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MIXED NORM SPACES OF DIFFERENCE SEQUENCES AND MATRIX TRANSFORMATIONS

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Abstract. In this paper, we generalise the definition of mixed norm spaces, define mixed norm spaces of difference sequences, determine their β-duals, and characterise matrix transformations on them. We obtain many known results as special cases.

1. Introduction

Let $1 \le p < \infty$. By ω we denote the set of all complex sequences $x = (x_k)_{k=1}^{\infty}$.

In 1968, *Maddox* [5] introduced and studied the sets

$$w_0^p = \left\{ x \in \omega : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |x_k|^p = 0 \right\} \text{ and } w_\infty^p = \left\{ x \in \omega : \sup_n \frac{1}{n} \sum_{k=1}^{n} |x_k|^p < \infty \right\}$$

of sequences that are strongly summable and bounded, respectively, with index p by the Cesàro method of order 1. He also observed that the *sections* $1/n \sum_{k=1}^{n}$ can be replaced by the *blocks* $1/2^{\nu+1} \sum_{k=2^\nu}^{2^{\nu+1}-1}$, and that the *section* and *block norms*

$$\|x\| = \sup_n \left(\frac{1}{n} \sum_{k=1}^{n} |x_k|^p \right)^{1/p} \text{ and } \|x\|' = \sup_{\nu\ge 0} \left(\frac{1}{2^{\nu+1}} \sum_{k=2^\nu}^{2^{\nu+1}-1} |x_k|^p \right)^{1/p}$$

are equivalent.

In 1974, *Jagers* [3] studied the *Cesàro sequence spaces*

$$ces(p) = \left\{ x \in \omega : \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p < \infty \right\}$$

which are Banach spaces with the norm

$$\|x\|_{ces(p)} = \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p \right)^{1/p}.$$

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It can be found in [1] that an equivalent norm on $ces(p)$ is

$$\|x\| = \left(\sum_{\nu=0}^{\infty} 2^{\nu(1-p)} \left(\sum_{k=2^\nu}^{2^{\nu+1}-1} |x_k| \right)^p \right)^{1/p}.$$

In 1969, *Hedlund* [2] introduced the *mixed norm spaces*

$$\ell(p,q) = \left\{ x \in \omega : \sum_{\nu=0}^{\infty} \left(\sum_{k=2^\nu}^{2^{\nu+1}-1} |x_k|^p \right)^{q/p} < \infty \right\} \text{ see also } Kellogg\ [4];$$

obviously the Cesàro sequence spaces $ces(p)$ are weighted $\ell(p,1)$ mixed norm spaces. Results on the equivalence of block and section norms on mixed norm spaces can also be found in [1].

In this paper, we generalise the definition of mixed norm spaces, define mixed norm spaces of difference sequences, determine their β-duals, and characterise matrix transformations on them. We obtain many known results as special cases.

2. Notations and Definitions

Let ℓ_∞, c, c_0 and ϕ be the sets of all bounded, convergent, null and finite sequences, cs and bs be the sets of all convergent and bounded series, and $\ell_p = \{x \in \omega : \sum_{k=1}^\infty |x_k|^p < \infty\}$ for $1 \le p < \infty$.

By e and $e^{(n)}$ $(n = 1, 2, \dots)$, we denote the sequences with $e_k = 1$ for all k, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \neq n$.

An *FK space* X is a complete linear metric sequence space with continuous *coordinates* $P_k : X \to \mathbb{C}$ where $P_k(x) = x_k$ for all $x \in X$ and $k = 1, 2, \dots$; a *BK space* is a normed FK space. We say that an FK space $X \supset \phi$ has AK if $x^{[m]} = \sum_{k=1}^m x_k e^{(k)} \to x$ $(m \to \infty)$ for every sequence $x = (x_k)_{k=1}^\infty \in X$; $x^{[m]}$ is called the *m-section of the sequence x*.

If X and Y are subsets of ω, and z is a sequence, we write $z^{-1} * Y = \{x \in \omega : xz = (x_k z_k)_{k=1}^\infty \in Y\}$ and $M(X,Y) = \bigcap_{x \in X} x^{-1} * Y = \{z \in \omega : zx \in Y \text{ for all } x \in X\}$ for the *multiplier of X and Y*. In the special cases when $Y = \ell_1$ or $Y = cs$, we write $z^\alpha = z^{-1} * \ell_1$ or $z^\beta = z^{-1} * cs$, and the sets $X^\alpha = M(X, \ell_1)$ and $X^\beta = M(X, cs)$ are called the α- or *Köthe-Toeplitz*- and β-*duals of* X.

Let $A = (a_{nk})_{n,k=1}^\infty$ be an infinite matrix of complex numbers, x be a sequence and X be a subset of ω. Then we write $A_n = (a_{nk})_{k=1}^\infty$ and $A^k = (a_{nk})_{n=1}^\infty$ for the sequences in the n-th row and the k-th column of A, respectively, A^T for the transpose of A, $A_n(x) = \sum_{k=1}^\infty a_{nk} x_k$ $(n = 1, 2, \dots)$ and $A(x) = (A_n(x))_{n=1}^\infty$, provided $A_n \in x^\beta$ for all n. The set $X_A = \{z \in \omega : A(z) \in X\}$ is called the *matrix domain of A in X*. Given any subsets X and Y of ω, then (X, Y) denotes the class of all matrices A that map X into Y, that is for which $A_n \in X^\beta$ for all n and $A(x) \in Y$ for all $x \in X$, or equivalently $A \in (X, Y)$ if and only if $X \subset Y_A$.

Throughout, let $(k(\nu))_{\nu=0}^\infty$ be a strictly increasing sequence of integers with $k(0) = 1$ and I_ν be the set of all integers k with $k(\nu) \le k \le k(\nu+1) - 1$ $(\nu = 0, 1, \dots)$. Given any sequence x, then, for each $\nu = 0, 1, \dots$, $x^{\langle \nu \rangle} = \sum_{k \in I_\nu} x_k e^{(k)}$ is the *ν-block of the sequence x*. Let $X, Y \supset \phi$ be sequence spaces, normed with $\|\cdot\|_X$ and $\|\cdot\|_Y$. We define the *generalised mixed norm spaces*

$$Z = [Y, X]^{\langle k(\nu)\rangle} = \left\{ z \in \omega : \left(\|z^{\langle \nu \rangle}\|_X \right)_{\nu=0}^{\infty} \in Y \right\}$$

and put

$$g(z) = \left\| \left(\|z^{\langle \nu \rangle}\|_X \right)_{\nu=0}^{\infty} \right\|_Y \quad (z \in Z). \tag{2.1}$$

Since $\phi \subset X$, $\|z^{\langle \nu \rangle}\|_X$ is defined for every $z \in \omega$ and for all $\nu = 0, 1, \dots$. Hence the sequence $y = (y_\nu)_{\nu=0}^{\infty}$ with $y_\nu = \|z^{\langle \nu \rangle}\|_X$ $(\nu = 0, 1, \dots)$ is defined. Furthermore, since $\phi \subset X, Y$, we obviously have $\phi \subset Z$.

Finally, let $\Delta = (\delta_{nk})_{n,k=1}^{\infty}$ be the matrix with $\delta_{nn} = 1$, $\delta_{n,n-1} = -1$ and $\delta_{nk} = 0$ otherwise. Then we define the *mixed norm spaces of difference sequences*

$$Z_\Delta = \left([Y, X]^{\langle k(\nu)\rangle} \right)_\Delta .$$

We consider a few special cases.

