MEAN VALUE THEOREMS IN q-CALCULUS

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Abstract. In this paper, some properties of continuous functions in q-analysis are investigated. The behavior of q-derivative in a neighborhood of a local extreme point is described. Two theorems are proved which are q-analogs of the fundamental theorems of the differential calculus. Also, two q-integral mean value theorems are proved and applied to estimating remainder term in q-Taylor formula. Finally, the previous results are used in considering some new iterative methods for equation solving.

1. Introduction

At the last quarter of the XX century, q-calculus appeared as a connection between mathematics and physics ([5], [6]). It has a lot of applications in different mathematical areas, such as number theory, combinatorics, orthogonal polynomials, basic hyper-geometric functions and other sciences—quantum theory, mechanics and theory of relativity.

Let $q \in \mathbb{R}^+ \setminus \{1\}$. A q-natural number $[n]_q$ is defined by

$$[n]_q := 1 + q + \cdots + q^{n-1}, \quad n \in \mathbb{N}.$$  

Generally, a q-complex number $[a]_q$ is $[a]_q := \frac{1 - q^a}{1 - q}$, $a \in \mathbb{C}$. The factorial of a number $[n]_q$ is $[0]_q! := 1$, $[n]_q! := [n]_q[n-1]_q\cdots[1]_q$, $n \in \mathbb{N}$.

Let q-derivative of a function $f(z)$ be

$$(D_q f)(z) := \frac{f(z) - f(qz)}{z - qz}, \quad z \neq 0, \quad (D_q f)(0) := \lim_{z \to 0} (D_q f)(z),$$

and high q-derivatives are

$$D_q^0 f := f, \quad D_q^n f := D_q(D_q^{n-1} f), \quad n = 1, 2, 3, \ldots$$

Notice, that a continuous function on an interval, which does not include 0, is continuous q-differentiable.

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2. Extreme values and $q$-derivative

We will consider relations between, on one side, extreme value of a continuous function and, on the other side, derivatives and $q$-derivatives.

**Theorem 2.1.** Let $f(x)$ be a continuous function on a segment $[a, b]$ and let $c \in (a, b)$ be a point of its local maximum.

(i) If $0 < a < b$, then there exists $\hat{q} \in (0, 1)$ such that

$$
(D_q f)(c) \begin{cases} 
\geq 0, & \forall q \in (\hat{q}, 1) \\
\leq 0, & \forall q \in (1, \hat{q}^{-1}).
\end{cases}
$$

(ii) If $a < b < 0$, then there exists $\hat{q} \in (0, 1)$ such that

$$
(D_q f)(c) \begin{cases} 
\leq 0, & \forall q \in (\hat{q}, 1) \\
\geq 0, & \forall q \in (1, \hat{q}^{-1}).
\end{cases}
$$

Furthermore, $(\forall q \in (\hat{q}, 1) \cup (1, \hat{q}^{-1})) \big( \exists \xi \in (a, b) \big) (D_q f)(\xi) = 0$.

**Proof.** Since the proofs of (i) and (ii) are very similar, we will expose only the first one. Since $c$ is a point of local maximum of the function $f(x)$, there exists $\varepsilon > 0$, such that $f(x) \leq f(c)$, for all $x \in (c - \varepsilon, c + \varepsilon) \subset (a, b)$. Let $q_0 \in (0, 1)$ such that $c - \varepsilon < q_0 c < c$. Now, for all $q \in (q_0, 1)$, it is valid $qc < c$ and $f(qc) \leq f(c)$, wherefrom $(D_q f)(c) \geq 0$. In a similar way, there exists $q_1 \in (0, 1)$ such that $c < c/q_1 < c + \varepsilon$ and for all $q \in (1, q_1^{-1})$ it holds $(D_q f)(c) \leq 0$. At last, denote by $\hat{q} = \max\{q_0, q_1\}$.

Let $q \in (\hat{q}, 1)$ be an arbitrary real number. Then $\eta = c/q \in (c, c + \varepsilon)$, wherefrom $f(c) \geq f(\eta)$, i.e., $f(q \eta) \geq f(\eta)$. From $q \eta < \eta$ we conclude $(D_q f)(\eta) \leq 0$. As $f(x)$ is a continuous function, $(D_q f)(x)$ is continuous in $(a, b)$, too. Since $(D_q f)(c) \geq 0$, $(D_q f)(\eta) \leq 0$, where $c, \eta \in (a, b)$, there exists $\xi \in (c, \eta) \subset (a, b)$, such that $(D_q f)(\xi) = 0$. Analogously, for an arbitrary $q \in (1, q_1^{-1})$, the number $\eta = c/q \in (c - \varepsilon, c)$, wherefrom $(D_q f)(\eta) \geq 0$. Since $(D_q f)(c) \leq 0$, we have proved the existence of a zero of $(D_q f)(x)$ for $q \in (1, q_1^{-1})$.

