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# INTEGRAL TRANSFORMS AND SUMMATION OF SOME SCHLÖMILCH SERIES

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**Abstract.** In this paper we present a survey of the results given in the papers [11, 12, 13, 14]. Connections between integral transforms and some Schlömilch series have also been considered. These series are represented in terms of the Riemann zeta and related functions of reciprocal powers and can be brought in so called closed form in certain cases, which means that the infinite series are represented by finite sums. As applications of our results, recursive relations of some related functions and the sums of new Schlömilch series involving the Neumann, MacDonald, Struve or Bessel functions are given as well.

## 1. Introduction

Bessel functions have been required in recent years in various problems of mathematical physics, acoustics, hydrodynamics, radio-physics, nuclear physics, etc. Numerical values of sums involving Bessel or Struve functions are particularly required in certain problems of telecommunication theory, electrostatics, etc. For this reason it is useful to have results in closed form. By combining some integral transforms (trigonometric, Laplace, Bessel, Mellin) and Bessel functions we can obtain the sums of new series.

We will consider certain types of the Schlömilch series (see [15]) in the form of $\sum_{n=1}^{\infty} q_n B_\nu((an-b)x)$, where $B_\nu$ denotes the Bessel $J_\nu$, Neumann $Y_\nu$, MacDonald $K_\nu$ or Struve $\mathbf{H}_\nu$ function, and $q_n$ is a sequence of reciprocal powers of natural numbers $an-b$, where $a=1$ and $b=0$ or $a=2$ and $b=1$. In more general cases $q_n$ is multiplied by $1/((an-b)^2-\omega^2)$, with $\omega \in R$ and $\omega \neq an-b$.

We first sum the series (see [11])

$$S_\alpha^\varphi = \sum_{n=1}^{\infty} \frac{(s)^{n-1}\varphi_\nu((an-b)x)}{(an-b)^\alpha}, \qquad S_{\alpha,\omega}^\varphi = \sum_{n=1}^{\infty} \frac{(s)^{n-1}\varphi_\nu((an-b)x)}{(an-b)^\alpha((an-b)^2-\omega^2)}, \tag{1}$$

where $\alpha \in R$, $a=\left\{\begin{smallmatrix}1\\2\end{smallmatrix}\right\}$ $b=\left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right\}$, $s=1$, or $-1$, $\omega \in R$, $\omega \neq an-b$, $\varphi_\nu$ denotes Bessel $J_\nu(x)$ or Struve $\mathbf{H}_\nu(x)$ functions of the first kind and of order $\nu$. We evaluate and represent these series as the series in terms of the Riemann zeta function and related functions of reciprocal powers. The obtained sums can be brought in certain cases in closed form. By applying integrals containing trigonometric or Bessel/Struve

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functions, by means of the above two series, we find the sums of new series (10) (see [12]) and (20) (see [15]), respectively. Applications of our results to finding the sums of new Schlömilch series are also given in the last section.

## 2. Sums of some trigonometric series

The method for obtaining the first sum in (1) relies on the summation of trigonometric series. We use the following formula (see [11])

$$T_\alpha^f = \sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an-b)x)}{(an-b)^\alpha} = \frac{c\pi x^{\alpha-1}}{2\Gamma(\alpha) f(\frac{\pi\alpha}{2})} + \sum_{i=0}^{\infty} \frac{(-1)^i F(\alpha-2i-\delta)}{(2i+\delta)!} x^{2i+\delta}, \tag{2}$$

where $f = \left\{ {\sin \atop \cos} \right\}$ $\delta = \left\{ {1 \atop 0} \right\}$, $\alpha \in R^+$, and all relevant parameters are given in Table I, in which $\zeta, \eta, \lambda$ and $\beta$ represent the Riemann zeta function and related functions (see [1]).

A special, but frequent case is a truncation of the right-hand side series and it is because functions $F$ take the value zero at certain points (see Table II), when we obtain a closed form formula

$$T_\alpha^f = (-1)^m \frac{c\pi x^{\alpha-1}}{2(\alpha-1)!} + \sum_{i=0}^{m} \frac{(-1)^i F(\alpha-2i-\delta)}{(2i+\delta)!} x^{2i+\delta}, \quad f = \left\{ {\sin \atop \cos} \right\} \delta = \left\{ {1 \atop 0} \right\}. \tag{3}$$