EXAMPLE 2.1. (a) Let $1 \le p < \infty$ and $1 \le r < \infty$. Then we obtain

$$[\ell_r, \ell_p]^{\langle k(\nu)\rangle} = \left\{ z \in \omega : \sum_{\nu=0}^{\infty} \left(\sum_{k \in I_\nu} |z_k|^p \right)^{r/p} < \infty \right\},$$

$$[l_r, \ell_\infty]^{\langle k(\nu)\rangle} = \left\{ z \in \omega : \sum_{\nu=0}^{\infty} \left(\max_{k \in I_\nu} |z_k| \right)^r < \infty \right\},$$

$$[c_0, \ell_p]^{\langle k(\nu)\rangle} = \left\{ z \in \omega : \lim_{\nu \to \infty} \sum_{k \in I_\nu} |z_k|^p = 0 \right\} \text{ and}$$

$$[\ell_\infty, \ell_p]^{\langle k(\nu)\rangle} = \left\{ z \in \omega : \sup_{\nu \ge 0} \sum_{k \in I_\nu} |z_k|^p < \infty \right\}.$$

In the special case of $r = p$ and $1 \le p \le \infty$, we have $[\ell_p, \ell_p]^{\langle k(\nu)\rangle} = \ell_p$.

If $k(\nu) = 2^\nu$ for $\nu = 0, 1, \dots$, then $[\ell_r, \ell_p]^{\langle k(\nu)\rangle} = \ell(r, p)$, the mixed norm spaces in [2, 4].

If $k(\nu) = \nu + 1$ for $\nu = 0, 1, \dots$, then we also obtain the classical sequence spaces $[\ell_r, \ell_1]^{\langle k(\nu)\rangle} = \ell_r$, $[c_0, \ell_1]^{\langle k(\nu)\rangle} = c_0$ and $[\ell_\infty, \ell_1]^{\langle k(\nu)\rangle} = \ell_\infty$.

(b) Let $1 \le p < \infty$ and $k(\nu) = 2^\nu$ for all ν. If $d_\nu = (1/k(\nu+1))^{1/p}$ for $\nu = 0, 1, \dots$ then

$$[d^{-1} * c_0, \ell_p]^{\langle k(\nu)\rangle} = w_0^p \quad \text{and} \quad [d^{-1} * \ell_\infty, \ell_p]^{\langle k(\nu)\rangle} = w_\infty^p \text{ [5]}.$$

If $d_\nu = 2^{\nu(1/p-1)}$ for $\nu = 0, 1, \dots$ then we obtain the Cesàro sequence spaces or weighted mixed norm spaces $[d^{-1} * \ell_p, \ell_1]^{\langle k(\nu)\rangle} = ces(p)$ [3].

EXAMPLE 2.2. (a) Let $1 \le p < \infty$ and $1 \le r < \infty$. Then we obtain

$$\left([\ell_r, \ell_p]^{\langle k(\nu)\rangle} \right)_\Delta = \left\{ z \in \omega : \sum_{\nu=0}^{\infty} \left(\sum_{k \in I_\nu} |z_k - z_{k-1}|^p \right)^{r/p} < \infty \right\} \text{ etc.}$$

If $k(\nu) = \nu + 1$ for $\nu = 0, 1, \dots$ then we obtain the *sets of sequences of bounded variation*

$$bv^p = \left([\ell_p, \ell_1]^{\langle k(\nu)\rangle} \right)_\Delta = \left\{ z \in \omega : \sum_{\nu=0}^{\infty} |z_{\nu+1} - z_\nu|^p < \infty \right\} \text{ [12]},$$

and the sets of difference sequences that are convergent to zero or bounded $([c_0,\ell_1]^{\langle k(\nu)\rangle})_\Delta = (c_0)_\Delta = c_0(\Delta)$ and $([\ell_\infty,\ell_1]^{\langle k(\nu)\rangle})_\Delta = (\ell_\infty)_\Delta = \ell_\infty(\Delta)$ [9].

(b) Let $(\mu_k)_{k=0}^\infty$ be an increasing sequence of positive reals tending to infinity and $d_\nu = 1/\mu_{k(\nu+1)}$ for $\nu = 0,1,\dots$. Then we obtain the sets of sequences that are *μ-strongly convergent to zero* or *bounded*, respectively, *with index* p

$$c_0(\mu) = \mu^{-1} * \left([d^{-1} * c_0, \ell_p]^{<k(\nu)>}\right)_\Delta$$
$$= \left\{z \in \omega : \lim_{\nu\to\infty} \frac{1}{\mu^p_{k(\nu+1)}} \sum_{k\in I_\nu} |\mu_k z_k - \mu_{k-1} z_{k-1}|^p = 0\right\}$$

and $c_\infty(\mu) = \mu^{-1} * ([d^{-1} * \ell_\infty, \ell_p]^{\langle k(\nu)\rangle})_\Delta$ [10].

3. The topological properties of the spaces Z and Z_Δ

Here we study the topological properties of $Z = [Y,X]^{\langle k(\nu)\rangle}$ and $Z_\Delta = ([Y,X]^{\langle k(\nu)\rangle})_\Delta$.

A norm $\|\cdot\|$ on a sequence space X is said to be *monotonous* if $|x_k| \le |\tilde{x}_k|$ $(k = 1,2,\dots)$ for $x, \tilde{x} \in X$ implies $\|x\| \le \|\tilde{x}\|$. A subset X of ω is called *normal* if $x \in X$ and $|y_k| \le |x_k|$ $(k = 1,2,\dots)$ for a sequence y together imply $y \in X$.

Given $z \in \omega$, we write $y = (y_\nu)_{\nu=0}^\infty$ for the sequence with $y_\nu = \|z^{\langle\nu\rangle}\|_X$ $(\nu = 0,1,\dots)$.

PROPOSITION 3.1. *Let $X \supset \phi$ and $Y \supset \phi$ be normed sequence spaces and $Z = [Y,X]^{\langle k(\nu)\rangle}$.*

(a) If Y is normal and $\|\cdot\|_X$ is monotonous then Z is normal.

(b) If $\|\cdot\|_Y$ is monotonous then Z is normed with respect to g defined in (2.1). If, however, $\|\cdot\|_Y$ is not monotonous, then g does not satisfy the triangle inequality in general.

Proof. (a) If $z \in Z$ and $\tilde{z} \in \omega$ with $|\tilde{z}_k| \le |z_k|$ for all k, then the monotony of $\|\cdot\|_X$ implies $|\tilde{y}_\nu| \le |y_\nu|$ for all ν. Since Y is normal, it follows that $\tilde{z} \in Z$.

(b) We show that g satisfies the triangle inequality, since it obviously satisfies the other properties of a norm. Let $z, \tilde{z} \in Z$. Then $\|(z+\tilde{z})^{<\nu>}\|_X = \|z^{\langle\nu\rangle} + \tilde{z}^{<\nu>}\|_X \le \|z^{\langle\nu\rangle}\|_X + \|\tilde{z}^{<\nu>}\|_X = y_\nu + \tilde{y}_\nu$ $(\nu = 0,1,\dots)$, so $z + \tilde{z} \in Z$, since Y is normal. Furthermore, by the monotony of $\|\cdot\|_Y$, we have $g(z+\tilde{z}) \le \|y+\tilde{y}\|_Y \le \|y\|_Y + \|\tilde{y}\|_Y = g(z) + g(\tilde{z})$.

To prove the last part, we choose $Y = (\ell_1)_\Delta$, $\|y\|_{bv} = \|\Delta(y)\|_1$, $k(\nu) = \nu+1$ $(\nu = 0,1,\dots)$ and $X = \ell_1$ with its natural norm. Then obviously $\|\cdot\|_Y$ is not monotonous. If we choose $z = e^{(1)} + e^{(2)} + e^{(3)}$ and $\tilde{z} = e^{(1)} - e^{(2)} + e^{(3)}$ then $g(z+\tilde{z}) = 2g(e^{(1)} + e^{(3)}) = 8 > 4 = g(z) + g(\tilde{z})$. ■

THEOREM 3.2. *Let $X \supset \phi$ be a normed sequence space, $Y \supset \phi$ be a normal BK space and $\|\cdot\|_Y$ be monotonous. Then Z is a BK space with $\|\cdot\|_Z = g$ where*

g is defined in (2.1). Furthermore, if Y has AK and $\|\cdot\|_X$ is monotonous then Z also has AK.

