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SPECTRAL STATES OF COMMUTATIVE L.M.C. ALGEBRAS

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Abstract. We characterize the commutative locally multiplicative convex (l.m.c.) algebras in terms of the spectral states. We also give a characterization of spectral states in terms of commutative semisimple l.m.c. algebras. Further, with the help of radicals of l.m.c. algebras we give a necessary and a sufficient condition for an algebra to be commutative modulo its radical.

1. Introduction

Let X be a locally m-convex (l.m.c.) algebra with unit e. We will follow the notations and terminologies of [4] and [6]. It is sufficient for our purpose to note that, for a given l.m.c. algebra X with unit e there exists a separating family of submultiplicative seminorms $\{P_\alpha\}$ on X which generates the topology and is such that $P_\alpha(e) = 1$ for all α in the index set I. Given such an aglebra, we denote by $P(X)$ the class of all such families of seminorms on X, and by $(X, \{P_\alpha\})$ the algebra X with a particular family of seminorms $\{P_\alpha\} \in P(X)$.

For every $\alpha \in I$, let X_α denote the unital Banach algebra. Using Bonsall and Duncan's notation [2], the spectral state of X_α is denoted by $\Omega(X_\alpha)$ and $\Omega(X_\alpha) = \{f \in X_\alpha^* : f(e) = 1, |f(x)| \le \rho_\alpha(x), x \in X_\alpha\}$, where $\rho_\alpha(\cdot)$ is the spectral radius of x_α and $\|x_\alpha\|_\alpha = P_\alpha(x)$. (See Michael [6]). $\Omega(X_\alpha)$ is a weak*-compact convex subset of the complex plane. The set of all spectral states of X is denoted by $\Omega(X)$. If $q_\alpha^*\colon X \to X_\alpha$ is the quotient map and q_α^* is the adjoint of q_α, then we define $\Omega(X) = \bigcup q_\alpha^*(\Omega(X_\alpha))$.

Given $(X, \{P_\alpha\})$, we define the set $D_\alpha(X, P_\alpha; e) = \{f \in X' : f(e) = 1 \text{ and } |f(x)| \le P_\alpha(X) \text{ for all } x \in X\}$ and we write

$$D(X, \{P_\alpha\}; e) = \bigcup \{D_\alpha(X, P_\alpha; e)\}.$$

Note that $D_\alpha(X, P_\alpha; e)$ is isomorphic to $D(X_\alpha, \|\cdot\|_\alpha; e_\alpha)$ and $D(X, \{P_\alpha\}; e)$ depends upon the particular family of seminorms $\{P_\alpha\} \in P(X)$ chosen to associate with X.

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2. Spectral states and commutative l.m.c. algebras

Using the idea in Giles and Koehler [4], we can say that $\Omega(X)$ is the state space of $(X, \{P_\alpha\})$ and $\Omega(X) = D(X, \{P_\alpha\}; e)$. Note that $\Omega(X)$ does not depend on the particular family of seminorms $\{P_\alpha\}$ chosen to generate the topology.

THEOREM 2.1. *Let* $D_X = \bigcap\{\, D(X, \{P_\alpha\}; e) \,:\, \{P_\alpha\} \in P(X)\,\}$. *Then* $\Omega(X) = D_X$.

Proof. Let $f \in \Omega(x)$ and $\{P_\alpha\} \in P(X)$. If $x \in X$ and $\alpha \in I$ with $|f(x)| \leq \rho_\alpha(x_\alpha) \leq P_\alpha(x)$, then there exists $M > 0$ and $\beta \in I$ such that $P_\alpha(x) \leq M P_\beta(x)$ and $|f(x)| \leq \sqrt[n]{P_\alpha(x^n)} \leq \sqrt[n]{M} P_\beta(x)$ for every natural number n and every $x \in X$. This shows that $f \in D_X$. Conversely, suppose that $f \in X'$ is not a spectral state and $f(e) = 1$. Then for each α, there exists $x_\alpha \in X_\alpha$ such that $|f(x_\alpha)| > \rho_\alpha(q_\alpha(x_\alpha))$. By Lemma 2.8, [2], there exist seminorms v_α on X_α equivalent to the usual norm $\|\cdot\|_\alpha$ such that $|f(x_\alpha)| > v_\alpha(q_\alpha(x_\alpha))$. Let $P_\alpha = v_\alpha$, $\alpha \in I$. Then $P_\alpha \in P(X)$, but $f \notin D(X, \{P_\alpha\}; e)$. This implies that $f \notin D_X$ and hence $\Omega(X) = D_X$. ∎