Table I

Table II: closed form case

Applying integral transforms one can obtain various trigonometric series. For instance, if in (3) we take $f = \cos, \delta = 0, s = 1, \alpha = 1, b = 0, F = \zeta,\ c = 1$, then apply the Laplace transform, we obtain

$$\sum_{n=1}^{\infty} \frac{p}{n^\alpha (p^2+n^2)} = (-1)^{\alpha/2} \frac{\pi}{2p^\alpha} + \sum_{i=0}^{m} \frac{(-1)^i \zeta(\alpha-2i)}{p^{2i+1}}. \tag{4}$$

Now we apply the inverse Mellin transform to this series, knowing that

$$M^{-1}\left(\frac{z}{z^2+n^2}\right) = \begin{cases} \cos(n \log x) & x < 1 \\ 0 & x > 1, \end{cases} \quad M^{-1}\left(\frac{1}{z^\nu}\right) = \begin{cases} \frac{(\log \frac{1}{x})^{\nu-1}}{\Gamma(\nu)} & x < 1 \\ 0 & x > 1, \end{cases}$$

(see [6], p. 166, 2.16 and p. 167, 2.25), coming to the sum of a trigonometric series

$$\sum_{n=1}^{\infty} \frac{\cos(n \log x)}{n^\alpha} = \frac{(-1)^{\alpha/2}\pi}{2(\alpha-1)!} \left(\log \frac{1}{x}\right)^{\alpha-1} + \sum_{i=0}^{M} \frac{(-1)^i \zeta(\alpha-2i)}{(2i)!} \left(\log \frac{1}{x}\right)^{2i}, \quad x < 1.$$

In order to sum the second series in (1), we require the following formula first obtained in [11]

$$\sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an-b)x)}{(an-b)^{\alpha}((an-b)^2-\omega^2)} = \frac{sd(1-b)}{2\omega^{2m+2}} - \frac{s\pi \sin^{b-1}\frac{\pi\omega}{2}}{4\omega^{2m+d}\cos\frac{\pi\omega}{2}} f\left(\omega x - \tfrac{\pi(s+1)(b+\omega)}{2a}\right)$$
$$+\frac{c\pi}{2}\sum_{i=1}^{m}\frac{(-1)^{i+d}x^{2i+d-2}}{\omega^{2m-2i+2}(2i+d-2)!} - \sum_{i=1}^{m}\sum_{k=0}^{M}\frac{(-1)^k F(2i-2k+d-1-\delta)}{\omega^{2m-2i+2}(2k+\delta)!}x^{2k+\delta}, \qquad (5)$$

where $\alpha = 2m+d-1$, $\omega \in R$, $\omega \neq an-b$, $f = \left\{\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\right\}$ $\delta = \left\{\begin{smallmatrix}1\\ 0\end{smallmatrix}\right\}$, $M = i-1+d(1-\delta)$ and the rest of the parameters are given in Table I.

## 3. Series over Bessel/Struve functions and some integrals

We find the sums of series over Bessel or Struve functions (1) as series in terms of the Riemann zeta and related functions. They reduce to closed form formulas in certain cases.

To obtain the sum of the first series in (1) we use the method described in [9] and [10], where the series involve only Bessel functions. However, in [11] a more general result comprising Struve functions was given. There it was started with the known integral representation of Bessel/Struve functions (see [1])

$$\varphi_\nu(z) = \frac{2\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^{\pi/2}\sin^{2\nu}\theta\, f(z\cos\theta)\,d\theta, \quad \mathrm{Re}\,\nu > -\frac{1}{2}, \qquad (6)$$

$\varphi_\nu = \left\{\begin{smallmatrix}J_\nu\\ \mathbf{H}_\nu\end{smallmatrix}\right\}$ $f = \left\{\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\right\}$. Substituting (6) in (1) and interchanging the order of summation and integration we came to the trigonometric series, the sum of which is given by (2) for $x\cos\theta$ instead of $x$ and for $\alpha-\nu$ in place of $\alpha$. Finally, we obtain the general formula for the summation of the first series (1)

$$S_\alpha^\varphi = \frac{c\pi\left(\frac{x}{2}\right)^{\alpha-1}}{2\Gamma\left(\frac{\alpha-\nu+1}{2}\right)\Gamma\left(\frac{\alpha+\nu+1}{2}\right)f\left(\pi\frac{\alpha-\nu}{2}\right)} + \sum_{i=0}^{\infty}\frac{(-1)^i F(\alpha-\nu-2i-\delta)\left(\frac{x}{2}\right)^{\nu+2i+\delta}}{\Gamma\left(i+1+\frac{\delta}{2}\right)\Gamma\left(\nu+i+1+\frac{\delta}{2}\right)}, \qquad (7)$$