Proof. By Proposition 3.1, $\|\cdot\|_Z = g$ is a norm. We write $\|\cdot\| = \|\cdot\|_Z$ for short. First, since Y is a BK space, $\|z^{(m)} - z\| \to 0$ $(m \to \infty)$ implies $\|(z^{(m)})^{<\nu>} - z^{\langle\nu\rangle}\|_X \to 0$ $(m \to \infty)$ for each ν, and it follows that $|z_k^{(m)} - z_k| \to 0$ $(m \to \infty)$ for each $k \in I_\nu$ $(\nu = 0, 1, \dots)$, since for each ν there are only finitely many $k \in I_\nu$. Thus the norm $\|\cdot\|$ is stronger than the metric of ω on Z.

To show that Z is complete with $\|\cdot\|$, let $(z^{(m)})_{m=1}^\infty$ be a Cauchy sequence in Z, hence in ω by what we have just shown. Thus there exists $z \in \omega$ such that

$$z^{(m)} \to z \ (m \to \infty) \text{ in } \omega. \tag{3.1}$$

Furthermore, by the completeness of Y, there is $y \in Y$ such that

$$y^{(m)} = \left(\|(z^{(m)})^{<\nu>}\|_X\right)_{\nu=0}^\infty \to y \ (m \to \infty) \text{ in } Y. \tag{3.2}$$

From (3.1), we conclude $z_k^{(m)} \to z_k$ $(m \to \infty)$ for each k, hence $(z^{(m)})^{<\nu>} \to z^{\langle\nu\rangle}$ $(m \to \infty)$ for each ν, and so

$$y_\nu^{(m)} = \|(z^{(m)})^{<\nu>}\|_X \to \|z^{\langle\nu\rangle}\|_X \ (m \to \infty) \text{ or each } \nu. \tag{3.3}$$

Since Y is a BK space, (3.2) implies $y_\nu^{(m)} \to y_\nu$ $(m \to \infty)$ for each ν, and so, by (3.3), $y_\nu = \|z^{\langle\nu\rangle}\|_X$ for each ν and $y = (\|z^{<\nu>}\|_X)_{\nu=0}^\infty \in Y$, hence $z \in Z$. This shows that Z is complete.

Finally, let Y have AK and $\|\cdot\|_X$ be monotonous. We show that Z as AK. Let $z = (z_k)_{k=1}^\infty \in Z$ and $\varepsilon > 0$ be given. For each $m \in \mathbb{N}$ let ν_m be the uniquely defined integer for which $m \in I_{\nu_m}$. We define the sequence $y = (y_\nu)_{\nu=0}^\infty$ by $y_\nu = \|z^{\langle\nu\rangle}\|_X$ for $\nu = 0, 1, \dots$, and write $y^{[\mu]} = \sum_{\nu=0}^\mu y_\nu e^{(\nu)}$ for $\mu = 0, 1, \dots$. Since Y has AK, there exists an integer μ_0 such that $\|y - y^{[\mu]}\|_Y < \varepsilon$ for all $\mu \geq \mu_0$.We choose $m_0 = k(\mu_0 + 1)$. Let $m \geq m_0$ be given. Then $\nu_m \geq \mu_0 + 1$ and

$$\tilde{y}_\nu = \|(z - z^{[m]})^{\langle\nu\rangle}\|_X = 0 = y_\nu - y_\nu^{[\nu_m - 1]} \text{ for } 0 \leq \nu \leq \nu_m - 1$$

$$\tilde{y}_{\nu_m} = \|(z - z^{[m]})^{\langle\nu_m\rangle}\|_X = \|(0, \dots, 0, z_{m+1}, \dots)^{\langle\nu_m\rangle}\|_X \leq \|z^{\langle\nu_m\rangle}\|_X = y_{\nu_m}$$

since $\|\cdot\|_X$ is monotonous, and

$$\tilde{y}_\nu = \|(z - z^{[m]})^{\langle\nu\rangle}\|_X = \|z^{\langle\nu\rangle}\|_X = y_\nu \text{ for all } \nu \geq \nu_m + 1.$$

Thus $|\tilde{y}_\nu| \leq |y_\nu - y_\nu^{[\nu_m - 1]}|$ for all ν, and so

$$\|\tilde{y}\|_Y = \left\|\left(\left\|(z - z^{[m]})^{<\nu>}\right\|_X\right)_{\nu=0}^\infty\right\|_Y \leq \|y - y^{[\nu_m - 1]}\|_Y < \varepsilon,$$

since $\|\cdot\|_Y$ is monotonous. Therefore $z^{[m]} \to z$ $(m \to \infty)$. ∎

As an immediate consequence of Theorem 3.2 and [14, Theorem 4.3.12, p. 63], we obtain

COROLLARY 3.3. *Let $X \supset \phi$ be a normed sequence space, $Y \supset \phi$ be a normal BK space and $\|\cdot\|_Y$ be monotonous. Then Z_Δ is a BK space with $\|z\|_\Delta = g(\Delta(z))$ $(z \in Z_\Delta)$ where g is defined in (2.1).*

We close this section with a few examples.

EXAMPLE 3.4. (a) Let $1 \leq p < \infty$ and $1 \leq r < \infty$. Then $[\ell_r, \ell_p]^{\langle k(\nu)\rangle}$ and $[c_0, \ell_p]^{\langle k(\nu)\rangle}$ are BK spaces with AK with

$$\|z\|_{(r,p)} = \left(\sum_{\nu=0}^{\infty}\left(\sum_{k\in I_\nu}|z_k|^p\right)^{r/p}\right)^{1/r} \text{ and } \|z\|_{(\infty,p)} = \sup_{\nu\geq 0}\left(\sum_{k\in I_\nu}|z_k|^p\right)^{1/p},$$

and $[\ell_\infty, \ell_p]^{\langle k(\nu)\rangle}$ is a BK space with $\|\cdot\|_{(\infty,p)}$; moreover, $[c_0, \ell_p]^{\langle k(\nu)\rangle}$ is a closed subspace of $[\ell_\infty, \ell_p]^{\langle k(\nu)\rangle}$ by [14, Corollary 4.2.4, p. 56]. The spaces $[l_r, \ell_\infty]^{\langle k(\nu)\rangle}$ are BK spaces with AK with

$$\|z\|_{(r,\infty)} = \left(\sum_{\nu=0}^{\infty}\left(\max_{k\in I_\nu}|z_k|\right)^{r}\right)^{1/r}.$$

(b) Let $1 \leq p < \infty$ and the sequences $(k(\nu))_{\nu=0}^{\infty}$ and $d = (d_\nu)_{\nu=0}^{\infty}$ be defined as in Example 2.1(b). Since c_0 and ℓ_∞ are BK spaces and c_0 has AK, and since $d_\nu \neq 0$ for all ν, the sets $Y_0 = d^{-1} * c_0$ and $Y_\infty = d^{-1} * \ell_\infty$ are BK spaces with $\|y\|_{Y_\infty} = \|yd\|_\infty$, and Y_0 has AK (cf. [14, Theorems 4.3.6 and 4.3.12, pp. 62 and 63]. Furthermore, obviously $\|\cdot\|_{Y_\infty}$ and $\|\cdot\|_p$ are monotonous. Therefore w_0^p and w_∞^p are BK spaces with

$$\|z\|' = \sup_{\nu=0}\left(\frac{1}{2^{\nu+1}}\sum_{k=2^\nu}^{2^{\nu+1}-1}|x_k|^p\right)^{1/p},$$

and w_0^p has AK; moreover w_0^p is a closed subspace of w_∞^p by [14, Corollary 4.2.4, p. 56].

