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A NEW VARIANT OF AN ITERATIVE METHOD FOR SOLVING THE COMPLETE PROBLEM OF EIGENVALUES OF MATRICES

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Abstract. In a complete problem of eigenvalues of matrices of the n-th order the essential role is played by the development of the characteristic determinant

$$D(\lambda) = \det(A - \lambda E)$$

or some other determinant which is essentially identical to this one. There is a series of different methods by which we come to the explicit form of this polynomial.

In this paper iterative formulas are derived for finding of all eigenvalues of a real matrix without developing the characteristic polynomial. The method is based on the Newton's method for solving systems of nonlinear equations.

In [1] iterative formulas are derived for finding of all eigenvalues of a real matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \tag{1}$$

without developing the characteristic determinant

$$D(\lambda) = \det(A - \lambda E) = \begin{bmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{bmatrix} \tag{2}$$

into the characteristic polynomial

$$D(\lambda) \equiv (-1)^n [\lambda^n - p_1 \lambda^{n-1} + p_2 \lambda^{n-2} - \dots + (-1)^n p_n]. \tag{3}$$

The following cases concerning matrix (1) were considered:

a) when it has real and distinct eigenvalues,

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b) when it has real eigenvalues some of which are equal,

c) when it has a pair of conjugate-complex eigenvalues.

Newton's method was used for solving of systems of nonlinear equations. We shall consider a new variant of this procedure.

Suppose the eigenvalues λ_j, $j = \overline{1,n}$, of matrix (1) be real and of distinct absolute values. Put

$$s_i = S_p A^i = \lambda_1^i + \lambda_2^i + \cdots + \lambda_n^i, \quad i = 1, 2, \ldots,$$

where s_i, $i = 1, 2, \ldots$, are the traces of matrices $A^i = A^{i-1} \cdot A$ and let us consider the system of equations

$$f_{jm}(\lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m) \equiv \lambda_1^{jm} + \lambda_2^{jm} + \cdots + \lambda_n^{jm} - s_{jm} = 0, \tag{4}$$

where $m \geqslant 1$ and $i = \overline{1,n}$.

We can consider the aggregate of arguments λ_1^m, λ_2^m, $\ldots$, λ_n^m, as an n-dimensional vector $\overline{\lambda^m} = (\lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m)^T$ and the aggregate of functions f_{jm}, $j = \overline{1,n}$ represents also an n-dimensional vector (vector-function) $\overline{f^m} = (f_m, f_{2m}, \ldots, f_{nm})^T$. The system of equations (4) may be shortly written as

$$\overline{f}(\overline{\lambda^m}) = \overline{0}. \tag{4$'$}$$

Theorem. *The characteristic equation*

$$\begin{aligned} D(\lambda^m) &\equiv \det(A^m - \lambda^m E) \\ &\equiv (-1)^n[(\lambda^m)^n - p_1(\lambda^m)^{n-1} + p_2(\lambda^m)^{n-2} - \cdots + (-1)^n p_n] = 0 \end{aligned} \tag{5}$$

and the system of equations (4) are connected in the following way: If λ_1^m, λ_2^m, $\ldots$, λ_n^m are the roots of equation (5), then the system of equations (4) has $n!$ solutions, namely, the solution $(\lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m)$ and all solutions derived from it by permuting the roots λ_1^m, λ_2^m, $\ldots$, λ_n^m. The system of equations (4) has no other solutions. Conversely, if $(\mu_1^m, \mu_2^m, \ldots, \mu_n^m)$ is a solution of system (4), then μ_1^m, μ_2^m, $\ldots$, μ_n^m are the roots of equation (5).

The proof of this theorem is the same as the proof of analogous theorem in [2].