where $\alpha,\nu \in R^+$, $\nu > -\frac{1}{2}$, $\alpha > \nu$, $\varphi_\nu = \left\{\begin{smallmatrix}J_\nu\\ \mathbf{H}_\nu\end{smallmatrix}\right\}$ $f = \left\{\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\right\}$ $\delta = \left\{\begin{smallmatrix}0\\ 1\end{smallmatrix}\right\}$, $s, a, b, c, F$ are given in Table I. When $\alpha-\nu = \left\{\begin{smallmatrix}2k+1\\ 2k\end{smallmatrix}\right\}$, $k \in N_0$, which corresponds to $f = \left\{\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\right\}$, one should work with limiting values or with principal values of gamma functions. Even in the case $\alpha-\nu = 1$ formula (7) is the correct one. Truncation of the second series in (7) due to vanishing of $F$ functions gives all closed form cases (see Table II for $\alpha-\nu$ instead of $\alpha$). If we take, for example, $\varphi_\nu = J_\nu$, there follows $f = \cos$, $\delta = 0$, and we have

$$\sum_{n=1}^{\infty}\frac{(s)^{n-1}J_\nu((an-b)x)}{(an-b)^\alpha} = \frac{c\pi x^{\alpha-1}}{2^\alpha\Gamma\left(\frac{\alpha-\nu+1}{2}\right)\Gamma\left(\frac{\alpha+\nu+1}{2}\right)\cos\left(\frac{\alpha-\nu}{2}\pi\right)} + \sum_{i=0}^{\infty}\frac{(-1)^i F(\alpha-\nu-2i)x^{\nu+2i}}{2^{\nu+2i}\,i!\,\Gamma(\nu+i+1)}.$$

By applying the Laplace transform we obtain

$$\sum_{n=1}^{\infty}\frac{(s)^{n-1}\left(\sqrt{p^2+(an-b)^2}-p\right)^\nu}{(an-b)^{\alpha+\nu}\sqrt{p^2+(an-b)^2}} = \frac{c\pi\Gamma(\alpha)}{G} + \sum_{i=0}^{\infty}\frac{(-1)^i F(\alpha-\nu-2i)\Gamma(\nu+2i+1)}{2^{\nu+2i}\,p^{\nu+2i+1}\,i!\,\Gamma(\nu+i+1)},$$

where $G = (2p)^\alpha\Gamma\left(\frac{\alpha-\nu+1}{2}\right)\Gamma\left(\frac{\alpha+\nu+1}{2}\right)\cos\left(\frac{\alpha-\nu}{2}\pi\right)$.

For the second series in (1), using (6) and then (5) for $\mu+\nu=1-d$, we derived the result

$$S^{\varphi}_{2m-\mu,\omega}=\frac{(x/2)^{\nu}}{\omega^{2m+2}}\Big[\frac{s(1-\mu-\nu)}{2\Gamma(\nu+1)}(1-b)+\frac{c\sqrt{\pi}}{2}\sum_{i=1}^{m}A_i-\frac{1}{\sqrt{\pi}}\sum_{i=1}^{m}\sum_{k=0}^{M}B_{ik}\Big]$$
$$-\frac{s\pi\omega^{\mu-2m-1}\sin^{b-1}\frac{\pi\omega}{2}}{4\cos\frac{\pi\omega}{2}}\Big[(-1)^{\delta}f\Big(\frac{\pi(\omega+b)(s+1)}{2a}\Big)J_{\nu}(\omega x)+\overline{f}\Big(\frac{\pi(\omega+b)(s+1)}{2a}\Big)\mathbf{H}_{\nu}(\omega x)\Big], \quad (8)$$

which was first obtained in [11]. The method was similar to that used for the first series in (1). For the sake of brevity, in the formula (8) we have put

$$A_i=\frac{(-1)^{i+1-\mu-\nu}\Gamma(i-\frac{\mu+\nu}{2})\omega^{2i}x^{2i-\mu-\nu-1}}{(2i-\mu-\nu-1)!\,\Gamma(i+\frac{\nu-\mu+1}{2})},\ B_{ik}=\frac{(-1)^{k}F(2i-2k-\mu-\nu-\delta)\Gamma(k+\frac{\delta+1}{2})\omega^{2i}x^{2k+\delta}}{(2k+\delta)!\,\Gamma(\nu+k+1+\frac{\delta}{2})}.$$

In (8), $\mathrm{Re}\,\nu>-\frac{1}{2}$, $\varphi_{\nu}=\left\{{J_{\nu} \atop \mathbf{H}_{\nu}}\right\}$ $f=\left\{{\cos \atop \sin}\right\}$ $\overline{f}=\left\{{\sin \atop \cos}\right\}$ $\delta=\left\{{0 \atop 1}\right\}$, $M=i-1+(1-\mu-\nu)(1-\delta)$ and the rest of the parameters are in Table I, where $d=1-\mu-\nu$.