EXAMPLE 3.5. (a) Let $1 \leq p < \infty$ and $1 \leq r < \infty$. Then $([\ell_r, \ell_p]^{\langle k(\nu)\rangle})_\Delta$, $([c_0, \ell_p]^{\langle k(\nu)\rangle})_\Delta$ and $([\ell_\infty, \ell_p]^{\langle k(\nu)\rangle})_\Delta$ are BK spaces with

$$\|z\|_{(r,p)_\Delta} = \left(\sum_{\nu=0}^{\infty}\left(\sum_{k\in I_\nu}|z_k - z_{k-1}|^p\right)^{r/p}\right)^{1/r} \qquad \text{and}$$

$$\|z\|_{(\infty,p)_\Delta} = \sup_{\nu\geq 0}\left(\sum_{k\in I_\nu}|z_k - z_{k-1}|^p\right)^{1/p},$$

and $([c_0, \ell_p]^{\langle k(\nu)\rangle})_\Delta$ is a closed subspace of $([\ell_\infty, \ell_p]^{\langle k(\nu)\rangle})_\Delta$ by Example 3.4(a) and [14, Theorem 4.3.14, p. 64].

(b) Let the sequences $(\mu_k)_{k=0}^{\infty}$ and $d = (d_\nu)_{\nu=0}^{\infty}$ be as in Example 2.2(b). Then, by a similar argument as that used in Example 3.4(b), $c_0(\mu)$ and $c_\infty(\mu)$ are BK spaces with

$$\|z\|_{c_\infty(\mu)} = \sup_{\nu\geq 0}\frac{1}{\mu_{k(\nu+1)}}\left(\sum_{k\in I_\nu}|\mu_k x_k - \mu_{k-1}x_{k-1}|^p\right)^{1/p},$$

and $c_0^p(\mu)$ is a closed subspace of $c_\infty^p(\mu)$ by Example 3.4 and [14, Theorem 4.3.14, p. 64].

4. The β-duals of the spaces Z and matrix transformations

In this section, we determine the β-duals of the spaces Z and characterise some classes of matrix transformations between them.

We denote the closed unit ball in a normed space X by $B_X = \{x \in X : \|x\| \leq 1\}$. If X is a normed sequence space and $a \in \omega$, we write $\|a\|_{X,\alpha} = \sup_{x \in B_X} \sum_{k=0}^{\infty} |a_k x_k|$ and $\|a\|_{X,\beta} = \sup_{x \in B_X} |\sum_{k=0}^{\infty} a_k x_k|$ provided the expressions exist and are finite which is the case whenever X is a BK space and $a \in X^{\alpha}$ or $a \in X^{\beta}$ (cf. [14, Theorems 4.3.15 and 7.2.9, pp. 64 and 107].

A norm on a sequence space X is said to be KB if the set $\mathcal{P} = \{P_k : X \to \mathbb{C} : P_k(x) = x_k \ (x \in X) \ k = 1, 2, \ldots\}$ of coordinates is equicontinuous, that is if there is a constant K such that $|x_k| \leq K\|x\|$ $(k = 1, 2, \ldots)$ for all $x \in X$. If X is a Banach sequence space with a norm which is KB then it is obviously a BK space. Conversely the norm of a BK space need not be KB in general. To see this, we choose $X = (\ell_\infty)_\Delta$ with $\|x\| = \sup_k |x_k - x_{k-1}|$, a BK space, and the sequence x with $x_k = k$ for $k = 1, 2, \ldots$.

If X is a normed sequence space then we write $X^{\delta} = \{a \in \omega : \|a\|_{X,\alpha} < \infty\}$.

THEOREM 4.1. *Let X and Y be normed sequence spaces with $X, Y \supset \phi$ and $\|\cdot\|_Y$ be monotonous.*

(a) Then $[Y^{\delta}, X^{\delta}]^{\langle k(\nu) \rangle} \subset ([Y, X]^{\langle k(\nu) \rangle})^{\delta}$.

(b) If, in addition, the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ are both KB, $\|\cdot\|_X$ is monotonous and Y is normal then $([Y, X]^{\langle k(\nu) \rangle})^{\delta} \subset [Y^{\delta}, X^{\delta}]^{\langle k(\nu) \rangle}$.

Proof. We write $Z = [Y, X]^{\langle k(\nu) \rangle}$ and $W = [Y^{\delta}, X^{\delta}]^{\langle k(\nu) \rangle}$. Since $\|\cdot\|_{X,\alpha}$ and $\|\cdot\|_{Y,\alpha}$ are norms on X^{δ} and Y^{δ}, respectively, and $\phi \subset X, Y$, the set $W = \{w \in \omega : (\|w^{\langle \nu \rangle}\|_{X,\alpha})_{\nu=0}^{\infty} \in Y^{\delta}\}$ is defined.

(a) First we observe that Z is a normed space with $\|\cdot\| = g$ by Proposition 3.1. Let $a \in W$ and $z \in B_Z$. Then $z^{\langle \nu \rangle} \in X$ for $\nu = 0, 1, \ldots$, and, by the definition of the norm $\|\cdot\|_{X,\alpha}$, we have

$$\sum_{k=1}^{\infty} |a_k^{<\nu>} z_k^{<\nu>}| \leq \|a^{\langle \nu \rangle}\|_{X,\alpha} \|z^{\langle \nu \rangle}\|_X \text{ for all } \nu = 0, 1, \ldots. \tag{4.1}$$

We define the sequences y and b by $y_\nu = \|z^{\langle \nu \rangle}\|_X$ and $b_\nu = \|a^{\langle \nu \rangle}\|_{X,\alpha}$ $(\nu = 0, 1, \ldots)$. Then $y \in B_Y$ and $b \in Y^{\delta}$, and it follows from (4.1) that $\sum_{k=1}^{\infty} |a_k z_k| = \sum_{\nu=0}^{\infty} \sum_{k=1}^{\infty} |a_k^{<\nu>} z_k^{<\nu>}| \leq \sum_{\nu=0}^{\infty} |b_\nu y_\nu| \leq \|b\|_{Y,\alpha}$ by the definition of the norm $\|\cdot\|_{Y,\alpha}$. Therefore $\|a\|_{Z,\alpha} = \sup_{z \in B_Z} \sum_{k=1}^{\infty} |a_k z_k| \leq \|b\|_{Y,\alpha} < \infty$, that is $a \in Z^{\delta}$.

(b) First we observe that Z is a BK space by Theorem 3.2. Let $a \in Z^{\delta}$ be given. Then

$$\sum_{k=1}^{\infty} |a_k z_k| \leq \|a\|_{Z,\alpha} = K_1 < \infty \text{ for all } z \in B_Z. \tag{4.2}$$

We have to show $a \in W$, that is

$$\sup_{y \in B_Y} \sum_{\nu=0}^{\infty} \|a^{\langle \nu \rangle}\|_{X,\alpha} |y_\nu| < \infty. \tag{4.3}$$

We note that $\|a^{\langle\nu\rangle}\|_{X,\alpha}$ is defined for each ν. For if $x \in B_X$ is given then $\sum_{k=1}^{\infty}|a_k^{<\nu>}x_k| = \sum_{k\in I_\nu}|a_k^{<\nu>}x_k|$, and since $\|\cdot\|_X$ is KB, there is a constant K_2 such that

$$\sum_{k=1}^{\infty}|a^{<\nu>}x_k| \le K_2 \sum_{k\in I_\nu}|a_k^{<\nu>}|\,\|x\|_X \le K_2 \sum_{k\in I_\nu}|a_k^{<\nu>}|,$$

hence $\|a^{\langle\nu\rangle}\|_{X,\alpha} = \sup_{x\in B_X}\sum_{k=1}^{\infty}|a_k^{<\nu>}x_k| \le K_2\sum_{k\in I_\nu}|a_k^{<\nu>}| < \infty$ for all ν. Now let $y \in B_Y$ be given. By the definition of $\|\cdot\|_{X,\alpha}$, for every ν, we can choose a sequence $x(\nu) = (x_k(\nu))_{k=1}^{\infty} \in B_X$ such that $\|a^{\langle\nu\rangle}\|_{X,\alpha} \le \sum_{k=1}^{\infty}|a_k^{<\nu>}x_k(\nu)| + 2^{-(\nu+1)}$, whence

$$\sum_{\nu=0}^{\infty}\|a^{\langle\nu\rangle}\|_{X,\alpha}\,|y_\nu| \le \sum_{\nu=0}^{\infty}\left(\sum_{k=1}^{\infty}|a_k^{<\nu>}x_k(\nu)y_\nu| + \frac{1}{2^{\nu+1}}|y_\nu|\right). \tag{4.4}$$

