DISPLACEMENT STRUCTURE OF GENERALIZED INVERSE $A^{(1,2)}_{T,S}$

Mel Qin and Yimin Wei

Abstract. It is well known that matrices with a UV-displacement structure possess generalized inverse with a VU-displacement structure. Estimation for the displacement rank of $A^{(1,2)}_{T,S}U - V A^{(1,2)}_{T,S}$ are presented, where $A^{(1,2)}_{T,S}$ is the (1,2)-inverse of $A$ with prescribed range $T$ and null space $S$. We extend the results due to G. Heinig and F. Hellinger, Wei and Ng, Cai and Wei for the Moore-Penrose inverse, group inverse and weighted Moore-Penrose inverse, respectively.

1. Introduction and Preliminaries

The aim of the present paper is to study the generalized inverse of a structured matrix. To begin with, we recall some facts concerning the regular inversion of structured matrices, which are motivated by [5, 6]. If the rank of a matrix’s displacement is small, fast algorithms for the matrix are available.

A matrix is called matrix with displacement structure [2, 3, 8] if and only if the rank of the matrix $AU - VA$ or $A - VAU$ is small compared with the order of the matrix $A$. The rank of $AU - VA$ is said to be Sylvester UV-displacement rank and the rank of $A - VAU$ is called the Stein UV-displacement rank of $A$, since $A$ is the solution of a Sylvester or Stein equation, respectively.

As is well known, fast inversion algorithms for matrix $A$ can be constructed if $A$ is a matrix with displacement structure. We are interested in the generalized inverse with smallest possible displacement rank. We present the estimate for the rank of $A^{(1,2)}_{T,S}U - V A^{(1,2)}_{T,S}$, where $A^{(1,2)}_{T,S}$ is the (1,2)-inverse of $A$, in Section 2. Then we give an explicit estimate for general displacement, which is defined in the paper, Section 3. Our results cover the previous results [2, 4, 14] for the Moore-Penrose inverse, weighted Moore-Penrose inverse and group inverse, respectively.

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A matrix $X$ is said to be a $(1,2)$-inverse of $A$ if the following condition is fulfilled:

$$XAX = X, \quad AXA = A.$$  

Let $A \in C_{r \times n}^m, T$ and $S$ are subspaces of $C^m$ and $C^n$, respectively, the dimensions of $T$ and $S$ are $r$ and $m-r$, then $A$ has a unique $(1,2)$-inverse $X$ satisfying $R(X) = T, \ N(X) = S$ if and only if $R(A) \bigoplus S = C^m, \ \text{Ker}(A) \bigoplus T = C^n$. We denote $X$ by $A^{(1,2)}_{T,S}$.

If $G \in C^{n \times m}$, and $G$ satisfies $R(G) = T, \ N(G) = S$, then we get [9]

$$A^{(1,2)}_{T,S} (GA)_g = G(AG)_g, \quad AA^{(1,2)}_{T,S} = AG(AG)_g, \quad A^{(1,2)}_{T,S} A = (GA)_g GA,$$

where $(GA)_g$ and $(GA)_g$ are group inverses of $AG$ and $GA$.

From Jordan canonical form theory, we get that for any complex $m \times n$ matrix $A$ with $\text{rank}(A) = r$ with prescribed range $T$ and null space $S$, there exist nonsingular matrices $R$ and $N$ [13] such that

$$A = R^{-1} \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} N, \quad G = N^{-1} \begin{bmatrix} G_{11} & 0 \\ 0 & 0 \end{bmatrix} R$$

(1)

where $R$ is an $m \times m$ nonsingular matrix and $N$ is $n \times n$. Note that $A_{11}, G_{11}$ are nonsingular matrices. Now we can write $A^{(1,2)}_{T,S}$ of $A$ in the form [13],

$$A^{(1,2)}_{T,S} = N^{-1} \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} R$$

(2)

Let

$$Q = A^{(1,2)}_{T,S} A, \quad P = I - Q, \quad Q_* = A^{(1,2)}_{T,S} A, \quad P_* = I - Q_*.$$  

(3)

Obviously, $Q$ and $P$ are oblique projections, where $\text{Im}(A)$ denotes the range of $A$ and $\text{Ker}(A)$ is the null space of $A$. It is easy to check that

$$\text{Im}(Q) = \text{Im}(A^{(1,2)}_{T,S}) = \text{Ker}(P) = \text{Im}(G), \quad \text{Im}(Q_*) = \text{Ker}(P_*) = \text{Im}(A),$$

$$\text{Im}(P) = \text{Ker}(A^{(1,2)}_{T,S}) = \text{Ker}(Q) = \text{Ker}(G), \quad \text{Im}(P_*) = \text{Ker}(Q_*).$$

(4)

For recent results on the generalized inverse $A^{(1,2)}_{T,S}$, we refer to [10, 11, 12, 15].

