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r-FUZZY STRONGLY PREOPEN SETS IN FUZZY TOPOLOGICAL SPACES

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Abstract. We introduce r-fuzzy strongly preopen and r-fuzzy strongly preclosed sets in fuzzy topological spaces in the sense of the definition of Šostak [8] and investigate some of their properties. Fuzzy strongly precontinuous, fuzzy strongly preopen and fuzzy strongly preclosed mappings between fuzzy topological spaces are defined. Their properties and the relationship between these mappings and other mappings introduced previously are investigated.

1. Introduction and preliminaries

A. P. Šostak [8] introduced a fuzzy topology as an extension of Chang's fuzzy topology [2]. It has been developed in many directions [3,4,7]. B. Krsteska [1] introduced fuzzy strongly preopen and fuzzy strongly preclosed sets in the Chang's fuzzy topology.

In this paper we define r-fuzzy strongly preopen and r-fuzzy strongly preclosed sets in a fuzzy topological space in view of the definition of Šostak [8] and investigate some of their properties. We show that fuzzy strong precontinuity implies fuzzy precontinuity, but the converse is not true. We obtain some properties of fuzzy strongly precontinuous mappings.

Throughout this paper, let X be a non-empty set, $I = [0,1]$ and $I_0 = (0,1]$. For $\alpha \in I$, $\bar{\alpha}(x) = \alpha$, for all $x \in X$. All other notations and definitions are standard in the fuzzy set theory.

DEFINITION 1.1. [8] A function $\tau\colon I^X \to I$ is called a fuzzy topology on X if it satisfies the following conditions:

(O1) $\tau(\bar{0}) = \tau(\bar{1}) = 1$,

(O2) $\tau(\mu_1 \wedge \mu_2) \geqslant \tau(\mu_1) \wedge \tau(\mu_2)$, for any $\mu_1, \mu_2 \in I^X$,

(O3) $\tau(\bigvee_{i\in\Gamma} \mu_i) \geqslant \bigwedge_{i\in\Gamma} \tau(\mu_i)$, for any $\{\mu_i\}_{i\in\Gamma} \subset I^X$.

The pair (X, τ) is called a fuzzy topological space (fts, for short).

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Remark 1.2. Let (X,τ) be an fts. Then, for each $\alpha \in I$, $\tau_\alpha = \{\mu \in I^X : \tau(\mu) \geqslant \alpha\}$ is a Chang's fuzzy topology on X.

Theorem 1.1. [7] *Let (X,τ) be an fts. For each $r \in I_0$, $\lambda \in I^X$, define an operator $C_\tau : I^X \times I_0 \to I^X$ as follows*

$$C_\tau(\lambda, r) = \bigwedge\{\, \mu \in I^X : \lambda \leqslant \mu,\ \tau(\bar{1} - \mu) \geqslant r \,\}.$$

For each $\lambda, \mu \in I^X$ and $r, s \in I_0$ it satisfies the following conditions:

(1) $C_\tau(\bar{0}, r) = \bar{0}$.

(2) $\lambda \leqslant C_\tau(\lambda, r)$.

(3) $C_\tau(\lambda, r) \vee C_\tau(\mu, r) = C_\tau(\lambda \vee \mu, r)$.

(4) $C_\tau(\lambda, r) \leqslant C_\tau(\lambda, s)$ *if* $r \leqslant s$.

(5) $C_\tau(C_\tau(\lambda, r), r) = C_\tau(\lambda, r)$.

Theorem 1.2. [6] *Let (X,τ) be an fts. For each $r \in I_0$, $\lambda \in I^X$, define an operator $I_\tau : I^X \times I_0 \to I^X$ as follows*

$$I_\tau(\lambda, r) = \bigvee\{\, \mu \in I^X : \lambda \geqslant \mu,\ \tau(\mu) \geqslant r \,\}.$$

For each $\lambda, \mu \in I^X$ and $r, s \in I_0$ it satisfies the following conditions:

(1) $I_\tau(\bar{1} - \lambda, r) = \bar{1} - C_\tau(\lambda, r)$ *and* $C_\tau(\bar{1} - \lambda, r) = \bar{1} - I_\tau(\lambda, r)$.

(2) $I_\tau(\bar{1}, r) = \bar{1}$.

(3) $I_\tau(\lambda, r) \leqslant \lambda$.

(4) $I_\tau(\lambda, r) \wedge I_\tau(\mu, r) = I_\tau(\lambda \wedge \mu, r)$.

(5) $I_\tau(\lambda, r) \geqslant I_\tau(\lambda, s)$ *if* $r \leqslant s$.

(6) $I_\tau(I_\tau(\lambda, r), r) = I_\tau(\lambda, r)$.

Definition 1.2. [6] Let (X,τ) be an fts. For $\lambda \in I^X$ and $r \in I_0$:

(1) λ is called an r-fuzzy semi-open (r-fso, for short) set if there exists $\nu \in I^X$ with $\tau(\nu) \geqslant r$ such that $\nu \leqslant \lambda \leqslant C_\tau(\nu, r)$. Equivalently, $\lambda \leqslant C_\tau(I_\tau(\lambda, r), r)$.

(2) λ is called an r-fuzzy semi-closed (r-fss, for short) set if there exists $\nu \in I^X$ with $\tau(\bar{1} - \nu) \geqslant r$ such that $I_\tau(\nu, r) \leqslant \lambda \leqslant \nu$. Equivalently, $I_\tau(C_\tau(\lambda, r), r) \leqslant \lambda$.