**Example 2.1.** Let us consider $f(x) = (x - 1)(3 - x) + 2$. Its maximum is at $c = 2$, but $q$-derivative is $(D_q f)(x) = (2|x - 1| + 4) x$ and it vanishes at the point $\xi = \frac{4}{1 + q}$. So, here is $\hat{q} = 1/3$. For $q = 3/4$, we have $(D_{3/4} f)(2) = 1/2, \eta = 22/3$ and $\xi = 22/7$.

In a similar way, we can prove the next theorem.

**Theorem 2.2.** Let $f(x)$ be a continuous function on a segment $[a, b]$ and let $c \in (a, b)$ be a point of its local minimum.

(i) If $0 < a < b$, then there exists $\hat{q} \in (0, 1)$ such that

$$
(D_q f)(c) \begin{cases} 
\leq 0, & \forall q \in (\hat{q}, 1) \\
\geq 0, & \forall q \in (1, \hat{q}^{-1}).
\end{cases}
$$
(ii) If $a < b < 0$, then there exists $\hat{q} \in (0,1)$ such that

$$
\begin{align*}
(D_q f)(c) &\geq 0, \quad \forall q \in (\hat{q}, 1) \\
&\leq 0, \quad \forall q \in (1, \hat{q}^{-1}).
\end{align*}
$$

Moreover, $(\forall q \in (\hat{q}, \hat{q}^{-1})) \left( \exists \xi \in (a,b) \right) (D_q f)(\xi) = 0.$

Remark. If $f(x)$ is differentiable for all $x \in (a,b)$, then $\lim_{q \uparrow 1} D_q f(x) = f'(x)$. So, if $c \in (a,b)$ is a point of local extreme of $f(x)$, we have $f'(c) = D_1 f(c) = 0.$

3. Some $q$-mean value theorems

By using the previous results, we can establish and prove analogons of well-known mean value theorems in $q$-calculus.

**Theorem 3.1.** ($q$-Rolle) Let $f(x)$ be a continuous function on $[a,b]$ satisfying $f(a) = f(b)$. Then there exists $\hat{q} \in (0,1)$ such that

$$
(\forall q \in (\hat{q}, 1) \cup (1, \hat{q}^{-1})) \left( \exists \xi \in (a,b) \right) : (D_q f)(\xi) = 0.
$$

**Proof.** If $f(x)$ is not a constant function on $[a,b]$, then it attains its extreme value in some point in $(a, b)$. But, according to Theorems 2.1–2, $(D_q f)(x)$ vanishes at a point $\xi \in (a,b).$ ■

**Theorem 3.2** ($q$-Lagrange) Let $f(x)$ be a continuous function on $[a,b]$. Then there exists $\hat{q} \in (0,1)$ such that

$$
(\forall q \in (\hat{q}, 1) \cup (1, \hat{q}^{-1})) \left( \exists \xi \in (a,b) \right) : f(b) - f(a) = (D_q f)(\xi)(b-a).
$$

**Proof.** The statement follows by applying the previous theorem to the function $f(x) - x(f(b) - f(a))/(b-a).$ ■

4. Mean value theorems for $q$-integrals

In $q$-analysis, we define $q$-integral by

$$
I_q(f) = \int_0^a f(t) d_q(t) := a(1-q) \sum_{n=0}^{\infty} f(aq^n)q^n.
$$

Notice that

$$
I(f) = \int_0^a f(t) dt = \lim_{q \uparrow 1} I_q(f).
$$

**Theorem 4.1** Let $f(x)$ be a continuous function on a segment $[0,a]$ ($a > 0$). Then

$$
(\forall q \in (0,1)) \left( \exists \xi \in [0,a] \right) : I_q(f) = \int_0^a f(t) d_q(t) = a f(\xi).
$$
Proof. Since \( f(x) \) is a continuous function on the segment \([0,q]\), it attains its minimum \( m \) and maximum \( M \) and takes all values between. According to assumption \( 0 < q < 1 \), we have \( 0 < aq^n < a \) and \( M \leq f(aq^n) \leq M \). Now,

\[
a(1-q) \sum_{n=0}^{\infty} mq^n \leq a(1-q) \sum_{n=0}^{\infty} f(aq^n)q^n \leq a(1-q) \sum_{n=0}^{\infty} Mq^n,
\]

wherefrom \( m \leq \frac{1}{a} I_0(f) \leq M \). So, there exists \( \xi \in [0,a] \) such that \( a^{-1} I_0(f) = f(\xi) \).