2. Sylvester displacement rank

Throughout the paper, $U \in C^{n \times n}$ and $V \in C^{m \times m}$ are some fixed matrices. The operator

$$d(U, V) = AU - VA$$

is called the $UV$-displacement of $A$. To distinguish this displacement concept from the more general case in Section 3 we call it the Sylvester $UV$-displacement, since $A$ is the solution of a certain Sylvester equation. We can easily find that for a nonsingular matrix $A$ with $UV$-displacement structure the inverse matrix $A^{-1}$ possesses a $VU$-displacement structure, and the relation

$$\text{rank}(A^{-1}V - UA^{-1}) = \text{rank}(AU - VA)$$

holds.
We want to get an estimate for the VU-displacement rank of the generalized inverse of a matrix with displacement structure. First we show that the displacement structure of $A_{T,S}^{(1,2)}$ is the following representation.

**Proposition 2.1.** Let $A \in C^{m \times n}$, $U \in C^{n \times n}$ and $V \in C^{m \times m}$, and $A_{T,S}^{(1,2)}$ is the $(1,2)$-inverse of $A$. Then

$$A_{T,S}^{(1,2)} V - U A_{T,S}^{(1,2)} = A_{T,S}^{(1,2)} V P_* - P U A_{T,S}^{(1,2)} (A U - V A) A_{T,S}^{(1,2)},$$

(5)

where $Q, P, Q_*$ and $P_*$ are defined in (4).

**Proof.** The relation is immediately proved by the following equation

$$A_{T,S}^{(1,2)} (A U - V A) A_{T,S}^{(1,2)} = (I - P) U A_{T,S}^{(1,2)} - A_{T,S}^{(1,2)} V (I - P_*).$$

(6)

From (4) we obtain the following Corollary.

**Corollary 2.1.** The VU-displacement rank of $A_{T,S}^{(1,2)}$ satisfies the following estimate:

$$\text{rank}(A_{T,S}^{(1,2)} V - U A_{T,S}^{(1,2)}) \leq \text{rank}(A U - V A) + \text{rank}(Q_* V P_*) + \text{rank}(P U Q).$$

(7)

**Proof.** We will prove $\text{dim}(P U A_{T,S}^{(1,2)}) = \text{rank}(P U Q)$, and $\text{rank}(A_{T,S}^{(1,2)} V P) = \text{rank}(Q V P)$.

$$\text{rank}(P U A_{T,S}^{(1,2)}) = \text{dim}[P U \text{Im}(A_{T,S}^{(1,2)})] = \text{dim}[\text{Im}(P U Q)] = \text{rank}(P U Q),$$

(8)

the second equality can be proved similarly.

Taking these into account, we obtain the above estimate. ■

Now we aim to obtain the estimate for the second and third terms on the right-hand side.

**Proposition 2.2.** With the notation above, we get the estimate

$$\text{rank}(Q_* V P_*) + \text{rank}(P U Q) \leq \text{rank}(U G - G V).$$

(9)

**Proof.** We set $F = U G - G V$ and make partitions

$$NU^{-1} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}, \quad RV^{-1} = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}. $$

(10)

From (1) we write the matrix $F$ in the following form

$$NFR^{-1} = NU^{-1} NGR^{-1} - NGR^{-1} RV^{-1}$$

$$= \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} G_{11} & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} G_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}$$

$$= \begin{bmatrix} U_{11} G_{11} - G_{11} V_{11} - G_{11} V_{12} \\ U_{21} G_{11} \end{bmatrix}.$$
From [7] we know that
\[
\begin{align*}
\text{rank} \begin{bmatrix}
U_{11}G_{11} - G_{11}V_{11} & -G_{11}V_{12} \\
U_{21}G_{11} & 0
\end{bmatrix} & \geq \text{rank}(-G_{11}V_{12}) + \text{rank}(U_{21}G_{11}) \\
& = \text{rank}(V_{12}) + \text{rank}(U_{21}) = \text{rank} \begin{bmatrix} 0 & V_{12} \\ 0 & 0 \end{bmatrix} + \text{rank} \begin{bmatrix} 0 & U_{21} \\ 0 & 0 \end{bmatrix} \\
& = \text{rank}(Q_4V_P) + \text{rank}(PU_Q).
\end{align*}
\]

where $G_{11}$ is nonsingular. Then the proof is over. \(\blacksquare\)

From Propositions 2.1 and 2.2 we derive the first main result.