Definition 1.3. [6] Let (X,τ) and (Y,τ^*) be fts's. A mapping $f : X \to Y$ is said to be:

(1) fuzzy continuous iff $\tau^*(\mu) \leqslant \tau(f^{-1}(\mu))$ for each $\mu \in I^Y$.

(2) fuzzy open iff $\tau^*(f(\mu)) \geqslant \tau(\mu)$ for each $\mu \in I^X$.

(3) fuzzy closed iff $\tau^*(\bar{1} - f(\mu)) \geqslant \tau(\bar{1} - \mu)$ for each $\mu \in I^X$.

DEFINITION 1.4. [6] Let (X,τ) and (Y,τ^*) be fts's. A mapping $f\colon X \to Y$ is said to be:

(1) fuzzy semi-continuous iff $f^{-1}(\mu)$ is r-fso for each $\mu \in I^Y$, $r \in I_0$ with $\tau^*(\mu) \geqslant r$.

(2) fuzzy semi-open iff $f(\mu)$ is r-fso for each $\mu \in I^X$, $r \in I_0$ with $\tau(\mu) \geqslant r$.

(3) fuzzy semi-closed iff $f(\mu)$ is r-fsc for each $\mu \in I^X$, $r \in I_0$ with $\tau(\bar{1}-\mu) \geqslant r$.

2. r-fuzzy strongly preopen and r-fuzzy strongly preclosed sets

DEFINITION 2.1. [6] Let (X,τ) be an fts. For $\lambda \in I^X$ and $r \in I_0$:

(1) λ is called r-fuzzy preopen (r-fpo, for short) iff

$$\lambda \leqslant I_\tau(C_\tau(\lambda,r),r).$$

(2) λ is called r-fuzzy preclosed (r-fpc, for short) iff

$$C_\tau(I_\tau(\lambda,r),r) \leqslant \lambda.$$

(3) λ is called r-fuzzy strongly semi-open (r-fsso, for short) iff

$$\lambda \leqslant I_\tau(C_\tau(I_\tau(\lambda,r),r),r).$$

(4) λ is called r-fuzzy strongly semi-closed (r-fssc, for short) iff

$$C_\tau(I_\tau(C_\tau(\lambda,r),r),r) \leqslant \lambda.$$

DEFINITION 2.2. [6] Let (X,τ) be an fts. For $\lambda \in I^X$ and $r \in I_0$:

(1) the r-fuzzy preinterior of λ, denoted by $PI_\tau(\lambda,r)$, is defined by

$$PI_\tau(\lambda,r) = \bigvee\{\, \nu \in I^X : \nu \leqslant \lambda,\ \nu \text{ is } r\text{-fpo}\,\},$$

(2) the r-fuzzy preclosure of λ, denoted by $PC_\tau(\lambda,r)$, is defined by

$$PC_\tau(\lambda,r) = \bigwedge\{\, \nu \in I^X : \nu \geqslant \lambda,\ \nu \text{ is } r\text{-fpc}\,\},$$

(3) the r-fuzzy strongly semi-interior of λ, denoted by $SSI_\tau(\lambda,r)$, is defined by

$$SSI_\tau(\lambda,r) = \bigvee\{\, \nu \in I^X : \nu \leqslant \lambda,\ \nu \text{ is } r\text{-fsso}\,\},$$

(4) the r-fuzzy strongly semi-closure of λ, denoted by $SSC_\tau(\lambda,r)$, is defined by

$$SSC_\tau(\lambda,r) = \bigwedge\{\, \nu \in I^X : \nu \geqslant \lambda,\ \nu \text{ is } r\text{-fssc}\,\}.$$

THEOREM 2.1. *Let (X,τ) be an fts, $\lambda \in I^X$ and $r \in I_0$. Then*
(1) $\lambda \vee C_\tau(I_\tau(\lambda,r),r) \leqslant PC_\tau(\lambda,r)$,
(2) $PI_\tau(\lambda,r) \leqslant \lambda \wedge I_\tau(C_\tau(\lambda,r),r)$,
(3) $I_\tau(PC_\tau(\lambda,r),r) \leqslant I_\tau(C_\tau(\lambda,r),r)$,
(4) $I_\tau(C_\tau(I_\tau(\lambda,r),r),r) \leqslant I_\tau(PC_\tau(\lambda,r),r)$,
(5) $PC_\tau(\bar{1}-\lambda,r) = \bar{1} - PI_\tau(\lambda,r)$ *and* $PI_\tau(\bar{1}-\lambda,r) = \bar{1} - PC_\tau(\lambda,r)$.

Proof. (1) Since $PC_\tau(\lambda,r)$ is an r-fpc set, we have

$$C_\tau(I_\tau(\lambda,r),r) \leqslant C_\tau(I_\tau(PC_\tau(\lambda,r),r),r) \leqslant PC_\tau(\lambda,r).$$

Thus, $\lambda \vee C_\tau(I_\tau(\lambda,r),r) \leqslant PC_\tau(\lambda,r)$.

(2) It can be shown as (1).

(3) It follows from the relation $PC_\tau(\lambda,r) \leqslant C_\tau(\lambda,r)$.