Moreover, if we define

\[
\int_a^b f(t) \, dq(t) := \int_0^b f(t) \, dq(t) - \int_0^a f(t) \, dq(t),
\]

then the next theorem is valid.

**Theorem 4.2.** Let \( f(x) \) be a continuous function on a segment \([a,b]\). Then there exists \( \tilde{q} \in (0,1) \) such that

\[
(\forall \tilde{q} \in (\tilde{q},1))(\exists \xi \in (a,b)) : \quad I_0(f) = \int_a^b f(t) \, dq(t) = f(\xi)(b-a).
\]

Proof. It is easy to prove that \( \lim_{q \uparrow 1} I_0(f) = I(f) \), i.e.

\[
(\forall \varepsilon > 0)(\exists q_0 \in (0,1))(\forall \tilde{q} \in (q_0,1)) : \quad I(f) - \varepsilon < I_0(f) < I(f) + \varepsilon.
\]

According to the well known mean value theorem for integrals, we have

\[
(\exists c \in (a,b)) : \quad I(f) = f(c)(b-a).
\]

Let \( \varepsilon \leq (b-a) \min\{M - f(c), f(c) - m\} \), where \( m \) and \( M \) are the minimum and maximum of \( f(x) \) on \([a,b]\). Now,

\[
(\exists \tilde{q} \in (0,1))(\forall \tilde{q} \in (\tilde{q},1)) : \quad f(c) - \frac{\varepsilon}{b-a} < \frac{1}{b-a} I_0(f) < f(c) + \frac{\varepsilon}{b-a},
\]

hence \( m < I_0(f)/(b-a) < M \). Since \( f(x) \) is a continuous function on the segment \([a,b]\), it takes all values between \( m \) and \( M \), i.e.

\[
(\exists \xi \in (a,b)) : \quad \frac{1}{b-a} I_0(f) = f(\xi),
\]

what we wanted to prove.

**Theorem 4.3** Let \( f(x) \) and \( g(x) \) be some continuous functions on a segment \([a,b]\). Then there exists \( \tilde{q} \in (0,1) \) such that

\[
(\forall \tilde{q} \in (\tilde{q},1))(\exists \xi \in (a,b)) : \quad I_0(fg) = g(\xi)I_0(f).
\]
Proof. According to the second mean value theorem for integrals, we have

\[ (\exists c \in (a, b)) : I(fg) = g(c)I(f). \]

Hence \( \lim_{q \uparrow 1} I_q(fg) = g(c)I(f) = g(c)\lim_{q \uparrow 1} I_q(f), \) i.e., \( \lim_{q \uparrow 1} \frac{I_q(fg)}{I_q(f)} = g(c) \). Now,

\[ (\exists q_0 \in (0, 1)) (\forall q \in (q_0, 1)) : g(c) - \varepsilon < \frac{I_q(fg)}{I_q(f)} < g(c) + \varepsilon. \]

Since \( g(x) \) is a continuous function on the segment \([a, b]\), it attains its minimum \( m_g \) and maximum \( M_g \). Let \( \varepsilon \leq \min\{M_g - g(c), g(c) - m_g\} \). Hence

\[ (\exists q \in (0, 1)) (\forall q \in (q_0, 1)) : m_g < \frac{I_q(fg)}{I_q(f)} < M_g. \]

Since \( f(x) \) takes all values between \( m_g \) and \( M_g \), we conclude that

\[ (\exists \xi \in (a, b)) : \frac{I_q(fg)}{I_q(f)} = g(\xi). \quad \blacksquare \]

5. Estimation of remainder term in \( q \)-Taylor formula

Let \( f(x) \) be a continuous function on some interval \((a, b)\) and \( c \in [a, b] \). Jackson’s \( q \)-Taylor formula (see [3], [4] and [2]) is given by

\[ f(z) = \sum_{k=0}^{\infty} \frac{(D_q^k f)(c)}{[k]_q!} (z - c)^{(k)}, \quad z \in (a, b), \]

where

\[ (z - c)^{(0)} = 1, \quad (z - c)^{(k)} = \prod_{i=0}^{k-1} (z - cq^i), \quad (k \in \mathbb{N}). \]