We are going to apply an integral transform of the function $\psi$, i.e. we will use trigonometric integrals, defined by

$$S(x)=\int_0^1\psi(y)\sin xy\,dy,\quad C(x)=\int_0^1\psi(y)\cos xy\,dy. \quad (9)$$

If we choose $\psi(y)=(1-y^2)^{\nu-1/2}$, than introduce $y=\cos\theta$ and then multiply integrals (9) by $\frac{2(x/2)^{\nu}}{\Gamma(1/2)\Gamma(\nu+1/2)}$, we obtain (6). Thus we consider integrals (9) a generalization of Bessel and Struve functions. If we put these integrals in place of Bessel or Struve functions $\varphi_{\nu}$ in both series (1), we come to the series (see [12])

$$S_{T_0}=\sum_{n=1}^{\infty}\frac{(s)^{n-1}T((an-b)x)}{(an-b)^{\alpha}},\quad S_T=\sum_{n=1}^{\infty}\frac{(s)^{n-1}T((an-b)x)}{(an-b)^{2m+p-1}((an-b)^2-\omega^2)}, \quad (10)$$

where $\alpha,\omega\in R$, $\omega\neq an-b$, $\alpha>0$, $m\in N$. Parameters $s,a,b$ are given in Table I, treating $p$ as $d$. Here $T(x)$ denotes $S(x)$ or $C(x)$ (zero in the index of $S_{T_0}$ denotes that $\omega=0$).

We are going to set out a method for representing the series (10) as series in terms of Riemann zeta and related functions. This method is based on the formulas (2) and (5). By substituting in the first series in (10) the integral of the form (9) and interchanging the order of summation and integration, we get an integral where a part of the integrand is the series of the type (2), which means that we can use (2) for $xy$ instead of $x$ (the boundaries for $xy$ are the same as those for $x$ in Table I, because of $0\le y\le 1$). Thus, we get the final formula for the summation of the first series in (10):

$$S_{T_0}=\frac{c\pi x^{\alpha-1}}{2\Gamma(\alpha)f(\frac{\pi\alpha}{2})}\int_0^1 y^{\alpha-1}\psi(y)\,dy+\sum_{i=0}^{\infty}\frac{(-1)^iF(\alpha-2i-\delta)x^{2i+\delta}}{(2i+\delta)!}\int_0^1 y^{2i+\delta}\psi(y)\,dy, \quad (11)$$

where $\alpha\in R^{+}, f=\left\{{\sin \atop \cos}\right\}$ $T(x)=\left\{{S(x) \atop C(x)}\right\}$ $\delta=\left\{{1 \atop 0}\right\}$. The other parameters are in Table I. In some cases, listed in Table II, when the right-hand side series truncate due to the vanishing of $F$ functions, representation (11) takes on a closed form where instead of $\Gamma(\alpha)f(\frac{\pi\alpha}{2})$ we have $(-1)^{\frac{\alpha-\delta}{2}}(\alpha-1)!$ and the sum goes to $m\in N_0$. For instance, if we choose $a=1, b=0, s=1$, from Table I there follows $c=1, F=\zeta$. If we choose $T=S$, then $f=\sin$, $\delta=1$ and from Table II, $\alpha=2m+1$. So we find

$$\sum_{n=1}^{\infty}\frac{S(nx)}{n^{2m+1}}=(-1)^m\frac{\pi x^{2m}}{2(2m)!}\int_0^1 y^{2m}\psi(y)\,dy+\sum_{i=0}^{m}\frac{(-1)^i\zeta(2m-2i)x^{2i+1}}{(2i+1)!}\int_0^1 y^{2i+1}\psi(y)\,dy,$$

which means the infinite series (first in (10)) can be represented by a finite sum.