(4) From (1) we have

$$I_\tau(PC_\tau(\lambda,r),r) \geqslant I_\tau(\lambda \vee C_\tau(I_\tau(\lambda,r),r),r) \geqslant I_\tau(C_\tau(I_\tau(\lambda,r),r),r).$$

(5) Straightforward. ■

DEFINITION 2.3. Let (X,τ) be an fts. For $\lambda \in I^X$ and $r \in I_0$:
(1) λ is called r-fuzzy strongly preopen (r-fspo, for short) iff

$$\lambda \leqslant I_\tau(PC_\tau(\lambda,r),r).$$

(2) λ is called r-fuzzy strongly preclosed (r-fspc, for short) iff

$$C_\tau(PI_\tau(\lambda,r),r) \leqslant \lambda.$$

(3) The r-fuzzy strong preinterior of λ, denoted by $SPI_\tau(\lambda,r)$, is defined by

$$SPI_\tau(\lambda,r) = \bigvee\{\, \nu \in I^X : \nu \leqslant \lambda,\ \nu \text{ is } r\text{-fspo}\,\}.$$

(4) The r-fuzzy strong preclosure of λ, denoted by $SPC_\tau(\lambda,r)$, is defined by

$$SPC_\tau(\lambda,r) = \bigwedge\{\, \nu \in I^X : \nu \geqslant \lambda,\ \nu \text{ is } r\text{-fspc}\,\}.$$

THEOREM 2.2. *Let (X,τ) be an fts, $\lambda \in I^X$ and $r \in I_0$. Then*
(1) if $\tau(\lambda) \geqslant r$, then λ is an r-fspo set;
(2) if λ is r-fsso, then λ is r-fspo;
(3) if l is r-fspo, then λ is r-fpo.

Proof. It follows from Theorem 2.1. ■

The following examples show that the converses in the above theorem are not true in general.

EXAMPLE 2.1. Let $X = \{a, b, c\}$. Define fuzzy sets $\lambda_1, \lambda_2, \mu \in I^X$ as follows:

$$\begin{aligned} \lambda_1(a) &= 0.3, & \lambda_1(b) &= 0.2, & \lambda_1(c) &= 0.7, \\ \lambda_2(a) &= 0.8, & \lambda_2(b) &= 0.8, & \lambda_2(c) &= 0.4, \\ \mu(a) &= 0.8, & \mu(b) &= 0.7, & \mu(c) &= 0.6. \end{aligned}$$

Define a fuzzy topology $\tau\colon I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda = \bar{0} \text{ or } \bar{1}, \\ \frac{1}{3}, & \text{if } \lambda = \lambda_1, \\ \frac{1}{2}, & \text{if } \lambda = \lambda_2, \\ \frac{2}{3}, & \text{if } \lambda = \lambda_1 \vee \lambda_2, \\ \frac{2}{3}, & \text{if } \lambda = \lambda_1 \wedge \lambda_2, \\ 0, & \text{otherwise.} \end{cases}$$

For the fts (X, τ) with $0 < r \leqslant \frac{1}{3}$, μ is an r-fspo set from

$$\mu \leqslant I_\tau(PC_\tau(\mu, r), r) = \bar{1}.$$

For $0 < r \leqslant \frac{1}{3}$, μ is neither r-fsso nor $\tau(\mu) \geqslant r$ from

$$\mu \nleqslant I_\tau(C_\tau(I_\tau(\mu, r), r), r) = \lambda_1 \wedge \lambda_2.$$

EXAMPLE 2.2. Let X be a non-empty set. We define fuzzy topology $\tau\colon I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda = \bar{0} \text{ or } \bar{1}, \\ \frac{2}{3}, & \text{if } \lambda = \overline{0.2}, \\ \frac{1}{3}, & \text{if } \lambda = \overline{0.4}, \\ \frac{1}{2}, & \text{if } \lambda = \overline{0.5}, \\ \frac{3}{4}, & \text{if } \lambda = \overline{0.6}, \\ 0, & \text{otherwise.} \end{cases}$$

For $0 < r \leqslant \frac{1}{3}$, $\overline{0.7}$ is an r-fpo set but it is not r-fspo, from the following:

$$\begin{aligned} \overline{0.7} &\leqslant I_\tau(C_\tau(\overline{0.7}, r), r) = \bar{1}, \\ \overline{0.7} &> I_\tau(PC_\tau(\overline{0.7}, r), r) = I_\tau(\overline{0.7}, r) = \overline{0.4}. \end{aligned}$$

For $\frac{1}{3} < r \leqslant \frac{2}{3}$, $\overline{0.3}$ is r-fso but it is not r-fspo, from the following:

$$\begin{aligned} \overline{0.3} &\leqslant C_\tau(I_\tau(\overline{0.3}, r), r) = \overline{0.4}, \\ \overline{0.3} &> I_\tau(PC_\tau(\overline{0.3}, r), r) = I_\tau(\overline{0.4}, r) = \overline{0.2}. \end{aligned}$$

Remark 2.1. From Example 2.1, for $0 < r \leqslant \frac{1}{3}$, μ is an r-fspo set but not an r-fso set because

$$\mu \nleqslant C_\tau(I_\tau(\mu, r), r) = \bar{1} - \lambda_1 \wedge \lambda_2.$$

Also, in Example 2.2, $\overline{0.3}$ is an r-fso but not r-fspo set for $\frac{1}{3} < r \leqslant \frac{2}{3}$. From the above discussion there is no relation between the concepts of r-fso sets and r-fspo sets.