T. Ernst [2] have found the next \( q \)-Taylor formula

\[ f(z) = \sum_{k=0}^{n-1} \frac{(D_q^k f)(c)}{[k]_q!} (z - c)^{(k)} + R_n(f, z, c, q), \quad (5.1) \]

where \( R_n(f, z, c, q) \) is the remainder term determined by

\[ R_n(f, z, c, q) = \int_{t=c}^{t=z} \frac{(z - t)^{(n)}}{z - t} \frac{(D_q^n f)(t)}{[n-1]_q!} d_q(t). \quad (5.2) \]

**Theorem 5.1.** Let \( f(x) \) be a continuous function on \([a, b]\) and \( R_n(f, z, c, q) \), \( z, c \in (a, b) \) be the remainder term in \( q \)-Taylor formula. Then there exists \( \hat{q} \in (0, 1) \) such that for all \( q \in (\hat{q}, 1), \xi \in (a, b) \) can be found between \( c \) and \( z \), which satisfies

\[ R_n(f, z, c, q) = \frac{(D_q^n f)(\xi)}{[n-1]_q!} \int_{t=c}^{t=z} \frac{(z - t)^{(n)}}{z - t} d_q(t). \quad (5.3) \]
Proof. Since \( f(x) \) is a continuous function on \([a, b]\), it can be expanded by \( q \)-Taylor formula (5.1) with the remainder term (5.2). Notice that the functions
\[
\frac{(z - t)^{(n)}}{z - t} = \prod_{i=1}^{n-1} (z - tq^i)
\]
and \((D_q^n f)(t)/[n - 1]_q!\) are continuous on the segment between \(c\) and \(z\) which is contained in \((a, b)\). According to Theorem 4.3., there exists \( \hat{q} \in (0, 1) \), such that for all \( q \in (\hat{q}, 1) \) can be found \( \xi \) between \(c\) and \(z\) such that (5.3) is valid. ■

**Theorem 5.2.** Let \( f(x) \) be a continuous function on \([a, b]\) and \( z, c \in (a, b)\). Then there exists \( \hat{q} \in (0, 1) \) such that for all \( q \in (\hat{q}, 1) \), \( \xi \in (a, b) \) can be found between \(c\) and \(z\), which satisfies
\[
f(z) = \sum_{k=0}^{n-1} \frac{(D_q^k f)(c)}{[k]_q!} (z - c)^{(k)} + \frac{(D_q^n f)(\xi)}{[n]_q!} (z - c)^{(n)}.
\]

**Proof.** Applying \( \frac{(z - t)^{(n)}}{z - t} = -D_q(t) \left( \frac{(z - t)^{(n)}}{[n]_q} \right) \) to the integral in (5.3) we have
\[
\int_{t=e}^{t=z} \frac{(z - t)^{(n)}}{z - t} d_q(t) = - \int_{t=e}^{t=z} D_q(t) \left( \frac{(z - t)^{(n)}}{[n]_q} \right) d_q(t)
\]
\[
= - \left. \frac{(z - t)^{(n)} (z - c)^{(n)}}{[n]_q} \right|_{t=e}^{t=z}.
\]
So, \( R_n(f, z, c, q) = \frac{(D_q^n f)(\xi)}{[n]_q!} (z - c)^{(n)} \). ■

**Theorem 5.3** Let \( f(x) \) be a continuous function on \([a, b]\) and \( R_n(f, z, c, q) \), \( z, c \in (a, b) \) be the remainder term in \( q \)-Taylor formula. Then there exists \( \hat{q} \in (0, 1) \) such that for all \( q \in (\hat{q}, 1) \), \( \xi \in (a, b) \) can be found between \(c\) and \(z\), which satisfies
\[
R_n(f, z, c, q) = \frac{(D_q^n f)(\xi)}{[n]_q!} (z - c)^{(n)}.
\]