Let X be a commutative l.m.c. algebra. Let Φ_X be the set of all multiplicative linear functionals on X and let Φ_α be the set of all multiplicative linear functionals on X_α. Also, suppose that $\psi_\alpha = q_\alpha^*(\Phi_\alpha)$. This means that for each α, Φ_α is homeomorphic to ψ_α. Let $\hat{X}$ and $\hat{X}_\alpha$ be the Gelfand transformations on X and X_α, respectively. Denote a compact Hausdorff space by E and suppose $\mu(E)$ denotes the set of all probability measures on E. (For more on these measures, see [1]).

PROPOSITION 2.1. *For a commutative l.m.c. algebra X with unit and for $f \in X'$, the following are equivalent.*

(a) For $\alpha \in I$ and $\mu \in \mu(E)\psi_\alpha$, $f(x) = \int \hat{X}(x)\, d\mu$, $x \in X$.

(b) There exists a probability measure μ on Φ_X with compact (equicontinuous) support K, (see [7]), with $f(x) = \int \hat{X}(x)\, d\mu$, $x \in X$.

Proof. If K is a compact subset of Φ_X, then K is contained in some ψ_α. Hence, (b) implies (a). The implication $(a) \implies (b)$ follows by the definitions involved. ∎

PROPOSITION 2.2. *There exists $\alpha \in I$ such that $|e^{f(x)}| \leq \|e^{x_\alpha}\|_\alpha$, $x \in X$ if and only if*

$$\operatorname{Re} f(x) \leq \sup\{\, \operatorname{Re} \eta_\alpha(x_\alpha) : \eta_\alpha \in \Phi_\alpha \,\}. \tag{$*$}$$

Proof. For $\alpha \in I$, $|e^{f(x)}| \leq \|e^{x_\alpha}\|_\alpha \Leftrightarrow \operatorname{Re} f(x) \leq \frac{1}{n} \ln \|e^{n x_\alpha}\|_\alpha$, where ln is the natural log function, n is a natural number, and $x \in X$. If $\operatorname{sp}(X_\alpha, x_\alpha)$ denotes the spectrum of x_α, see [4], then by Theorem 8, page 32 of [2], we have $\sup\{\, \operatorname{Re} \lambda \,:\, \lambda \in \operatorname{sp}(X_\alpha, x_\alpha)\,\} = \inf\{\, \frac{1}{n} \ln \|e^{n x_\alpha}\|_\alpha \,:\, n \text{ is a natural number}\,\}$ and hence $\|e^{f(x)}\| \leq \|e^{x_\alpha}\|_\alpha$ is equivalent to condition $(*)$. ∎

REMARK 2.1. The above propositions provide us with a characterization for spectral states of l.m.c. algebras. Further, these characterizations also show that $\Omega(X)$ does not depend on the particular family of seminorms.

It is clear that $\Omega(X)$ contains all non-zero multiplicative linear functionals. Also, if X is a commutative l.m.c. algebra, then every probability measure on the Carrier space [8, p. 261] of X provides a spectral state on X.

For commutative X, $\Omega(X)$ is nonempty, but if H is an infinite dimensional complex Hilbert space and $B(H)$ is the set of all bounded linear operators on H, $\Omega(B(H)) = \emptyset$, see example 5, page 115, [2]. In fact for C*-algebra X, $\Omega(X) = \emptyset$. On the other hand, if $B(H)$ is the set of all compact operators on H, then $\Omega(B(H)) = \{0\}$.

