D. Mitrović *Uniform Description of Singularity Interaction in Scalar Conservation Laws*, Faculty of Mathematics and Natural Sciences in Podgorica, PhD thesis, 2004.

[**7**] A. Y. Goritskii, S. N. Kruzhkov, G. A. Ceckin *Partial Differential Equation of the First Order*, Russian edition, Moscow State University, Faculty of Mathematics and Mechanics, Moscow, 1999.

[**8**] A. Bressan, *Hyperbolic System of Conservation Laws*, Preprints S.I.S.S.A., Trieste, Italy.

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