Theorem 2.3. *Let (X, τ) be an fts and $r \in I_0$.*

(1) Any union of r-fspo sets is an r-fspo set.

(2) Any intersection of r-fspc sets is an r-fspc set.

Proof. (1) Let $\{\lambda_\alpha : \alpha \in \Gamma\}$ be a family of r-fspo sets. For each $\alpha \in \Gamma$, $\lambda_\alpha \leqslant I_\tau(PC_\tau(\lambda_\alpha, r), r)$. Hence we have

$$\bigvee_{\alpha \in \Gamma} \lambda_\alpha \leqslant \bigvee_{\alpha \in \Gamma} I_\tau(PC_\tau(\lambda_\alpha, r), r) \leqslant I_\tau\left(PC_\tau\left(\bigvee_{\alpha \in \Gamma} \lambda_\alpha, r\right), r\right).$$

So, $\bigvee_{\alpha \in \Gamma} \lambda_\alpha$ is an r-fspo set.

(2) It is easily proved in the same manner. ■

Remark 2.2. The intersection of two r-fspo sets need not be an r-fspo set. And the union of two r-fspc sets need not be an r-fspc set. We will show it in the next example.

Example 2.3. Consider the fts (X, τ) form Example 2.1. The fuzzy set ρ defined as:

$$\rho(a) = 0.4, \quad \rho(b) = 0.2, \quad \rho(c) = 0.8,$$

is a $\frac{1}{3}$-fspo set, but $\lambda_2 \wedge \rho$ is not a $\frac{1}{3}$-fspo set. Also, $(\bar{1} - \lambda_2) \vee (\bar{1} - \rho)$ is not a $\frac{1}{3}$-fspc set in (X, τ).

Theorem 2.4. *Let (X, τ) be an fts. For $\lambda \in I^X$ and $r \in I_0$ the following statements hold:*

(1) $C_\tau(\lambda, r)$ *is r-fspc.*

(2) λ *is r-fspo iff* $\lambda = SPI_\tau(\lambda, r)$.

(3) λ *is r-fspc iff* $\lambda = SPC_\tau(\lambda, r)$.

(4) $I_\tau(\lambda, r) \leqslant SPI_\tau(\lambda, r) \leqslant PI_\tau(\lambda, r) \leqslant \lambda \leqslant PC_\tau(\lambda, r) \leqslant SPC_\tau(\lambda, r) \leqslant C_\tau(\lambda, r)$.

(5) $SPI_\tau(\bar{1} - \lambda, r) = \bar{1} - SPC_\tau(\lambda, r)$ *and* $SPC_\tau(\bar{1} - \lambda, r) = \bar{1} - SPI_\tau(\lambda, r)$.

(6) $C_\tau(SPC_\tau(\lambda, r), r) = SPC_\tau(C_\tau(\lambda, r), r) = C_\tau(\lambda, r)$.

Proof. (1), (2), (3) and (4) follow from the definitions.

(5) For $\lambda \in I^X$, $r \in I_0$ we have the following:

$$\begin{aligned}
\bar{1} - SPI_\tau(\lambda, r) &= \bar{1} - \bigvee\{\, \nu : \nu \leqslant \lambda,\ \nu \text{ is } r\text{-fspo} \,\} \\
&= \bigwedge\{\, \bar{1} - \nu : \bar{1} - \lambda \leqslant \bar{1} - \nu,\ \bar{1} - \nu \text{ is } r\text{-fspc} \,\} \\
&= SPC_\tau(\bar{1} - \lambda, r).
\end{aligned}$$

(6) From (1) and (3), $SPC_\tau(C_\tau(\lambda,r),r)=C_\tau(\lambda,r)$. We only show that

$$C_\tau(SPC_\tau(\lambda,r),r)=C_\tau(\lambda,r).$$

Since $\lambda\leqslant SPC_\tau(\lambda,r)$, $C_\tau(\lambda,r)\leqslant C_\tau(SPC_\tau(\lambda,r),r)$. Suppose that

$$C_\tau(\lambda,r)\not\geqslant C_\tau(SPC_\tau(\lambda,r),r).$$

There exist $x\in X$ and $r\in I_0$ such that

$$C_\tau(\lambda,r)(x)<C_\tau(SPC_\tau(\lambda,r),r)(x).$$

By the definition of C_τ, there exists $\rho\in I^X$ with $\lambda\leqslant\rho$ and $\tau(\bar{1}-\rho)\geqslant r$ such that

$$C_\tau(SPC_\tau(\lambda,r),r)(x)>\rho(x)\geqslant C_\tau(\lambda,r)(x).$$

On the other hand, since $\rho=C_\tau(\rho,r)$, $\lambda\leqslant\rho$ implies

$$SPC_\tau(\lambda,r)\leqslant SPC_\tau(\rho,r)=SPC_\tau(C_\tau(\rho,r),r)=C_\tau(\rho,r)=\rho.$$

Thus,

$$C_\tau(SPC_\tau(\lambda,r),r)\leqslant\rho.$$

It is a contradiction. Hence, $C_\tau(SPC_\tau(\lambda,r),r)\leqslant C_\tau(\lambda,r)$. ■

3. Fuzzy strong precontinuity

Definition 3.1. Let (X,τ) and (Y,τ^*) be fts's and $f\colon X\to Y$ be a mapping. Then, f is called fuzzy strongly precontinuous (resp. fuzzy strongly semi-continuous, fuzzy precontinuous) if $f^{-1}(\mu)$ is an r-fspo (resp. r-fsso, r-fpo) set in X for each $\mu\in I^Y$ and $r\in I_0$ with $\tau^*(\mu)\geqslant r$.