For solving of system (4), respectively (4$'$), we use the Newton's method [3]. Suppose that we already have the k-th approximation $(\overline{\lambda^m})^{(k)}$, then we calculate the next approximation from

$$(\overline{\lambda^m})^{(k+1)} = (\overline{\lambda^m})^{(k)} - W_n^{-1}\big((\overline{\lambda^m})^{(k)}\big) \cdot \overline{f}\big((\overline{\lambda^m})^{(k)}\big), \quad k = 0, 1, \ldots, \tag{6}$$

where

$$\begin{aligned} \overline{f'}(\overline{\lambda^m}) \equiv W_n(\overline{\lambda^m}) &= \begin{bmatrix} \dfrac{\partial f_m}{\partial \lambda_1^m} & \dfrac{\partial f_m}{\partial \lambda_2^m} & \cdots & \dfrac{\partial f_m}{\partial \lambda_n^m} \\ \vdots & & & \\ \dfrac{\partial f_{nm}}{\partial \lambda_1^m} & \dfrac{\partial f_{nm}}{\partial \lambda_2^m} & \cdots & \dfrac{\partial f_{nm}}{\partial \lambda_n^m} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \vdots & & & \\ n(\lambda_1^m)^{n-1} & n(\lambda_2^m)^{n-1} & \cdots & n(\lambda_n^m)^{n-1} \end{bmatrix} \end{aligned} \tag{7}$$

is the Jacobi matrix and $W_n^{-1}\big((\overline{\lambda^m})^{(k)}\big)$ is its inverse matrix for $\overline{\lambda^m} = (\overline{\lambda^m})^{(k)}$. The determinant of matrix $W_n(\overline{\lambda^m})$ is

$$\det W_n(\overline{\lambda^m}) = n! \prod_{n\geqslant i>j\geqslant 1} (\lambda_i^m - \lambda_j^m) \neq 0.$$

The special structure of matrix (7) enables a simple calculation of the inverse matrix $W_n^{-1}(\overline{\lambda^m})$.

As an illustration we take a matrix of order three. Namely, let us consider the following system of equations

$$\begin{aligned}
f_m(\lambda_1^m, \lambda_2^m, \lambda_2^m) &\equiv \lambda_1^m + \lambda_2^m + \lambda_3^m - s_m = 0,\\
f_{2m}(\lambda_1^m, \lambda_2^m, \lambda_3^m) &\equiv (\lambda_1^m)^2 + (\lambda_2^m)^2 + (\lambda_3^m)^2 - s_{2m} = 0,\\
f_{3m}(\lambda_1^m, \lambda_2^m, \lambda_3^m) &\equiv (\lambda_1^m)^3 + (\lambda_2^m)^3 + (\lambda_3^m)^3 - s_{3m} = 0.
\end{aligned}$$

Here

$$W_3(\overline{\lambda^m}) = \begin{bmatrix} 1 & 1 & 1\\ 2\lambda_1^m & 2\lambda_2^m & 2\lambda_3^m\\ 3(\lambda_1^m)^2 & 3(\lambda_2^m)^2 & 3(\lambda_3^m)^2 \end{bmatrix}$$

and

$$\det W_3(\overline{\lambda^m}) = 3!\,(\lambda_3^m - \lambda_2^m)(\lambda_3^m - \lambda_1^m)(\lambda_2^m - \lambda_1^m) \neq 0.$$

The inverse matrix is

$$W_3^{-1}(\overline{\lambda^m}) = \frac{1}{\det W_3(\overline{\lambda^m})}\times$$
$$\begin{bmatrix} 2\cdot 3\lambda_2^m\lambda_3^m(\lambda_3^m-\lambda_2^m) & -1\cdot 3(\lambda_2^m+\lambda_3^m)(\lambda_3^m-\lambda_2^m) & 1\cdot 2(\lambda_3^m-\lambda_2^m)\\ -2\cdot 3\lambda_1^m\lambda_3^m(\lambda_3^m-\lambda_1^m) & 1\cdot 3(\lambda_1^m+\lambda_3^m)(\lambda_3^m-\lambda_1^m) & -1\cdot 2(\lambda_3^m-\lambda_1^m)\\ 2\cdot 3\lambda_1^m\lambda_2^m(\lambda_2^m-\lambda_1^m) & -1\cdot 3(\lambda_1^m+\lambda_2^m)(\lambda_2^m-\lambda_1^m) & 1\cdot 2(\lambda_2^m-\lambda_1^m) \end{bmatrix}.$$