We will apply the same procedure to the summation of a special type of the second series in (10), when $m=0$. In the paper [11] we stated a general formula for the summation of the series (5) for $m=0$ (it is formula (5) for $m=0$). We use this formula for $xy$ instead of $x$. Further we apply the well known trigonometric formulas, which can be written in the form of $f(\alpha-\beta)=f(\alpha)\cos\beta\mp\overline{f}(\alpha)\sin\beta$, where $f=\left\{{\sin\atop\cos}\right\}$ $\overline{f}=\left\{{\cos\atop\sin}\right\}$, and obtain finally

$$\sum_{n=1}^{\infty}\frac{(s)^{n-1}T((an-b)x)}{(an-b)^{p-1}(an-b)^2-\omega^2}=\frac{sp(1-b)}{2\omega^2}C(0)-\frac{s\pi\sin^{b-1}(\pi\omega/2)}{4\omega^p\cos(\pi\omega/2)}\left(T(\omega x)\cos\omega_1\mp\overline{T}(\omega x)\sin\omega_1\right), \tag{12}$$

where $p=0$ or $p=1$ and $\omega_1=\frac{\pi(s+1)(b+\omega)}{2a}$. Here $T(x)$ and $\overline{T}(x)$ are integrals (9).

Similarly, using (5) this time and denoting by $\Phi$ the right-hand side of the formula (12), we finally get the following formula for the summation of the second series in (10):

$$S_T=\frac{\Phi}{\omega^{2m}}+\sum_{i=1}^{m}\frac{1}{\omega^{2m-2i+2}}\left(\frac{(-1)^{i+p}c\pi x^{2i+p-2}}{2(2i+p-2)!}I_{2i+p-2}-\sum_{k=0}^{L}\frac{(-1)^kF(2i+p-1-2k-\delta)x^{2k+\delta}}{(2k+\delta)!}I_{2k+\delta}\right) \tag{13}$$

where $I_\nu=\int_0^1 y^\nu\psi(y)dy$, $m\geq 1, \omega\in R, \omega\neq an-b, \delta=\left\{{1\atop 0}\right\}, T(x)=\left\{{S(x)\atop C(x)}\right\}$, $\overline{T}(x)=\left\{{C(x)\atop S(x)}\right\}$, $L=i-1+p(1-\delta)$ and the other relevant parameters are given in Table I, treating $p$ as $d$. Note that (12) can be obtained from the general formula (13) for $m=0$.

By setting the Bessel $J_\nu$ and Struve function $\mathbf{H}_\nu$ in place of sin and cos in (9), we have

$$B_\nu(x)=\int_0^1 J_\nu(xy)\psi(y)dy,\quad S_\nu(x)=\int_0^1\mathbf{H}_\nu(xy)\psi(y)dy. \tag{14}$$

Now we can obtain the sums of both series (1), where the Bessel or Struve functions are replaced by integrals (14). Namely, using formulas (7) and (8) for the summation of two series (1), we can sum the following series (see [13])

$$I_\alpha^D=\sum_{n=1}^{\infty}\frac{(s)^{n-1}D_\nu((an-b)x)}{(an-b)^\alpha},\quad I_{\alpha,\omega}^D=\sum_{n=1}^{\infty}\frac{(s)^{n-1}(an-b)^\mu D_\nu((an-b)x)}{(an-b)^{2m}((an-b)^2-\omega^2)}, \tag{15}$$

where $D_\nu(x)$ is $B_\nu(x)$ or $S_\nu(x)$, $a=\left\{{1\atop 2}\right\}$ $b=\left\{{0\atop 1}\right\}$, $s=1$ or $-1$, $\mu,\nu,\omega\in R$, $\omega\neq an-b, m\in N$, $\alpha=2m-\mu$.

Thus we get the final formula for the summation of the first series in (15)

$$I_\alpha^D=\frac{c\pi x^{\alpha-1}}{2^\alpha\Gamma(\frac{\alpha-\nu+1}{2})\Gamma(\frac{\alpha+\nu+1}{2})f(\frac{\pi(\alpha-\nu)}{2})}I_{\alpha-1}+\sum_{i=0}^{\infty}\frac{(-1)^iF(\alpha-\nu-2i-\delta)x^{\nu+2i+\delta}}{2^{\nu+2i+\delta}\Gamma(i+1+\frac{\delta}{2})\Gamma(\nu+i+1+\frac{\delta}{2})}I_{\nu+2i+\delta} \tag{16}$$

where $D_\nu=\left\{{B_\nu\atop S_\nu}\right\}$ $f=\left\{{\cos\atop\sin}\right\}$ $\delta=\left\{{0\atop 1}\right\}$ and $I_\nu$ is the same as in (13). The other parameters you can look up in Table I. This formula takes a closed form in the same cases as the formula (7).