The implications contained in the following diagram are true.

fuzzy continuity   fuzzy strong semi-continuity
⇓   ⇓
fuzzy strong precontinuity
⇓
fuzzy precontinuity

The following examples show that the reverse may not be true in general.

Example 3.1. Let $X=\{a,b\}$. Define $\lambda_1,\lambda_2,\lambda_3,\lambda_4\in I^X$ as follows:

$$\lambda_1(a)=0.2,\quad \lambda_1(b)=0.4,\quad \lambda_2(a)=0.6,\quad \lambda_2(b)=0.5;$$
$$\lambda_3(a)=0.3,\quad \lambda_3(b)=0.4,\quad \lambda_4(a)=0.6,\quad \lambda_4(b)=0.7.$$

Let $\tau,\tau^*\colon I^X\to I$ be fuzzy topologies on X defined as:

$$\tau(\lambda)=\begin{cases}1, & \text{if } \lambda=\bar{0} \text{ or } \bar{1},\\ \frac{1}{2}, & \text{if } \lambda=\lambda_1 \text{ or } \overline{0.4},\\ \frac{2}{3}, & \text{if } \lambda=\lambda_2 \text{ or } \overline{0.7},\\ 1, & \text{otherwise.}\end{cases}\qquad \tau^*(\lambda)=\begin{cases}1, & \text{if } \lambda=\bar{0} \text{ or } \bar{1},\\ \frac{1}{2}, & \text{if } \lambda=\lambda_3,\\ \frac{1}{2}, & \text{if } \lambda=\lambda_4,\\ 0, & \text{otherwise.}\end{cases}$$

Then the identity mapping $id_X : (X,\tau) \to (X,\tau^*)$ is fuzzy precontinuous but not fuzzy strongly precontinuous because: for $0 < r \leqslant \frac{1}{2}$, λ_3 is r-fpo and λ_4 is r-fpo but not r-fspo because

$$\lambda_3 \leqslant I_\tau(C_\tau(\lambda_3, r), r) = I(\tau)(\bar{1} - \lambda_2, r) = \overline{0.4},$$
$$\lambda_4 \leqslant I_\tau(C_\tau(\lambda_4, r), r) = \bar{1},$$
$$\lambda_4 > I_\tau(PC_\tau(\lambda_4, r), r) = I_\tau(\lambda_4, r) = \lambda_2.$$

Example 3.2. We consider Example 2.1 and put

$$\tau^*(\lambda) = \begin{cases} 1, & \text{if } \lambda = \bar{0} \text{ or } \bar{1}, \\ \frac{1}{3}, & \text{if } \lambda = \mu, \\ 0, & \text{otherwise.} \end{cases}$$

For fts (X,τ^*) the identity mapping $id_X : (X,\tau) \to (X,\tau^*)$ is fuzzy strongly precontinuous but it is neither fuzzy continuous nor fuzzy strongly semi-continuous.

Theorem 3.1. *Let (X,τ) and (Y,τ^*) be fts's and $f : X \to Y$. The following statements are equivalent:*

(1) The mapping f is fuzzy strongly precontinuous.

(2) $f^{-1}(\mu)$ is r-fspc in X for each $\mu \in I^Y$, $r \in I_0$ with $\tau^*(\bar{1} - \mu) \geqslant r$.

(3) $f(SPC_\tau(\lambda, r)) \leqslant C_{\tau^*}(f(\lambda), r)$, $\forall \lambda \in I^X$, $r \in I_0$.

(4) $SPC_\tau(f^{-1}(\mu), r) \leqslant f^{-1}(C_{\tau^*}(\mu, r))$, $\forall \mu \in I^Y$, $r \in I_0$.

(5) $f^{-1}(I_{\tau^*}(\mu, r)) \leqslant SPI_\tau(f^{-1}(\mu), r)$, $\forall \mu \in I^Y$, $r \in I_0$.

Proof. (1) $\iff$ (2). Clear.

(1) $\implies$ (3). For all $\lambda \in I^X$, $r \in I_0$, since $\tau^*(\bar{1} - C_{\tau^*}(f(\lambda), r)) \geqslant r$ from the definition of C_{τ^*}, and Definition 1.1 (O3),

$$f^{-1}(\bar{1} - C_{\tau^*}(f(\lambda), r)) = \bar{1} - f^{-1}(C_{\tau^*}(f(\lambda), r))$$

is r-fspo. By Definition 2.3.(2), $f^{-1}(C_{\tau^*}(f(\lambda), r))$ is an r-fspc set in X. Since

$$\lambda \leqslant f^{-1}(f(\lambda)) \leqslant f^{-1}(C_{\tau^*}(f(\lambda), r)),$$

we have $SPC_\tau(\lambda, r) \leqslant f^{-1}(C_{\tau^*}(f(\lambda), r))$. Hence

$$f(SPC_\tau(\lambda, r)) \leqslant f(f^{-1}(C_{\tau^*}(f(\lambda), r))) \leqslant C_{\tau^*}(f(\lambda), r).$$

(3) $\implies$ (4). For $\mu \in I^Y$, $r \in I_0$, let $\lambda = f^{-1}(\mu)$. By (3)

$$f(SPC_\tau(f^{-1}(\mu), r)) \leqslant C_{\tau^*}(f(f^{-1}(\mu)), r) \leqslant C_{\tau^*}(\mu, r).$$

It implies $SPC_\tau(f^{-1}(\mu), r) \leqslant f^{-1}(C_{\tau^*}(\mu, r))$.