Example 2.1. Let E be a compact Hausdorff space and $C(E)$ be the l.m.c. algebra of all complex-valued continuous functions on E. The topology on $C(E)$ is of the uniform convergence. Then $\Phi_{C(E)}$ is isomorphic to E. The countable compact subsets of E are the compact subsets of $\Phi_{C(E)}$. Let $\phi \in C(E)$ and for each natural number n, $a_n \in E$. Suppose $\lambda_n \in [0, 1]$, then a linear functional f on $C(E)$ is given by $f(\phi) = \sum_{n=1}^{\infty} \lambda_n \phi(a_n)$. These linear functionals define the spectral states of $C(E)$. Let μ be a probability measure on E which vanishes at singletons. Then f is defined by integration with respect to μ such that $f(e) = 1$. Further, $|f(\phi)| \leq \rho_{C(E)}(\phi)$ and $f(\phi) \in \operatorname{co}\operatorname{sp}(C(E), \phi)$ for each $\phi \in C(E)$, where *co* is the convex hull. Since f is defined by integration with respect to a probability measure μ with an uncountable support, f is not a spectral state.

Remark 2.2. If A is a finite dimensional complex Banach algebra with unit and Wedderburn decomposition $A = A_1 \oplus A_2 \oplus \cdots \oplus A_m \oplus R$ (where R is the radical of A and each A_i is a subalgebra of A that is isomorphic to a matrix algebra over the complex numbers), then $\Omega(A)$ is the convex hull of the normalized traces $T_i (i = 1, 2, \ldots, m)$, see Theorem 11, p. 119 [2]. Also, if R is the Jacobsen radical of A, then $f(R) = \{0\}$ for each f in $\Omega(A)$.

3. Commutative semisimple algebra and spectral states

Definition 3.1. X_α is semisimple if the Gelfand transformation on X_α is one-to-one.

A commutative Banach algebra A is simple if $Rad(A) = \{0\}$.

So if we have a semisimple l.m.c. algebra, then a rich supply of spectral states is possible. We prove the following theorem which characterizes such algebras.

Theorem 3.1. *Let X be an l.m.c. algebra with unit. Then X is commutative and semisimple if and only if $\Omega(X)$ separates the points of X.*

Proof. Let X be a commutative semisimple l.m.c. algebra with unit. Then the complex homomorphisms of X separate the points of X, see Corollary 3.5.1, [5]. Hence, $\Omega(X)$ separates the points of X.

Conversely, suppose that $\Omega(X)$ separates the points of X. If $f \in \Omega(X)$, then $f(ab) = f(ba)$, $a, b \in X$, by Theorem 4, p. 114 [2]. Hence, X is commutative. Since every $f \in \Omega(X)$ vanishes on the kernel of the Gelfand transformation $\hat{X}$ on X, Proposition 2.1 proves that $\hat{X}$ is one-to-one and X is semisimple by Definition 3.1 above. ■

In [4], it is shown that if X is a complex l.m.c. algebra with unit, then for each $x \in X$,

$$\operatorname{co}\operatorname{sp}(X,x) \subseteq \bigcap \{\, V(X,\{P_\alpha\};x) : \{P_\alpha\} \in P(X)\,\} \subseteq \overline{\operatorname{co}}\operatorname{sp}(X,x)$$

where $V(X,\{P_\alpha\};x)$ is the numerical range of x in X.

If X is commutative modulo its radical, then $\operatorname{co}\operatorname{sp}(X,x) = \{f(x) : f \in \Omega(X)\}$. This follows from the fact that the following condition in [4]

$$\bigcap \{\, f(x) : f \in D(X,\{P_\alpha\};e)\,\}$$

can be replaced by

$$\Big\{\, f(x) : f \in \bigcap D(X,\{P_\alpha\};e)\,\Big\}$$

Inspired by this observation, we have the following theorem.