Further we give the formula for the summation of the second series in (15)

$$I^D_{\alpha,\omega} = \frac{(x/2)^\nu}{\omega^{2m+2}}\Big[K(1-b) + \frac{c\sqrt{\pi}}{2}\sum_{i=1}^{m} W_i - \frac{1}{\sqrt{\pi}}\sum_{i=1}^{m}\sum_{k=0}^{M} Z_{ik}\Big] - P\frac{s\pi\omega^{\mu-2m-1}\sin^{b-1}\frac{\pi\omega}{2}}{4\cos\frac{\pi\omega}{2}}, \qquad (17)$$

where we have denoted $P = (-1)^\delta f(\omega_1)B_\nu(\omega x) + \overline{f}(\omega_1)S_\nu(\omega x)$, $K = \frac{s(1-\mu-\nu)}{2\Gamma(\nu+1)}I_\nu$, $W_i = A_i I_{2i-\mu-\nu-1}$, $Z_{ik} = B_{ik}I_{2k+\delta}$, with $A_i$ and $B_{ik}$ introduced previously in the formula (8), $\omega_1$ in (12), $I_\nu$ in (13), $D_\nu(x) = \left\{\begin{smallmatrix} B_\nu(x) \\ S_\nu(x)\end{smallmatrix}\right\}$ $f = \left\{\begin{smallmatrix}\cos \\ \sin\end{smallmatrix}\right\}$ $\overline{f} = \left\{\begin{smallmatrix}\sin \\ \cos\end{smallmatrix}\right\}$ $\delta = \left\{\begin{smallmatrix}0 \\ 1\end{smallmatrix}\right\}$, $M = i - 1 + (1-\mu-\nu)(1-\delta), m \in N_0$, $\alpha = 2m - \mu$. The other parameters you can look up in Table I, where $d = 1 - (\mu+\nu)$.

## 4. Applications

Many particular cases of our formula (7) are cited in the literature. However, they include only those cases with $\alpha = \nu + m$, $m \in N$, while (7) holds true for $\alpha > \nu > -\frac{1}{2}$, $\alpha > 0$. For instance, formula (7) for $a = 0$, $b = 0$, $s = \pm 1$, $\alpha = \nu + 2k$, $\varphi_\nu = J_\nu$, gives the sums 13. and 14. from [8], p. 678. These two particular cases were proved by induction in [lorch], although they were already known. Notice that the general formula (7) for $\varphi_\nu = J_\nu$ is the formula (6) in [10], p. 384. For $\varphi_\nu = \mathbf{H}_\nu$ in [4] were proved by induction two particular cases of (7), for $\alpha = \nu + 2k, k \in N, a = 1, b = 0, s = \pm 1$. In [8] there is only one particular case of (8), formula 24, p. 679. Note that this is $S^J_{\nu,p}, a = 1, b = 0, s = \pm 1$ in our notation, and there are no cases for $m \neq 0$. In [3] is given the sum $S^J_{2m+\nu,p}, a = 1, b = 0, s = -1, m \neq 0$ and in [4] four particular cases, $S^\varphi_{2m+\nu,c}, a = 1, b = 0, s = \pm 1, m \neq 0$.

For certain values of $x$ formulas (3) and (5) become formulas for the summation of numerical series and that fact we used for obtaining some recursion relations for the Riemann zeta and other functions. Moreover, the derived summation formulas (3) and (5) also make it possible to find relations between any two functions of reciprocal powers.

Example 1. By taking $f = \cos$ in (3) and putting $x = \pi$ if $F = \zeta, \eta, \lambda$ and $x = \pi/2$ if $F = \beta$ one obtains recursive relations

$$F(2m+d) = c\frac{(-1)^{m+1}\pi^{2m}}{2(2m)!} + \sum_{i=1}^{m} \frac{(-1)^{i+1}F(2m-2i+d)\pi^{2i}}{2^{2id}(2i+1-d)!}, \quad m \geq 1$$

where $d = 0$ for $F = \zeta, \eta, \lambda$ and $d = 1$ for $F = \beta$. Parameter $c$ is in the Table I.

Example 2. Next, we use formula (5) where we take $a = 1, b = 0, s = -1, f = \cos, p = 1$ and put $x = \pi$. So we get

$$\sum_{n=1}^{\infty} \frac{1}{n^{2m}(n^2-\omega^2)} = \frac{1}{2\omega^{2m+2}} - \frac{\pi \operatorname{ctg} \omega\pi}{2\omega^{2m+1}} + \sum_{i=1}^{m} \frac{1}{\omega^{2m-2i+2}} \sum_{j=0}^{i} \frac{(-1)^j \eta(2i-2j)}{(2j)!}\pi^{2j}. \qquad (18)$$

Here we have used the summation formula (3) for the same choice of parameters. Using the partial fractions decomposition of the left-hand side summation term and formula 4 in [7], p. 685, the left-hand side of (18) becomes

$$\sum_{n=1}^{\infty} \frac{1}{n^{2m}(n^2-\omega^2)} = \frac{1}{2\omega^{2m+2}} - \frac{\pi \operatorname{ctg} \pi\omega}{2\omega^{2m+1}} - \sum_{i=1}^{m} \frac{\zeta(2i)}{\omega^{2m-2i+2}}.$$

Comparing it to formula (18) we obtain $\zeta(2i) = \sum_{j=0}^{i} \frac{(-1)^{j+1}\eta(2i-2j)}{(2j)!}\pi^{2j}$.