(4) $\Longrightarrow$ (5). For $\mu \in I^Y$, $r \in I_0$, by (4) we have

$$f^{-1}(C_{\tau^*}(\bar{1}-\mu,r)) \geqslant SPC_\tau(f^{-1}(\bar{1}-\mu),r) = SPC_\tau(\bar{1}-f^{-1}(\mu),r).$$

Then by Theorem 2.4.(5) we have

$$\begin{aligned} f^{-1}(C_{\tau^*}(\bar{1}-\mu,r)) &\geqslant \bar{1}-SPI_\tau(f^{-1}(\mu),r),\\ \bar{1}-f^{-1}(\bar{1}-I_{\tau^*}(\mu,r)) &\leqslant SPI_\tau(f^{-1}(\mu),r),\\ f^{-1}(I_{\tau^*}(\mu,r)) &\leqslant SPI_\tau(f^{-1}(\mu),r). \end{aligned}$$

(5) $\Longrightarrow$ (1). For each $\mu \in I^Y$, $r \in I_0$ with $\tau^*(\mu) \geqslant r$, since $I_{\tau^*}(\mu,r)=\mu$,

$$f^{-1}(\mu) = f^{-1}(I_{\tau^*}(\mu,r)) \leqslant SPI_\tau(f^{-1}(\mu),r).$$

Hence, by definition of $SPI_\tau(f^{-1}(\mu),r)$, $f^{-1}(\mu) = SPI_\tau(f^{-1}(\mu),r)$. By Theorem 2.4.(2), $f^{-1}(\mu)$ is r-fspo. Therefore f is a fuzzy strongly precontinuous mapping. ■

THEOREM 3.2. *Let (X,τ) and (Y,τ^*) be fts's and $f\colon X \to Y$. The following statements are equivalent:*

(1) The mapping f is fuzzy strongly precontinuous.

(2) $C_\tau(PI_\tau(f^{-1}(\mu),r),r) \leqslant f^{-1}(C_{\tau^*}(\mu,r))$, $\forall \mu \in I^Y$, $r \in I_0$.

(3) $f(C_\tau(PI_\tau(\lambda,r),r)) \leqslant C_{\tau^*}(f(\lambda),r)$, $\forall \lambda \in I^X$, $r \in I_0$.

Proof. The proof is standard and therefore omitted. ■

THEOREM 3.3. *Let (X,τ) and (Y,τ^*) be fts's and $f\colon X \to Y$. The following statements are equivalent:*

(1) The mapping f is fuzzy precontinuous.

(2) $f^{-1}(\mu)$ is r-fpc in X for each $\mu \in I^Y$, $r \in I_0$ with $\tau^*(\bar{1}-\mu) \geqslant r$.

(3) $f(PC_\tau(\lambda,r)) \leqslant C_{\tau^*}(f(\lambda),r)$, $\forall \lambda \in I^X$, $r \in I_0$.

(4) $PC_\tau(f^{-1}(\mu),r) \leqslant f^{-1}(C_{\tau^*}(\mu,r))$, $\forall \mu \in I^Y$, $r \in I_0$.

(5) $f^{-1}(I_{\tau^*}(\mu,r)) \leqslant PI_\tau(f^{-1}(\mu),r)$, $\forall \mu \in I^Y$, $r \in I_0$.

Proof. Similar to the proof of Theorem 3.1. ■

THEOREM 3.4. *Let (X,τ) and (Y,τ^*) be fts's and $f\colon X \to Y$ be a bijective mapping. The following statements are equivalent:*

(1) The mapping f is fuzzy strongly precontinuous.

(2) $I_{\tau^*}(f(\lambda),r) \leqslant f(SPI_\tau(\lambda,r))$, $\forall \lambda \in I^X$, $r \in I_0$.

Proof. (1) $\Longrightarrow$ (2). Let f be a fuzzy strongly precontinuous mapping and $\lambda \in I^X$, $r \in I_0$. Then, $f^{-1}(I_{\tau^*}(f(\lambda),r))$ is an r-fspo set in X. By Theorem 3.1.(5), and the fact that f is injective, we have

$$f^{-1}(I_{\tau^*}(f(\lambda),r)) \leqslant SPI_\tau(f^{-1}(f(\lambda)),r) = SPI_\tau(\lambda,r).$$

Since f is surjective, we have

$$I_{\tau^*}(f(\lambda), r) = f(f^{-1}(I_{\tau^*}(f(\lambda), r))) \leqslant f(SPI_\tau(\lambda, r)).$$

(2) $\Longrightarrow$ (1). Let $\mu \in I^Y$, $r \in I_0$ with $\tau^*(\mu) \geqslant r$. Then $I_{\tau^*}(\mu, r) = \mu$ and by (2) we have

$$\mu = I_{\tau^*}(\mu, r) \leqslant f(SPI_\tau(f^{-1}(\mu), r)).$$

Since f is injective we have

$$f^{-1}(\mu) \leqslant f^{-1}f(SPI_\tau(f^{-1}(\mu), r)) = SPI_\tau(f^{-1}(\mu), r).$$