Since $\|\cdot\|_Y$ is KB, there is a constant K_3 such that $|y_\nu| \le K_3\|y\|_Y \le K_3$ for all $\nu = 0,1,\dots$, and it follows from (4.4) that

$$\sum_{\nu=0}^{\infty}\|a^{\langle\nu\rangle}\|_{X,\alpha}\,|y_\nu| \le \sum_{\nu=0}^{\infty}\left(\sum_{k=1}^{\infty}|a_k^{<\nu>}x_k(\nu)y_\nu|\right) + K_3. \tag{4.5}$$

We define the sequence z by $z_k = x_k(\nu)y_\nu$ ($k \in I_\nu$; $\nu = 0,1,\dots$). Then $\|z^{\langle\nu\rangle}\|_X = |y_\nu|\|(x(\nu))^{<\nu>}\|_X$ for all $\nu = 0,1,\dots$. Since, for each ν, we have $|(x_k(\nu))^{<\nu>}| \le |x_k(\nu)|$ ($k = 1,2,\dots$), the monotony of $\|\cdot\|_X$ implies $\|(x(\nu))^{<\nu>}\|_X \le \|x(\nu)\|_X = 1$ ($\nu = 0,1,\dots$), hence $\|z^{\langle\nu\rangle}\|_X \le |y_\nu|$ ($\nu = 0,1,\dots$). Since Y is normal, this implies $(\|z^{\langle\nu\rangle}\|_X)_{\nu=0}^{\infty} \in Y$, that is $z \in Z$. Furthermore, $|\,|y_\nu|\,| \le |y_\nu|$ for $\nu = 0,1,\dots$ implies $(|y_\nu|)_{\nu=0}^{\infty} \in Y$, since Y is normal, and the monotony of $\|\cdot\|_Y$ yields $\|z\|_Z \le \|(|y_\nu|)_{\nu=0}^{\infty}\|_Y \le \|y\|_Y$. Now (4.5) and (4.2) together imply

$$\sum_{\nu=0}^{\infty}\|a^{\langle\nu\rangle}\|_{X,\alpha}\,|y_\nu| \le \sum_{\nu=0}^{\infty}\sum_{k\in I_\nu}|a_k^{<\nu>}z_k| + K_3 = \sum_{k=1}^{\infty}|a_k z_k| \le K_1\|z\|_Z + K_3 \le K_1 + K_3.$$

Since $y \in B_Y$ was arbitrary, condition (4.3) follows. ∎

If X is a BK space then $X^\alpha = X^\delta$ by [14, Theorem 4.3.15, p. 64], and if X is normal then $X^\alpha = X^\beta$. Therefore we obtain from Proposition 3.1 and Theorems 3.2 and 4.1

Corollary 4.2. *Let X be a normed sequence space, Y be a normal BK space and the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ be monotonous and KB. Then $Z^\alpha = ([Y,X]^{\langle k(\nu)\rangle})^\alpha = [Y^\alpha, X^\alpha]^{\langle k(\nu)\rangle}$. If, in addition, X is normal then $Z^\beta = [Y^\beta, X^\beta]^{\langle k(\nu)\rangle}$.*

Example 4.3. (a) Let $1 \le p < \infty$, $1 \le r < \infty$, q and s be the *conjugate numbers of* p and r, that is $q = \infty$ for $p = 1$ and $q = p/(p-1)$ for $1 < p < \infty$ and s defined similarly. Since the norms $\|\cdot\|_{\ell_p,\beta}$ and $\|\cdot\|_q$ and $\|\cdot\|_{\ell_\infty,\alpha}$ and $\|\cdot\|_1$ are equivalent on ℓ_p^β and on $\ell_\infty^\beta = c_0^\beta$, we have $([\ell_r,\ell_p]^{\langle k(\nu)\rangle})^\beta = [\ell_s,\ell_q]^{\langle k(\nu)\rangle}$ and $([c_0,\ell_p]^{\langle k(\nu)\rangle})^\beta = ([\ell_\infty,\ell_p]^{\langle k(\nu)\rangle})^\beta = [\ell_1,\ell_q]^{\langle k(\nu)\rangle}$.

(b) Let $\mathcal{U}$ denote the set of all sequences u with $u_k \neq 0$ for all k. If $u \in \mathcal{U}$ then we write $1/u = (1/u_k)_{k=1}^{\infty}$, and it is obvious that $(u^{-1} * X)^{\beta} = (1/u)^{-1} * X^{\beta}$ for arbitrary subsets X of ω. Let the sequences $k(\nu)$ and d be defined as in Example 2.1(b). Then

$$(w_0^p)^{\beta} = (w_{\infty}^p)^{\beta} = \mathcal{M}_p = \begin{cases} \left\{a \in \omega : \sum_{\nu=0}^{\infty} 2^{\nu+1} \max_{k \in I_{\nu}} |a_k| < \infty\right\} & (p = 1) \\ \left\{a \in \omega : \sum_{\nu=0}^{\infty} 2^{\nu+1} \left(\sum_{k \in I_{\nu}} |a_k|^q\right)^{1/q} < \infty\right\} & (1 < p < \infty). \end{cases}$$

Now we characterise some classes of matrix transformations between mixed norm spaces.

Let $(m(\mu))_{\mu=0}^{\infty}$ be a strictly increasing sequence of integers with $m(0) = 1$ and $M_{\mu} = \{m \in \mathbb{N} : m(\mu) \leq m \leq m(\mu+1) - 1\}$ $(\mu = 0, 1, \ldots)$. Furthermore, let T denote the set of all sequences $(t_{\mu})_{\mu=0}^{\infty}$ of integers such that for each μ there is one and only one $t_{\mu} \in M_{\mu}$.

First we give a result that characterises the classes (X, Y) where X is any BK space and Y is any of the spaces ℓ_{∞}, c_0, ℓ_1, $[\ell_{\infty}, \ell_1]^{\langle m(\mu)\rangle}$, $[\ell_1, \ell_{\infty}]^{\langle m(\mu)\rangle}$ or $[c_0, \ell_1]^{\langle m(\mu)\rangle}$.

THEOREM 4.4. *Let X be a BK space, or a BK space with AK in the cases marked $*$. We write $\sup_N$ for the supremum taken over all finite subsets N of $\mathbb{N}_0$. Then the conditions for $A \in (X, Y)$ when Y is any of the spaces ℓ_{∞}, c_0, ℓ_1, $[\ell_{\infty}, \ell_1]^{\langle m(\mu)\rangle}$, $[\ell_1, \ell_{\infty}]^{\langle m(\mu)\rangle}$ or $[c_0, \ell_1]^{\langle m(\mu)\rangle}$ can be read from the table*

From \ To	ℓ_{∞}	c_0	ℓ_1	$[\ell_{\infty}, \ell_1]^{\langle m(\mu)\rangle}$	$[\ell_1, \ell_{\infty}]^{\langle m(\mu)\rangle}$	$[c_0, \ell_1]^{\langle m(\mu)\rangle}$
X	*(1.)*	**(2.)*	*(3.)*	*(4.)*	*(5.)*	**(6.)*

where

(1.) *(1.1)* *where (1.1)* $\sup_n \|A_n\|_{X,\beta} < \infty$

(2.) *(1.1) and (2.1)* *where (2.1)* $\lim_{n\to\infty} a_{nk} = 0$ *for each* k

(3.) *(3.1)* *where (3.1)* $\sup_N \left\|\sum_{n\in N} A_n\right\|_{X,\beta} < \infty$

(4.) *(4.1)* *where (4.1)* $\sup_{\mu}\left(\max_{M(\mu)\subset M_{\mu}} \left\|\sum_{m\in M(\mu)} A_m\right\|_{X,\beta}\right) < \infty$

(5.) *(5.1)* *where (5.1)* $\sup_N\left(\sup_{t\in T} \left\|\sum_{\mu\in N} A_{t_{\mu}}\right\|_{X,\beta}\right) < \infty$

(6.) *(4.1) and (6.1)* *where (6.1)* $\lim_{\mu\to\infty} \sum_{n\in M_{\mu}} |a_{nk}| = 0$ *for each* k.