EXAMPLE 3. One can apply our summation formula (7) to finding sums of new series. So we consider, for instance (see formula 35 in [8], p. 179)

$$\int_0^\infty \frac{x^{\nu+1}}{x^2-y^2} J_\nu(nx)dx = -\frac{\pi}{2} y^\nu Y_\nu(ny),\ n\in N, y>0,\ -1<\mathrm{Re}\,\nu<\frac{3}{2},$$

where $Y_\nu$ is Bessel function of the second kind, also called the Neumann function. By multiplying both sides by $1/n^\alpha$, then taking the sum, we have

$$\int_0^\infty \frac{x^{\nu+1}}{x^2-y^2} \sum_{n=1}^\infty \frac{J_\nu(nx)}{n^\alpha}\,dx = -\frac{\pi}{2} y^\nu \sum_{n=1}^\infty \frac{Y_\nu(ny)}{n^\alpha}.$$

Now we apply (7) to the left-hand side sum, where $\alpha-\nu=2m$, $m\in N$, $\varphi_\nu = J_\nu$ which implies $f=\cos, \delta=0$. This is the case when $a=1, b=0, s=1$ whence we must take $c=1, F=\zeta$. In further evaluation we need the integral $\int_0^{+\infty} \frac{x^\mu}{x^2-y^2} = \frac{\pi|y|^{\mu+1}\,\mathrm{tg}\,\frac{\mu\pi}{2}}{2y^2}$ and finally get

$$\sum_{n=1}^\infty \frac{Y_\nu(ny)}{n^{2m+\nu}} = \frac{(-1)^{m-1}\pi|y|^{2m+2\nu+1}\,\mathrm{tg}(m+\nu)\pi}{2^{2m+\nu}y^{\nu+2}\Gamma(m+\frac{1}{2})\Gamma(m+\nu+\frac{1}{2})} - \sum_{i=0}^m \frac{(-1)^i\zeta(2m-2i)|y|^{2\nu+2i+2}\,\mathrm{tg}(\nu+i+\frac{1}{2})\pi}{2^{\nu+2i}y^{\nu+2}i!\Gamma(\nu+i+1)}.$$

EXAMPLE 4. Similarly, if we take the formula 28 in [8], p. 179, and repeat the preceding procedure, we have

$$\int_0^{+\infty} \frac{x^\nu+1}{(x^2+z^2)^\rho} \sum_{n=1}^\infty \frac{J_\nu(nx)}{n^\alpha}\,dx = \frac{z^{\nu-\rho+1}}{2^{\rho-1}\Gamma(\rho)} \sum_{n=1}^\infty \frac{K_{\nu-\rho+1}(nz)}{n^{\alpha-\rho+1}}.$$

where $K_\nu(z)$ is MacDonald function. By using in the process the integral

$$\int_0^{+\infty} \frac{x^\mu\,dx}{(x^2+z^2)^\rho} = \frac{|z|^{\mu+1}\Gamma(\frac{1}{2}+\frac{\mu}{2})\Gamma(-\frac{1}{2}-\frac{\mu}{2}+\rho)}{2z^{2\rho}\Gamma(\rho)},$$

we obtain

$$\sum_{n=1}^\infty \frac{K_{\nu-\rho+1}(nz)}{n^{2m+\nu-\rho+1}} = \frac{(-1)^m\pi|z|^{2m+2\nu+1}\Gamma(\rho-m-\nu-\frac{1}{2})}{2^{2m+\nu-\rho+2}z^{\nu+\rho+1}\Gamma(m+\frac{1}{2})} + \sum_{i=0}^m \frac{(-1)^i\zeta(2m-2i)\Gamma(\rho-\nu-i-1)}{2^{\nu+2i-\rho+2}z^{\nu+\rho+1}\,i!\,|z|^{-2\nu-2i-2}}.$$

Some series of the type (15) containing special types of functions instead of Bessel or Struve integrals can take analogous representations over Riemann zeta and related functions by specifying the function $\psi(y)$. We give some examples.