Then, by the definition of $SPI_\tau(f^{-1}(\mu), r)$ we have $f^{-1}(\mu) = SPI_\tau(f^{-1}(\mu), r)$. By Theorem 2.4.(2) we have $f^{-1}(\mu)$ is an r-fspo set in X. Thus, f is a fuzzy strongly precontinuous mapping. ■

4. Fuzzy strongly preopen and fuzzy strongly preclosed mappings

Definition 4.1. Let (X, τ) and (Y, τ^*) be fts's and $f\colon X \to Y$ be a mapping. Then f is called:

(1) fuzzy strongly preopen (resp. fuzzy strongly semi-open, fuzzy preopen) if $f(\lambda)$ is an r-fspo (resp. r-fsso, r-fpo) set in Y for each $\lambda \in I^X$, $r \in I_0$ with $\tau(\lambda) \geqslant r$.

(2) fuzzy strongly preclosed (resp. fuzzy strongly semi-closed, fuzzy preclosed) if $f(\lambda)$ is an r-fspc (resp. r-fssc, r-fpc) set in Y for each $\lambda \in I^X$, $r \in I_0$ with $\tau(\bar{1} - \lambda) \geqslant r$.

The implications contained in the following diagram are true.

$$\begin{array}{ccc} \text{fuzzy open (closed)} & & \text{fuzzy strongly semi-open (semi-closed)} \\ \Downarrow & & \Downarrow \\ & \text{fuzzy strongly preopen (preclosed)} & \\ & \Downarrow & \\ & \text{fuzzy preopen (preclosed)} & \end{array}$$

The following examples show that the reverse may not be true.

Example 4.1. In Example 3.1, the identity mapping $id_X\colon (X, \tau^*) \to (X, \tau)$ is a fuzzy preopen mapping but not a fuzzy strongly preopen mapping.

Example 4.2. In Example 3.2, the identity mapping $id_X\colon (X, \tau^*) \to (X, \tau)$ is a fuzzy strongly preopen mapping but it is neither fuzzy open nor fuzzy strongly semi-open mapping.

THEOREM 4.1. *Let $f\colon (X,\tau) \to (Y,\tau^*)$ be a mapping. Then the following statements are equivalent:*

(1) f is a fuzzy strongly preopen mapping.

(2) $f(I_\tau(\lambda,r)) \leqslant SPI_{\tau^*}(f(\lambda),r)$ for each $\lambda \in I^X$, $r \in I_0$.

(3) $I_\tau(f^{-1}(\mu),r) \leqslant f^{-1}(SPI_{\tau^*}(\mu,r))$ for each $\mu \in I^Y$, $r \in I_0$.

Proof. (1) $\Longrightarrow$ (2). For all $\lambda \in I^X$, $r \in I_0$, since $\tau(I_\tau(\lambda,r)) \geqslant r$, $f(I_\tau(\lambda,r))$ is r-(τ^*,τ^*)-fspo. From Theorem 2.4.(2),

$$f(I_\tau(\lambda,r)) = SPI_{\tau^*}(f(I_\tau(\lambda,r),r)) \leqslant SPI_{\tau^*}(f(\lambda),r).$$

(2) $\Longrightarrow$ (1). For all $\lambda \in I^X$, $r \in I_0$, with $\tau(\lambda) \geqslant r$ we have $I_\tau(\lambda,r) = \lambda$. From (2), we have

$$f(\lambda) = f(I_\tau(\lambda,r)) \leqslant SPI_{\tau^*}(f(\lambda),r) \leqslant f(\lambda).$$

Then, $f(\lambda) = SPI_{\tau^*}(f(\lambda),r)$, and by Theorem 2.4.(2) we have that $f(\lambda)$ is an r-fspo set in Y. Therefore f is a fuzzy strongly preopen mapping.

(2) $\Longrightarrow$ (3). For all $\mu \in I^Y$, $r \in I_0$, by (2) we have

$$f(I_\tau(f^{-1}(\mu),r)) \leqslant SPI_{\tau^*}(ff^{-1}(\mu),r) \leqslant SPI_{\tau^*}(\mu,r).$$

It implies that $I_\tau(f^{-1}(\mu),r) \leqslant f^{-1}f(I_\tau(f^{-1}(\mu),r)) \leqslant f^{-1}(SPI_{\tau^*}(\mu,r))$.

(3) $\Longrightarrow$ (2). For all $\lambda \in I^X$, $r \in I_0$, by (3) we have

$$I_\tau(\lambda,r) \leqslant I_\tau(f^{-1}(f(\lambda)),r) \leqslant f^{-1}(SPI_{\tau^*}(f(\lambda),r)).$$

It implies that $f(I_\tau(\lambda,r)) \leqslant f(f^{-1}(SPI_{\tau^*}(f(\lambda),r))) \leqslant SPI_{\tau^*}(f(\lambda),r)$. ■

THEOREM 4.2. *Let $f\colon (X,\tau) \to (Y,\tau^*)$ be a mapping. Then the following statements are equivalent:*

(1) f is a fuzzy strongly preclosed mapping.

(2) $SPC_{\tau^*}(f(\lambda),r) \leqslant f(C_\tau(\lambda,r))$ for each $\lambda \in I^X$, $r \in I_0$.

Proof. Similar to the proof of Theorem 4.1. ■

THEOREM 4.3. *Let $f\colon (X,\tau) \to (Y,\tau^*)$ be a bijective mapping. Then the following statements are equivalent:*

(1) f is a fuzzy strongly preclosed mapping.