Proof. **(1.)** is [11, Theorem 1.23, p. 155], **(2.)** follows from **(1.)** and [14, 8.3.6, p. 123], since c_0 is a closed subspace of ℓ_{∞}, and **(3.)** is [8, Satz 1].

(4.) We have $A \in (X, [\ell_\infty, \ell_1]^{\langle m(\mu)\rangle})$ if and only if $A_n \in X^\beta$ for all n and

$$(\|(A(x))^{\langle m(\mu)\rangle}\|_1)_{\mu=0}^\infty \in \ell_\infty \text{ for all } x \in X. \tag{4.6}$$

Since by a well-known inequality [13]

$$\max_{M(\mu)\subset M_\mu} \left|\sum_{m\in M(\mu)} A_m(x)\right| \le \sum_{m\in M_\mu} |A_m(x)| = \|(A(x))^{\langle \mu\rangle}\|_1 \le$$
$$\le 4\cdot \max_{M(\mu)\subset M_\mu} \left|\sum_{m\in M(\mu)} A_m(x)\right| \text{ for all } \mu \text{ and all } x \in X,$$

it follows by condition (1.1) that (4.6) holds if and only if condition (4.1) is satisfied.

(5.) First we assume that condition (5.1) holds. Then obviously $A_n \in X^\beta$ for all n. Let $x \in X$ be given. For each $\mu = 0, 1, \dots$, let $m_\mu \in M_\mu$ be such that $|A_{m_\mu}(x)| = \max_{m\in M_\mu} |A_m(x)|$. Let μ_0 be an arbitrary nonnegative integer. Then we have by the definition of the norm $\|\cdot\|_{X,\beta}$

$$\sum_{\mu=0}^{\mu_0} \|(A(x))^{\langle m(\mu)\rangle}\|_\infty = \sum_{\mu=0}^{\mu_0} |A_{m_\mu}(x)| \le 4\cdot \sup_{\substack{N\subset\mathbb{N}_0\\ N \text{finite}}} \left|\sum_{\mu\in N} A_{m_\mu}(x)\right|$$
$$\le 4\cdot\left(\sup_{\substack{N\subset\mathbb{N}_0\\ N \text{finite}}} \left\|\sum_{\mu\in N} A_{m_\mu}\right\|_{X,\beta}\right)\|x\| \le 4\cdot \sup_{\substack{N\subset\mathbb{N}_0\\ N \text{finite}}} \left(\sup_{t\in T}\left\|\sum_{\mu\in N} A_{t_\mu}\right\|_{X,\beta}\right)\|x\| < \infty.$$

Since μ_0 was arbitrary, it follows that $(\|(A(x))^{\langle m(\mu)\rangle}\|_\infty)_{\mu=0}^\infty \in \ell_1$, that is $A(x) \in [\ell_1, \ell_\infty]^{\langle m(\mu)\rangle}$.

Conversely we assume $A \in [\ell_1, \ell_\infty]^{\langle m(\mu)\rangle}$. Since X and $[\ell_1, \ell_\infty]^{\langle m(\mu)\rangle}$ are BK spaces, the map $f_A : X \to [\ell_1, \ell_\infty]^{\langle m(\mu)\rangle}$ with $f_A(x) = A(x)$ $(x \in X)$ is continuous (cf. [14, Theorem 4.2.8, p. 57]. Hence there is a constant K such that

$$\|f_A(x)\|_{(1,\infty)} = \|A(x)\|_{(1,\infty)} \le K\|x\| \text{ for all } x \in X. \tag{4.7}$$

We observe that $A_m \in X^\beta$ for all m implies $\sum_{\mu\in N} A_{t_\mu} \in X^\beta$ for all finite subsets N of $\mathbb{N}_0$ and for all sequences $t \in T$, and so by (4.7), $|\sum_{\mu\in N} A_{t_\mu}(x)| \le \sum_{\mu=0}^\infty |A_{t_\mu}(x)| \le \|f_A(x)\|_{(1,\infty)} \le K\|x\|$. Now condition (5.1) follows from the definition of the norm $\|\cdot\|_{X,\beta}$.

(6.) By Example 3.4(a), $[c_0, \ell_1]^{\langle m(\mu)\rangle}$ is a closed subspace of $[\ell_\infty, \ell_1]^{\langle m(\mu)\rangle}$. Thus **(6.)** is an immediate consequence of **(4.)** and [14, 8.3.6, p. 123]. ∎

We obtain as an immediate consequence of Example 4.3 and Theorem 4.4

COROLLARY 4.5. *Let $1 < r < \infty$ and $1 < p \le \infty$ and s and q be the conjugate numbers of r and p. Then the conditions for $A \in ([\ell_r, \ell_p]^{\langle k(\nu)\rangle}, Y)$ where Y is any of the spaces in Theorem 4.4 can be read from the table*

To / *From*	ℓ_∞	c_0	ℓ_1	$[\ell_\infty, \ell_1]^{\langle m(\mu)\rangle}$	$[\ell_1, \ell_\infty]^{\langle m(\mu)\rangle}$	$[c_0, \ell_1]^{\langle m(\mu)\rangle}$
$[\ell_r, \ell_p]^{\langle k(\nu)\rangle}$	*(1.)*	*(2.)*	*(3.)*	*(4.)*	*(5.)*	*(6.)*

where

(1.) *(1.1)* *where (1.1)* $\sup_n \sum_{\nu=0}^{\infty} \Big(\sum_{k \in I_\nu} |a_{nk}|^q \Big)^{s/q} < \infty$

(2.) *(1.1) and (2.1)* *where (2.1) is (2.1) in Theorem 4.4*

(3.) *(3.1)* *where (3.1)* $\sup_N \sum_{\nu=0}^{\infty} \Big(\sum_{k \in I_\nu} \Big| \sum_{n \in N} a_{nk} \Big|^q \Big)^{s/q} < \infty$

(4.) *(4.1)* *where (4.1)*

$$\sup_\mu \Big(\max_{M(\mu) \subset M_\mu} \sum_{\nu=0}^{\infty} \Big(\sum_{k \in I_\nu} \Big| \sum_{m \in M(\mu)} a_{mk} \Big|^q \Big)^{s/q} \Big) < \infty$$

(5.) *(5.1)* *where (5.1)* $\sup_N \Big(\sup_{t \in T} \sum_{\nu=0}^{\infty} \Big(\sum_{k \in I_\nu} \Big| \sum_{\mu \in N} a_{t_\mu,k} \Big|^q \Big)^{s/q} \Big) < \infty$

(6.) *(4.1) and (6.1)* *where (6.1) is (6.1) in Theorem 4.4.*

If $r = 1$ or $p = 1$ replace $\sum_{\nu=0}^{\infty}$ or $\sum_{k \in I_\nu}$ by $\sup_{\nu \geq 0}$ or $\max_{k \in I_\nu}$ in conditions (1.1), (3.1), (4.1) and (5.1) in **(1.)–(6.)**. *The conditions for $A \in ([c_0, \ell_p]^{\langle k(\nu) \rangle}, Y)$ are those in* **(1.)–(6.)** *with $s = 1$ in (1.1), (3.1), (4.1) and (5.1). Finally $([\ell_\infty, \ell_p]^{\langle k(\nu) \rangle}, Y) = ([c_0, \ell_p]^{\langle k(\nu) \rangle}, Y)$ for $Y \neq c_0, [c_0, \ell_1]^{\langle m(\mu) \rangle}$.*

Now we give the *dual result* of Theorem 4.4. We write T' for the set of all strictly increasing sequences $t = (t_\nu)_{\nu=0}^{\infty}$ of integers such that for each ν there is one and only one $t_\nu \in I_\nu$.