We define briefly: $f_{jm}^{(k)} = f_{jm}\big((\lambda_1^m)^{(k)}, (\lambda_2^m)^{(k)}, (\lambda_3^m)^{(k)}\big)$, $j = 1,2,3$, and apply formula (6). We get

$$\begin{aligned}
(\lambda_1^m)^{(k+1)} &= (\lambda_1^m)^{(k)} - \frac{6(\lambda_2^m)^{(k)}(\lambda_3^m)^{(k)}f_m^k - 3\big((\lambda_2^m)^{(k)} + (\lambda_3^m)^{(k)}\big)f_{2m}^{(k)} + 2f_{3m}^{(k)}}{6\big((\lambda_3^m)^{(k)} - (\lambda_1^m)^{(k)}\big)\big((\lambda_2^m)^{(k)} - (\lambda_1^m)^{(k)}\big)}\\
(\lambda_2^m)^{(k+1)} &= (\lambda_2^m)^{(k)} - \frac{-6(\lambda_1^m)^{(k)}(\lambda_3^m)^{(k)}f_m^k + 3\big((\lambda_1^m)^{(k)} + (\lambda_3^m)^{(k)}\big)f_{2m}^{(k)} - 2f_{3m}^{(k)}}{6\big((\lambda_3^m)^{(k)} - (\lambda_2^m)^{(k)}\big)\big((\lambda_2^m)^{(k)} - (\lambda_1^m)^{(k)}\big)}\\
(\lambda_3^m)^{(k+1)} &= (\lambda_3^m)^{(k)} - \frac{6(\lambda_1^m)^{(k)}(\lambda_2^m)^{(k)}f_m^k - 3\big((\lambda_1^m)^{(k)} + (\lambda_2^m)^{(k)}\big)f_{2m}^{(k)} + 2f_{3m}^{(k)}}{6\big((\lambda_3^m)^{(k)} - (\lambda_2^m)^{(k)}\big)\big((\lambda_3^m)^{(k)} - (\lambda_1^m)^{(k)}\big)}.
\end{aligned}$$

In the same way we would treat the case of a matrix of order n.

Example 1. Calculate the eigenvalues of the matrix

$$A = \begin{bmatrix} 2.54 & 3.11 & 3.11\\ 2.00 & 3.65 & 3.11\\ 2.00 & 2.00 & 4.76 \end{bmatrix}$$

knowing that they are real and distinct.

Solution. Let us take $m = 2$. The traces of matrices A^2, A^4, A^6 are resp. $s_2 = 79.7517$, $s_4 = 5896.1562$, $s_6 = 451901.76$. The corresponding system is

$$f_2 \equiv \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 79.7517 = 0,$$
$$f_4 \equiv \lambda_1^4 + \lambda_2^4 + \lambda_3^4 - 5896.1562 = 0,$$
$$f_6 \equiv \lambda_1^6 + \lambda_2^6 + \lambda_3^6 - 451901.76 = 0.$$

For initial approximation let us take $\lambda_1^{2(0)} = 76.7$, $\lambda_2^{2(0)} = 2.7$, $\lambda_3^{2(0)} = 0.3$. By applying of previous formulas we successively get:

$$\lambda_1^{2(1)} = 76.73762, \quad \lambda_2^{2(1)} = 2.72233, \quad \lambda_3^{2(1)} = 0.29340,$$
$$\lambda_1^{2(2)} = 76.73760, \quad \lambda_2^{2(2)} = 2.72252, \quad \lambda_3^{2(2)} = 0.29158,$$
$$\lambda_1^{2(3)} = 76.73760, \quad \lambda_2^{2(3)} = 2.72252, \quad \lambda_3^{2(3)} = 0.29158.$$

In that way we get $\lambda_1 = 8.7600$, $\lambda_2 = 1.6500$, $\lambda_3 = 0.5400$.

EXAMPLE 2. Calculate the eigenvalues of the matrix

$$A = \begin{bmatrix} 1.25 & 0.95 & 0.95 \\ 0.95 & 1.25 & 0.95 \\ 0.95 & 0.95 & 1.25 \end{bmatrix}$$

knowing they are real and $\lambda_1 > \lambda_2 = \lambda_3$.