(2) $f^{-1}(SPC_{\tau^*}(\mu,r)) \leqslant C_\tau(f^{-1}(\mu),r)$ for each $\mu \in I^Y$, $r \in I_0$.

Proof. (1) $\Longrightarrow$ (2). For all $\mu \in I^Y$, $r \in I_0$, since $\tau(\bar{1} - C_\tau(f^{-1}(\mu),r)) \geqslant r$, from (1) we have $f(C_\tau(f^{-1}(\mu),r))$ is an r-fspc set in Y. Then by Theorem 2.4.(3) we have

$$f(C_\tau(f^{-1}(\mu),r)) = SPC_{\tau^*}(f(C_\tau(f^{-1}(\mu),r)),r) \geqslant SPC_{\tau^*}(ff^{-1}(\mu),r).$$

By the surjectivity of f we have

$$f(C_\tau(f^{-1}(\mu),r)) \geqslant SPC_{\tau^*}(\mu,r).$$

Also, by injectivity of f we have

$$f^{-1}(SPC_{\tau}(\mu,r)) \leqslant f^{-1}(f(C_{\tau}(f^{-1}(\mu),r))) = C_{\tau}(f^{-1}(\mu),r).$$

(2) $\Longrightarrow$ (1). For all $\lambda \in I^X$, $r \in I_0$ with $\tau(\bar{1}-\lambda) \geqslant r$, from (2) we have

$$f^{-1}(SPC_{\tau^*}(f(\lambda),r)) \leqslant C_{\tau}(f^{-1}(f(\lambda)),r).$$

By the injectivity of f we have

$$f^{-1}(SPC_{\tau^*}(f(\lambda),r)) \leqslant C_{\tau}(\lambda,r) = \lambda$$

and by surjectivity of f we have

$$SPC_{\tau^*}(f(\lambda),r) \leqslant f(\lambda) \leqslant SPC_{\tau^*}(f(\lambda),r).$$

Then $f(\lambda) = SPC_{\tau^*}(f(\lambda),r)$. Hence, $f(\lambda)$ is an r-fspc set in Y. So, f is a fuzzy strongly preclosed mappimng. ∎

Theorem 4.4. *Let $f\colon (X,\tau) \to (Y,\tau^*)$ be a mapping. Then f is fuzzy strongly preopen iff for each $\nu \in I^Y$ and each $\lambda \in I^X$, $r \in I_0$ with $\tau(\bar{1}-\lambda) \geqslant r$, when $f^{-1}(\nu) \leqslant \lambda$, there exists an r-fspc set μ in Y such that $\nu \leqslant \mu$ and $f^{-1}(\mu) \leqslant \lambda$.*

Proof. Suppose that f is a fuzzy strongly preopen mapping, $\nu \in I^Y$ and $\lambda \in I^X$, $r \in I_0$ with $\tau(\bar{1}-\lambda) \geqslant r$ such that $f^{-1}(\nu) \leqslant \lambda$. Then, $f(\bar{1}-\lambda) \leqslant f(f^{-1}(\bar{1}-\nu)) \leqslant \bar{1}-\nu$. Since $\tau(\bar{1}-\lambda) \geqslant r$ and f is a fuzzy strongly preopen mapping, then $f(\bar{1}-\lambda)$ is an r-fspo set in Y. So,

$$\begin{gathered} f(\bar{1}-\lambda) = SPI_{\tau^*}(f(\bar{1}-\lambda),r) \leqslant SPI_{\tau^*}(\bar{1}-\nu,r), \\ \bar{1}-\lambda \leqslant f^{-1}(f(\bar{1}-\lambda)) \leqslant f^{-1}(SPI_{\tau^*}(\bar{1}-\nu,r)), \\ \bar{1}-f^{-1}(SPI_{\tau^*}(\bar{1}-\nu,r)) = f^{-1}(\bar{1}-SPI_{\tau^*}(\bar{1}-\nu,r)) \leqslant \lambda, \\ f^{-1}(SPC_{\tau^*}(\nu,r)) \leqslant \lambda. \end{gathered}$$

Let $\mu = SPC_{\tau^*}(\nu,r)$. Then, μ is an r-fspc set in Y, $\nu \leqslant SPC_{\tau^*}(\nu,r) = \mu$ and $f^{-1}(\mu) \leqslant \lambda$.

Conversely, for all $\eta \in I^X$, $r \in I_0$ with $\tau(\eta) \geqslant r$, take $\lambda = \bar{1}-\eta \in I^X$ and $\nu = \bar{1}-f(\eta) \in I^Y$. Then, $\tau(\bar{1}-\lambda) \geqslant r$ and since $\eta \leqslant f^{-1}f(\eta)$ we have

$$\bar{1}-f^{-1}(f(\eta)) = f^{-1}(\bar{1}-f(\eta)) \leqslant \bar{1}-\eta,$$

which implies that $f^{-1}(\nu) \leqslant \lambda$. Then there exists an r-fspc set in Y such that $\nu = \bar{1}-f(\eta) \leqslant \mu$ and $f^{-1}(\mu) \leqslant \bar{1}-\eta$. Then, $\bar{1}-\mu = f(\eta)$. But μ is an r-fspc set in Y and so $\bar{1}-\mu = f(\eta)$ is an r-fspo set in Y. Hence, f is a fuzzy strongly preopen mapping. ∎

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