EXAMPLE 5. For instance, we consider $\psi(y) = y^{1-\nu}(1-y^2)^{-1/2}$ in (14). Since (see [2], p. 702) $\int_0^1 J_\nu(xy)y^{1-\nu}(1-y^2)^{-1/2}\,dy = \sqrt{\pi/2x}\,\mathbf{H}_{\nu-1/2}(x)$, we have $B_\nu(x) = \sqrt{\pi/2x}\,\mathbf{H}_{\nu-1/2}(x)$. Replacing this in the first series in (15) we get

$$I_\alpha^B = \sum_{n=1}^\infty \frac{(s)^{n-1}B_\nu(nx)}{n^\alpha} = \sqrt{\frac{\pi}{2x}}\cdot\sum_{n=1}^\infty \frac{(s)^{n-1}\mathbf{H}_{\nu-1/2}(nx)}{n^{\alpha+1/2}}.$$

$I_\alpha^B$ is given by (16), where $a=1, b=0, f=\cos, \delta=0$ (now $D_\nu = B_\nu$), so we can evaluate

$$\sum_{n=1}^\infty \frac{(s)^{n-1}\mathbf{H}_{\nu-1/2}(nx)}{n^{\alpha+1/2}} = \frac{c\pi x^{\alpha-1/2}\ 2^{-(\alpha+1/2)}}{\Gamma(\frac{\alpha-\nu+2}{2})\Gamma(\frac{\alpha+\nu+1}{2})\cos(\frac{\pi(\alpha-\nu)}{2})} + \sum_{i=0}^\infty \frac{(-1)^iF(\alpha-\nu-2i)x^{\nu+2i+1/2}}{2^{\nu+1/2+2i}\Gamma(i+\frac{3}{2})\Gamma(\nu+i+1)},$$

where the other parameters we read from Table I.

EXAMPLE 6. We can take $\psi(y) = y^\nu(1-y^2)^{\nu-1/2}$. Then $B_\nu(x) = 2^{\nu-1}x^{-\nu}\sqrt{\pi}\times\Gamma(\nu+1/2)J_\nu^2(x/2)$ (see [2], p. 702). Replacing this in the second series (15) for $a=1, b=0, s=-1, m=0$, we have

$$I^B_{-\mu,\omega} = \frac{2^{\nu-1}\sqrt{\pi}\Gamma(\nu+\frac12)}{x^\nu}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n^{\mu-\nu}}{n^2-\omega^2}J_\nu^2\Big(\frac{nx}{2}\Big).$$

Using (17) for $\varphi_\nu = B_\nu, m=0, a=1, b=0, s=-1$ which implies $\mu+\nu=0$ (see Table I where $\mu+\nu = 1-d$) we evaluate the series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}J_\nu^2\left(\frac{nx}{2}\right)}{n^{2\nu}(n^2-\omega^2)} = \frac{J_\nu^2\left(\frac{\omega x}{2}\right)}{2\omega^{2\nu+1}\sin\pi\omega} - \frac{x^{2\nu}}{2^{4\nu+1}\omega^2\Gamma^2(\nu+1)}.$$

EXAMPLE 7. If we want to apply the Bessel instead of the Mellin transform to the series (4), we first recall that there holds (see [5], p. 36, 4.23 and p. 33, 4.6)

$$B(x^{\nu+1/2}(a^2+x^2)^{-\mu}) = \frac{a^{\nu-\mu+1}y^{\mu-1/2}}{2^{\mu-1}\Gamma(\mu)}K_{\nu-\mu+1}(ay), \qquad B(x^\mu) = \frac{2^{\mu+\frac12}}{y^{\mu+1}}\frac{\Gamma(\frac{\nu}{2}+\frac{\mu}{2}+\frac34)}{\Gamma(\frac{\nu}{2}-\frac{\mu}{2}+\frac14)},$$

where $K_\nu$ is McDonald function. Thus we obtain the sum of a Schlömilch series

$$\sum_{n=1}^{\infty}\frac{K_{1/2}(ny)}{n^{\alpha-1/2}} = (-1)^{\alpha/2}\frac{\pi y^{\alpha-3/2}\Gamma(1-\frac{\alpha}{2})}{2^{\alpha+1/2}\Gamma(\frac{\alpha+1}{2})} + \sum_{i=0}^{M}\frac{(-1)^i\zeta(\alpha-2i)y^{2i-1/2}\Gamma(\frac12-i)}{2^{2i+1/2}\,i!}.$$

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