**Theorem 2.1.** Let $A \in C^{m \times n}$ and $A^{(1,2)}_{T,S}$ be its $(1,2)$-inverse. Then
\[
\text{rank}(A^{(1,2)}_{T,S}V - UA^{(1,2)}_{T,S}) \leq \text{rank}(AU - VA) + \text{rank}(UG - GV).
\]

**Proposition 2.3.** If $AU = UG, VA = GV$, then we get the estimate:
\[
\text{rank}(A^{(1,2)}_{T,S}V - UA^{(1,2)}_{T,S}) \leq 2 \text{rank}(AU - VA).
\] (11)

### 3. Displacement structure for generalized displacement

In order to generalize Theorem 2.1 we introduce a generalized displacement concept [2]. Let $a = [a_{ij}]^1$ denote a nonsingular $2 \times 2$ matrix. We associate $a$ with the polynomial in two variables
\[
a(\lambda, \mu) = \sum_{i,j=0}^1 a_{ij} \lambda^i \mu^j
\]
and the linear fractional function
\[
f_a(\lambda) = \frac{a_{10} + a_{11} \lambda}{a_{00} + a_{01} \lambda}.
\] (12)

For any fixed $U \in C^{m \times n}$ and $V \in C^{m \times m}$, the generalized $(a, U, V)$ displacement of $A \in C^{m \times n}$ generated by $a(\lambda, \mu)$ is defined by
\[
a(V, U)A = \sum_{i,j=0}^1 a_{ij}V^iAU^j.
\]

If $a = d \equiv \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, we just get Sylvester displacement that we have discussed. If $a = d \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, we get Stein displacement.

**Lemma 3.1.** [2] Let $a = [a_{ij}]^1, b = [b_{ij}]^1, c = [c_{ij}]^1, d = [d_{ij}]^1$ be nonsingular $2 \times 2$ matrices such that
\[
a = b^Tdc.
\] (13)

Then
\[
(b_{00} + b_{01} \lambda)^{-1}a(\lambda, \mu)(c_{00} + c_{01} \mu)^{-1} = d(f_b(\lambda), f_c(\mu))
\] (14)
for all $\lambda, \mu$ with $b_{00} + b_{01} \lambda \neq 0$ and $c_{00} + c_{01} \mu \neq 0$. 
Lemma 3.2. [2] If \( d = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \), then there exist \( 2 \times 2 \) matrices \( b, c \) such that (13) holds and \( b_{00} + b_{01} V \) and \( c_{00} + c_{01} U \) are invertible.

Taking Lemma 3.1 and Lemma 3.2 together, we obtain the following

Proposition 3.1. [2] If \( b \) and \( c \) be matrices satisfying the conditions in Lemma 3.2, then for \( A \in C_{m \times n} \),
\[
a(V, U)A = (b_{00} + b_{01} V)\left[ A_f c(U) - f_0(V) A(c_{00} + c_{01} U) \right].
\]

The following is very important to generalize Theorem 2.1 for general \( (a, U, V) \) displacement.

Proposition 3.2. (a) If \( \psi = [\psi_{ij}]_0^1 \) is nonsingular and \( \psi_{00} + \psi_{01} V \) is invertible, then \( \text{rank}(Q, V P_i) = \text{rank}(Q, \tilde{V} P_i) \), where \( \tilde{V} \equiv f_0(V) \) is defined in (12).

(b) If \( \phi = [\phi_{ij}]_0^1 \) is nonsingular and \( \phi_{00} + \phi_{01} U \) is invertible, then
\[
\text{rank}(P U Q) = \text{rank}(P \tilde{U} Q),
\]
where \( \tilde{U} \equiv f_0(U) \) is defined in (12).