THEOREM 4.6. *Let W be a BK space with AK and $Y = W^\beta$. Then the conditions for $A \in (X, Y)$ where X is any of the spaces ℓ_∞, c_0, ℓ_1, $[\ell_1, \ell_\infty]^{\langle k(\nu) \rangle}$, $[\ell_\infty, \ell_1]^{\langle k(\nu) \rangle}$ or $[c_0, \ell_1]^{\langle k(\nu) \rangle}$ can be read from the table*

From / *To*	ℓ_∞	c_0	ℓ_1	$[\ell_\infty, \ell_1]^{\langle k(\nu) \rangle}$	$[\ell_1, \ell_\infty]^{\langle k(\nu) \rangle}$	$[c_0, \ell_1]^{\langle k(\nu) \rangle}$
Y	*(1.)*	*(2.)*	*(3.)*	*(4.)*	*(5.)*	*(6.)*

where

(1.) *(1.1)* *where (1.1)* $\sup_N \| \sum_{n \in N} A^n \|_Y < \infty$

(2.) *(1.1)*

(3.) *(3.1)* *where (3.1)* $\sup_n \|A^n\|_Y < \infty$

(4.) *(4.1)* *where (4.1)* $\sup_N \Big(\sup_{t \in T'} \| \sum_{\nu \in N} A^{t_\nu} \|_Y \Big) < \infty$

(5.) *(5.1)* *where (5.1)* $\sup_N \Big(\max_{K(\nu) \subset K_\nu} \| \sum_{m \in K(\nu)} A^m \|_Y \Big) < \infty$

(6.) *(4.1).*

Proof. Since X is a BK space with AK when X is any of the spaces c_0, ℓ_1, $[\ell_1, \ell_\infty]^{\langle k(\nu)\rangle}$ and $[c_0, \ell_1]^{\langle k(\nu)\rangle}$, we have $A \in (X, Y)$ if and only if $A^T \in (W, X^\beta)$ by [14, Theorem 8.3.9, p. 124], and **(2.)**, **(3.)**, **(5.)** and **(6.)** are immediate consequences of Theorem 4.4 **(3.)**, **(1.)**, **(4.)** and **(5.)**. Furthermore, since $c_0^{\beta\beta} = \ell_\infty$ and $([c_0, \ell_1]^{\langle k(\nu)\rangle})^{\beta\beta} = [\ell_\infty, \ell_1]^{\langle k(\nu)\rangle}$, and $(X, Y) = (X^{\beta\beta}, Y)$ by [14, Theorem 8.3.9, p. 124], **(1.)** and **(4.)** follow from **(2.)** and **(6.)**. ■

We obtain as an immediate consequence of Theorem 4.6

COROLLARY 4.7. *Let $1 < r < \infty$ and $1 < p < \infty$. Then the conditions for $A \in (X, [\ell_r, \ell_p]^{\langle m(\mu)\rangle})$ where X is any of the spaces in Theorem 4.4 can be read from the table*

From / To	ℓ_∞	c_0	ℓ_1	$[\ell_\infty, \ell_1]^{\langle k(\nu)\rangle}$	$[\ell_1, \ell_\infty]^{\langle k(\nu)\rangle}$	$[c_0, \ell_1]^{\langle k(\nu)\rangle}$
$[\ell_r, \ell_p]^{\langle m(\mu)\rangle}$	*(1.)*	*(2.)*	*(3.)*	*(4.)*	*(5.)*	*(6.)*

where

(1.) *(1.1) where (1.1)* $\sup_N \sum_{\mu=0}^\infty \Big(\sum_{k \in M_\mu} \Big|\sum_{n \in N} a_{kn}\Big|^p\Big)^{r/p} < \infty$

(2.) *(1.1)*

(3.) *(3.1) where (3.1)* $\sup_n \sum_{\mu=0}^\infty \Big(\sum_{k \in M_\mu} |a_{kn}|^p\Big)^{r/p} < \infty$

(4.) *(4.1) where (4.1)* $\sup_N \Big(\sup_{t \in T'} \sum_{\mu=0}^\infty \Big(\sum_{k \in M_\mu} \Big|\sum_{\nu \in N} a_{k,t_\nu}\Big|^p\Big)^{r/p}\Big) < \infty$

(5.) *(5.1) where (5.1)* $\sup_N \Big(\max_{k(\nu) \in K_\nu} \sum_{\nu=0}^\infty \Big(\sum_{k \in M_\mu} \Big|\sum_{m \in K(\nu)} a_{km}\Big|^p\Big)^{r/p}\Big) < \infty$

(6.) *(4.1).*

5. The β-duals of the spaces Z_Δ and matrix transformations

In this section, we determine the β-duals of the sets Z_Δ and characterise some matrix transformations between them.

First we prove a general result which reduces the determination of $(X_\Delta)^\beta$ for arbitrary BK spaces with AK to that of X^β and the characterisation of the class (X, c_0).

If X is a normed space, we write X^* its continuous dual, that is the set of all continuous linear functionals f on X with the norm $\|f\| = \sup_{x \in B_X} |f(x)|$.

Let $\Sigma = (\sigma_{nk})_{n,k=1}^\infty$ be the matrix with $\sigma_{nk} = 1$ for $1 \le k \le n$ and $\sigma_{nk} = 0$ for $k > n$ $(n = 1, 2, \dots)$. Then $x = \Delta(\Sigma(x)) = \Sigma(\Delta(x))$ for all $x \in \omega$. Let $X \subset \omega$

and $Y = X_\Delta$. Then $x \in X$ if and only if $y = \Sigma(x) \in Y$, and $y \in Y$ if and only if $x = \Delta(y) \in X$. If X is a BK space then so is Y and $B_X = B_Y$ by [14, Theorem 4.3.12, p. 63].

Given any sequence a, we write B^a for the matrix with the rows $B^a_n = a_n e^{[n]}$ $(n = 1, 2, \dots)$. Then $B^a_n(x) = a_n \Sigma_n(x) = a_n y_n$ for all $x \in X$, $y = \Sigma(x)$ and all n, that is

$$a \in M(X_\Delta, W) \text{ if and only if } B^a \in (X, W) \text{ for arbitrary subsets } X \text{ and } W \text{ of } \omega. \tag{5.1}$$

THEOREM 5.1. *Let $E = \Sigma^T$. If X is a BK space with AK then $a \in (X_\Delta)^\beta$ if and only if $a \in (X^\beta)_E$ and $V^a \in (X, c_0)$ where V^a is the matrix with the rows $V^a_n = E_n(a)e^{[n]}$ $(n = 1, 2, \dots)$. Furthermore if $a \in (X_\Delta)^\beta$ then*

$$\sum_{k=1}^{\infty} a_k y_k = \sum_{k=1}^{\infty} E_k(a)\Delta_k(y) \textit{ for all } y \in X_\Delta. \tag{5.2}$$

Proof. We write $Y = X_\Delta$ and $V = V^a$ for short.