**Proof.** Since \( f(x) \) is a continuous function on \([a, b]\), it can be expanded by \( q \)-Taylor formula (5.1) with the remainder term (5.2). Notice that the functions
\[
\frac{(z - t)^{(n)}}{z - t} = \prod_{i=1}^{n-1} (z - tq^i)
\]
and \((D_q^n f)(t)/[n - 1]_q!\) are continuous on the segment between \(c\) and \(z\) which is contained in \((a, b)\). According to Theorem 4.3., there exists \( \hat{q} \in (0, 1) \), such that for all \( q \in (\hat{q}, 1) \), \( \xi \) between \(c\) and \(z\) can be found such that
\[
R_n(f, z, c, q) = \frac{(D_q^n f)(\xi)}{[n - 1]_q!} \int_{t=c}^{t=z} \frac{(z - t)^{(n)}}{z - t} d_q(t).
\]
Applying
\[
\frac{(z - t)^{(n)}}{z - t} = -D_{q,t} \frac{(z - t)^{(n)}}{[n]_q}
\]

we have
\[
\int_{t=c}^{z} \frac{(z - t)^{(n)}}{z - t} d_q(t) = -\int_{t=c}^{z} D_{q,t} \frac{(z - t)^{(n)}}{[n]_q} d_q(t)
\]
\[
= -\frac{(z - t)^{(n)} |_{t=c}^{z}}{[n]_q} = \frac{(z - c)^{(n)}}{[n]_q}.
\]

So, \( R_n(f, z, c, q) = \frac{(D^q f)(\xi)}{[n]_q!} (z - c)^{(n)} \). \( \blacksquare \)

6. Application

Here we will apply the previous theorems in analyzing an iterative method for solving equations.

Suppose that an equation \( f(x) = 0 \) has a unique isolated solution \( x = \tau \). If \( x_n \)
is an approximation for the exact solution \( \tau \), using Jackson’s \( q \)-Taylor formula, we have
\[
0 = f(\tau) \approx f(x_n) + (D_q f)(x_n) (\tau - x_n),
\]
hence \( \tau \approx x_n - \frac{f(x_n)}{(D_q f)(x_n)} \). So, we can construct \( q \)-Newton method
\[
x_{n+1} = x_n - \frac{f(x_n)}{(D_q f)(x_n)}. \tag{6.1}
\]

More simply, it looks like \( x_{n+1} = x_n \left\{ 1 - \frac{1 - q}{1 - q f(x_n)} \right\} \). This method written in the form
\[
x_{n+1} = x_n - \frac{x_n - q f(x_n)}{f(x_n) - f(q x_n)} f(x_n)
\]
reminds to the method of chords (secants).

**Theorem 6.1.** Suppose that a function \( f(x) \) is continuous on a segment \([a, b]\) and that the equation \( f(x) = 0 \) has a unique isolated solution \( \tau \in (a, b) \). Let the conditions
\[
|(D_q f)(x)| \geq M_1 > 0, \quad |(D_q^2 f)(x)| \leq M_2
\]
are satisfied for all \( x \in (a, b) \). Then there exists \( \bar{q} \in (0, 1) \), such that for all \( q \in (\bar{q}, 1) \), the iterations obtained by \( q \)-Newton method satisfy
\[
|\tau - x_{n+1}| \leq \frac{M_2}{(1 + q)M_1} |(\tau - x_n)^{(2)}|.
\]
\begin{proof}
From the formulation of q-Newton method (6.1), we have
\[ x_{k+1} - \tau = x_k - \tau - \frac{f(x_k)}{(D_q f)(x_k)}, \]
hence \( f(x_k) + (D_q f)(x_k)(\tau - x_k) = (D_q f)(x_k)(\tau - x_{k+1}). \) By using q-Taylor formula at the point \( x_k \) of order \( n = 2 \) for \( f(\tau) \) we have
\[ f(\tau) = f(x_k) + (D_q f)(x_k)(\tau - x_k) + R_2(f, \tau, x_k, q). \]
Since \( f(\tau) = 0 \), we obtain \( (D_q f)(x_k)(\tau - x_{k+1}) = -R_2(f, \tau, x_k, q) \), i.e.
\[ |\tau - x_{k+1}| = \frac{|R_2(f, \tau, x_k, q)|}{|(D_q f)(x_k)|}. \]
According to Theorem 5.1., there exists \( \bar{q} \in (0, 1) \) such that for all \( q \in (\bar{q}, 1), \) \( \xi \in (a, b) \) can be found such that
\[ R_2(f, \tau, x_k, q) = \frac{(D_q^2 f)(\xi)}{2! q} (\tau - x_k)^2. \]
Now,
\[ |\tau - x_{k+1}| = \frac{|(D_q^2 f)(\xi)|}{|(D_q f)(x_k)|} \frac{|(\tau - x_k)^2|}{1 + q}. \]
Using the conditions which function \( f(x) \) and its q-derivatives satisfy we obtain the statement of the theorem. \( \blacksquare \)

**Remark.** In our papers [7] and [8] we have discussed q-iterative methods in details.

**References**


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