Proof. We define
\[
\mathcal{S} = \text{Ker}(G) \cap \text{Ker}(GV), \quad \mathcal{S}_1 = \text{Ker}(G) \ominus \text{CalS}.
\]

We show that \( Q, V P_i \) is one-to-one on \( \mathcal{S}_1 \). If \( Q, V P_i, x = 0 \) and \( x \in \mathcal{S}_1 \), then \( V P_i, x \in \text{Ker}(Q_i) = \text{Ker}(G) \). That means \( G V P_i, x = G V x = 0 \). Noting that \( x \in \text{Ker}(G) \), we conclude \( x \in \mathcal{S} \). Thus \( x = 0 \).

Furthermore, \( Q, V P_i \) vanishes on \( \mathcal{S} \). Since \( Q, V P_i, x = Q, V x = (AG)_g A G V x = 0 \) for all \( x \in \mathcal{S} \) we have
\[
\text{rank}(Q, V P_i) = \text{dim}(\mathcal{S}_1). \tag{15}
\]

Analogously we define \( \tilde{\mathcal{S}} = \text{Ker}(G) \cap \text{Ker}(G \tilde{V}) \), \( \tilde{\mathcal{S}}_1 = \text{Ker}(G) \ominus \tilde{\mathcal{S}} \), and we get
\[
\text{rank}(Q, \tilde{V} P_i) = \text{dim}(\tilde{\mathcal{S}}_1). \tag{16}
\]

Now we show that the invertible matrix \( \tilde{\psi}_{00} + \tilde{\psi}_{01} V \) bijectively maps \( \mathcal{S} \) onto \( \tilde{\mathcal{S}} \). Suppose that \( x \in \mathcal{S} \). Then \( x, V x \in \text{Ker}(G) \). Hence \( y \equiv (\tilde{\psi}_{10} + \tilde{\psi}_{11} V) x \) and \( z \equiv (\tilde{\psi}_{00} + \tilde{\psi}_{01} V) x \) are all contained in \( \text{Ker}(G) \). Thus \( y = V z \) and we conclude that \( z, \tilde{V} z \in \text{Ker}(G) \), which implies \( z \in \tilde{\mathcal{S}} \). Conversely, with the same arguments we get \( (\tilde{\psi}_{00} + \tilde{\psi}_{01} V)^{-1} z \in \mathcal{S} \) for \( z \in \tilde{\mathcal{S}} \).

This implies
\[
\text{dim}(\mathcal{S}_1) = \text{dim}[\text{Ker}(G)] - \text{dim}(\mathcal{S}) = \text{dim}[\text{Ker}(G)] - \text{dim}(\tilde{\mathcal{S}}) = \text{dim}(\tilde{\mathcal{S}}_1). \tag{15}
\]
According to (15) and (16), we get assertion (a).

Assertion (b) is proved analogously. ■

Now we can generalize Theorem 2.1 for general \( (a, U, V) \) displacement.

Theorem 3.1. If \( a, b \) are \( 2 \times 2 \) nonsingular matrices, then
\[
\text{rank}[a(U, V) A_{I_{f, \tilde{S}}}^{(3, 2)}] \leq \text{rank}[a_{I_{f, \tilde{S}}}^{(3, 2)} + \text{rank}[b(U, V) G]. \tag{17}
\]
Proof. According to Lemma 3.2 there exist $2 \times 2$ matrices $w, x, y, z$ such that $w_00 + w_01U, x_00 + x_01V, y_00 + y_01U, z_00 + z_01V$ are invertible and $a = w^T dz, b = z^T dy$. Hence,

\[
\text{rank}[a(U,V)A^{(1,2)}_{T,S}] - \text{rank}[a^T(V,U)A] \\
= \text{rank}[f_a(U)A^{(1,2)}_{T,S} - A^{(1,2)}_{P,S} f_a(V)] - \text{rank}[f_a(V)A - Af_a(U)] \\
\leq \text{rank}[Pf_a(U)Q] + \text{rank}[Q, f_a(V)P] = \text{rank}[Pf_a(U)Q] + \text{rank}[Q, f_a(V)P] \\
\leq \text{rank}[f_a(U)G - Gf_a(V)] = \text{rank}[b(U,V)G].
\]

4. Concluding remarks

In this paper we study the displacement structure of $(1,2)$-inverse of a singular matrix. It is natural to ask if we can extend our results to linear operators in Hilbert space. This will be the future research.

REFERENCES


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Mei Qin, Institute of Mathematics, Fudan University, Shanghai, 200433, P.R.of China

Yimin Wei, Department of Mathematics, Fudan University, Shanghai, 200433, P.R.of China

E-mail: ymwei@fudan.edu.cn