Solution. It may be shown that

$$A^i = \frac{1}{3}(1.25 + 2 \cdot 0.95)^i \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \frac{1}{3}(1.25 + 2 \cdot 0.95)^i \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix},$$

$i = 1, 2, \ldots$. Let us take $m = 2$. Then we have $s_2 = 10.1025$, $s_4 = 98.472206$. The corresponding system is

$$f_2 \equiv \lambda_1^2 + 2\lambda_2^2 - 10.1025 = 0,$$
$$f_4 \equiv \lambda_1^4 + 2\lambda_2^4 - 98.472206 = 0.$$

In this case the formulas are

$$\lambda_1^{2(k+1)} = \lambda_1^{2(k)} - \frac{2\lambda_2^{2(k)} f_2^{(k)} - f_4^{(k)}}{2!\,(\lambda_2^{2(k)} - \lambda_1^{2(k)})},$$
$$\lambda_2^{2(k+1)} = \lambda_2^{2(k)} - \frac{-2\lambda_1^{2(k)} f_2^{(k)} + f_4^{(k)}}{2 \cdot 2!\,(\lambda_2^{2(k)} - \lambda_1^{2(k)})}, \quad k = 0, 1, \ldots$$

For initial approximation let us take $\lambda_1^{2(0)} = 9$, $\lambda_2^{2(0)} = 0$. By applying these formulas we successively get:

$$\lambda_1^{2(1)} = 9.97068, \quad \lambda_1^{(1)} = 3.15764, \qquad \lambda_2^{2(1)} = 0.065911, \quad \lambda_2^{(1)} = 0.25673,$$
$$\lambda_1^{2(2)} = 9.92268, \quad \lambda_1^{(2)} = 3.15003, \qquad \lambda_2^{2(2)} = 0.089913, \quad \lambda_2^{(2)} = 0.29985,$$
$$\lambda_1^{2(3)} = 9.92250, \quad \lambda_1^{(3)} = 3.15000, \qquad \lambda_2^{2(3)} = 0.090000, \quad \lambda_2^{(3)} = 0.30000,$$
$$\lambda_1^{2(4)} = 9.92250, \quad \lambda_1^{(4)} = 3.15000, \qquad \lambda_2^{2(4)} = 0.090000, \quad \lambda_2^{(4)} = 0.30000.$$

In that way we get $\lambda_1 = 3.1500$, $\lambda_2 = \lambda_3 = 0.3000$.

The method may be used also in the case of complex eigenvalues. For every pair of conjugate-complex eigenvalues one should take

$$\lambda_s + \overline{\lambda_s} = (a+ib) + (a-ib) = 2a = \tau_1, \quad \lambda_s \cdot \overline{\lambda_s} = a^2 + b^2 = \tau_2.$$

EXAMPLE 3. Calculate the eigenvalues of the matrix

$$A = \begin{bmatrix} 4 & -5 & 7 \\ 1 & -4 & 9 \\ -4 & 0 & 5 \end{bmatrix}$$

knowing that λ_1 is real and $\lambda_2 = a \pm ib$.

Solution. Let us take again $m = 2$. Here we have $s_2 = -9$, $s_4 = -237$, $s_6 = 4071$. The corresponding system is

$$\begin{aligned} g_2(\lambda_1, \tau_1, \tau_2) &\equiv \lambda_1^2 + \tau_1^2 - 2\tau_2 - 9 = 0, \\ g_4(\lambda_1, \tau_1, \tau_2) &\equiv \lambda_1^4 + \tau_1^4 - 4\tau_1^2\tau_2 + 2\tau_2^2 + 237 = 0, \\ g_6(\lambda_1, \tau_1, \tau_2) &\equiv \lambda_1^6 + \tau_1^6 - 6\tau_1^4\tau_2 - 9\tau_1^2\tau_2^2 - 22\tau_2^3 - 4071 = 0. \end{aligned}$$

For initial approximation let us take $\lambda_1^{2(0)} = 0.5$, $\tau_1^{2(0)} = 3.5$, $\tau_2^{(0)} = 12.5$. In this case we get $\lambda_1 = 1$, $\lambda_{2,3} = 2 \pm 3i$.

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