Theorem 3.2. *Let X be a complete l.m.c. algebra with unit. Then X is commutative modulo $Rad(X)$ if and only if $\operatorname{co}\operatorname{sp}(X,x) = \{f(x) : f \in \Omega(X)\}$ for every $x \in X$.*

Proof. Let X be commutative modulo $Rad(X)$. By Proposition 24.16 in [3], it follows that for $a,x,y \in X$, $a(xy-yx)$ is quasi-regular or quasi-invertible, see [5, p.13]. This implies that $\rho_X(xy-yx)=0$. Thus, for each $\alpha \in I$, $\rho_\alpha(x_\alpha y_\alpha - y_\alpha x_\alpha) = 0$, which proves that X_α is commutative modulo $Rad(X_\alpha)$.

For each $x \in X$, $\{\, f_\alpha(x_\alpha) : f_\alpha \in \Omega(X_\alpha)\,\} \subset \operatorname{co}\operatorname{sp}(X_\alpha,x_\alpha)$ and since X_α is commutative modulo $Rad(X_\alpha)$, we have $\operatorname{sp}(X_\alpha,x_\alpha) = \{\, \phi_\alpha(x_\alpha) : \phi_\alpha \in \Phi_\alpha\,\}$. Further, since $\Phi_\alpha \subset \Omega(X_\alpha)$, and $\Omega(X_\alpha)$ is convex, we have $\operatorname{co}\operatorname{sp}(X_\alpha,x_\alpha) \subset \{\, f_\alpha(x_\alpha) : f_\alpha \in \Omega(X_\alpha)\,\}$. Hence we have established that if X is commutative modulo $Rad(X)$, then $\operatorname{co}\operatorname{sp}(X_\alpha,x_\alpha) = \{\, f_\alpha(x_\alpha) : f_\alpha \in \Omega(X_\alpha)\,\}$.

Since the family of spectra is a well directed family, we have $\operatorname{co}\operatorname{sp}(X,x) = \bigcup \operatorname{co}\operatorname{sp}(X_\alpha,x_\alpha)$, see Theorem 1 [4]. By the definition of $\Omega(X)$, we prove that $\operatorname{co}\operatorname{sp}(X_\alpha,x_\alpha) = \{\, f_\alpha(x_\alpha) : f_\alpha \in \Omega(X_\alpha)\,\}$ implies that $\operatorname{co}\operatorname{sp}(X,x) = \{\, f(x) : f \in \Omega(X)\,\}$ for every $x \in X$.

Conversely, suppose that for each $x \in X$, $\operatorname{co}\operatorname{sp}(X,x) = \{\, f(x) : f \in \Omega(X)\,\}$. Since $\operatorname{sp}(X,xy-yx) = \{0\}$, we have the commutativity of each X_α modulo its radical. Hence, $a(xy-yx)$ is quasi-regular in X. This shows that $xy-yx$ belongs to $Rad(X)$ and hence X is commutative modulo $Rad(X)$. ■

Corollary 3.1. *If X is a complete l.m.c. algebra with unit and X is commutative modulo $Rad(X)$, then*

$$\begin{aligned}\Gamma_X &= \{\, x \in X : \{\sup|f(x)| : f \in \Omega(X)\} < \infty\,\} \\ &= \{\, x \in X : V(X,\{P_\alpha\};x) \ \text{ is bounded}\} = U_X \\ &= \{\, x \in X : \rho_X(x) < \infty\,\} = R_X.\end{aligned}$$

Proof. $\Gamma_X = U_X$ follows from Theorem 4, [4]. By Theorem 3.2, $\{f(X) : f \in \Omega(X)\} = \operatorname{co}\operatorname{sp}(X, x)$, hence $U_X = R_X$. ■

COROLLARY 3.2. *$\Omega(X)$ is weak*-bounded.*

Proof. The result follows from Corollary 3.1 and the fact that $\Omega(X)$ is weak*-bounded if and only if $\operatorname{sp}(X, x)$ is bounded for each $x \in X$. ■

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