First we assume $a \in Y^\beta$. Then $B^a \in (X, cs)$ by (5.1), and so $C = \Sigma B^a \in (X, c)$ by [11, Theorem 3.8, p. 180]. Since c is a closed subspace of ℓ_∞, we have by [14, 8.3.6, p. 123]

$$\lim_{n\to\infty} c_{nk} = \sum_{j=k}^{\infty} a_j = E_k(a) \text{ exists for all } k \tag{5.3}$$

and

$$C \in (X, \ell_\infty). \tag{5.4}$$

From (5.3), we obtain that the matrix V is defined and

$$\lim_{n\to\infty} v_{nk} = \lim_{n\to\infty} \sum_{j=n}^{\infty} a_j = 0. \tag{5.5}$$

We also have

$$\sum_{k=1}^{m-1} a_k y_k = \sum_{k=1}^{m} E_k(a)\Delta_k(y) - \sum_{k=1}^{m} v_{mk}\Delta_k(y) \text{ for all } m \text{ and all } y. \tag{5.6}$$

Since X is a BK space with AK, condition (5.4) implies $C^T \in (\ell_1, X^\beta)$ by [14, Theorem 8.3.9, p. 124]. Now X^β is a BK space with

$$\|b\|^\beta = \sup_m \sup_{x\in B_X} \left| \sum_{k=1}^{m} b_k x_k \right| = \sup_m \|b^{[m]}\|_{X,\beta} \ (b \in X^\beta)$$

by [14, Example 4.3.16, p. 65]. Therefore, by [14, Example 8.4.1, p. 126], the columns of the matrix C^T, that is the rows of C are a bounded set in X^β. Thus there is a constant K_1 such that

$$\left| \sum_{k=1}^{m} c_{nk} x_k \right| \le K_1 \text{ for all } m \text{ and } n \text{ and for all } x \in B_X. \tag{5.7}$$

Now (5.3) implies $|\sum_{k=1}^m E_k(a)x_k| \le K_1$ for all m and all $x \in B_X$. It follows from this and (5.6) that

$$|V_m(x)| \le K_1 + \left| \sum_{k=1}^{m-1} a_k y_k \right| \text{ for all } x \in B_X,\ y \in B_Y \text{ and all } m. \tag{5.8}$$

We define the linear functionals f_m $(m = 1, 2, \dots)$ on Y by $f_m(y) = \sum_{k=1}^{m-1} a_k y_k$ $(y \in Y)$. We note that $f_m \in Y^*$ for all m, since Y is a BK space. Furthermore $a \in Y^\beta$ implies that $f(y) = \lim_{m\to\infty} f_m(y)$ exists for every $y \in Y$, that is the sequence $(f_m)_{m=1}^\infty$ is pointwise convergent, hence pointwise bounded, and so uniformly bounded by the uniform boundedness principle. Thus there exists a constant K_2 such that $|f_m(y)| = |\sum_{k=1}^{m-1} a_k y_k| \le K_2$ for all $y \in B_Y$ and all m, and it follows from (5.8) that $|V_m(x)| \le K_1 + K_2$ for all m and for all $x \in B_X$, hence $\sup_m \|V_m\|_{X,\beta} < \infty$. This and (5.5) imply $V \in (X, c_0)$ by Theorem 4.4(**2.**); and then (5.6) implies $E(a) \in X^\beta$, that is $a \in (X^\beta)_E$.

If $a \in Y^\beta$ then $E(a) \in X^\beta$ and $V \in (X, c_0)$, as we have just shown, and so (5.2) follows from (5.6).

Conversely, if $a \in (X^\beta)$ and $V \in (X, c_0)$ then $a \in Y^\beta$ by (5.6). ■

Now we give the $(Z_\Delta)^\beta$ in some special cases.

EXAMPLE 5.2. (a) Let $1 \le p \le \infty$, $1 \le r \le \infty$ and q and s be the conjugate numbers of p and r. The conditions for $E(a) \in ([\ell_r, \ell_p]^{\langle k(\nu)\rangle})^\beta$ and $E(a) \in ([c_0, \ell_p]^{\langle k(\nu)\rangle})^\beta$ are given in Example 4.3(a). Corollary 4.5 yields the conditions for $V^a \in ([\ell_r, \ell_p]^{\langle k(\nu)\rangle}, c_0)$ and $V^a \in ([c_0, \ell_p]^{\langle k(\nu)\rangle}, c_0)$, the condition $\lim_{n\to\infty} v_{nk}^a = 0$ for each k being redundant. For each positive integer n, let $\nu(n)$ denote the uniquely defined integer such that $n \in I_{\nu(n)}$. We define the sequence $b^{s,q}$ by

$$b_n^{s,q} = \begin{cases} \left(\sum_{\nu=0}^{\nu(n)-1} ((k(\nu+1) - k(\nu))^{s/q} + (n+1-k(\nu(n))^{s/q}\right)^{1/s} & (1 < r \le \infty, 1 < p \le \infty) \\ (\nu(n)+1)^{1/s} & (1 < r \le \infty, p = 1) \\ \max\left\{\max_{0\le\nu\le\nu(n)-1} (k(\nu+1) - k(\nu))^{1/q}, (n+1-k(\nu(n)))^{1/q}\right\} & (r = 1, 1 < p \le \infty). \end{cases}$$

It is easy to see that condition (1.1) for $A = V^a$ in Corollary 4.5 is equivalent to $E(a) \in (b^{s,q})^{-1} * \ell_\infty$; in the case of $[c_0, \ell_p]^{\langle k(\nu)\rangle}$ $(1 \le p < \infty)$, we use the sequence $b^{1,q}$.

Let us mention that the condition $E(a) \in (b^{s,q})^{-1} * \ell_\infty$ becomes redundant is some cases. As in Example 2.2, let $k(\nu) = \nu+1$ $(\nu = 0, 1, \dots)$. Then, for $1 < p < \infty$, we have $bv^p = ([\ell_p, \ell_1]^{\langle k(\nu)\rangle})_\Delta$, and $a \in (bv^p)^\beta$ if and only if $\sum_{\nu=0}^\infty |\sum_{k=\nu}^\infty a_k|^q < \infty$ and $\sup_n (n+1)^{1/q} |\sum_{k=n}^\infty a_k| < \infty$, and it is easy to see that, in general neither condition implies the other. If, however, $p = 1$, then $bv = ([\ell_1, \ell_1]^{\langle k(\nu)\rangle})_\Delta = ([\ell_1, \ell_\infty]^{\langle k(\nu)\rangle})_\Delta$, and the conditions $E(a) \in [\ell_\infty, \ell_1]^{\langle k(\nu)\rangle}$ and $E(a) \in b^{\infty,1} * \ell_\infty$ are the same, namely $\sup_n |\sum_{k=n}^\infty a_k| < \infty$ that is $a \in cs$.

(b) Let the sequences μ and d be defined as in Example 2.2(b). First we observe that $a \in (c_0^p(\mu))^\beta$ if and only if $a/u = (a_k/u_k)_{k=1}^\infty \in (([d^{-1} * c_0, \ell_p]^{\langle k(\nu)\rangle})_\Delta)^\beta$. Also $E(a/u) \in [(1/d)^{-1} * \ell_1, \ell_q]^{\langle k(\nu)\rangle}$ if and only if

$$c \in [\ell_1, \ell_q]^{\langle k(\nu)\rangle} \text{ where } c_k = 1/d_\nu E_k(a/\mu) = \mu_{k(\nu+1)} \sum_{j=k}^\infty \frac{a_j}{\mu_j} \ (k \in I_\nu; \nu = 0, 1, \dots). \tag{5.10}$$

Since obviously, for all $u \in \mathcal{U}$ and for all $X, Y \subset \omega$, we have $A \in (u^{-1} * X, Y)$ if and only if $B \in (X, Y)$ where $b_{nk} = a_{nk}/u_k$ for all n and k, it follows that $V^a \in (\mu^{-1} * [d^{-1} * c_0, \ell_p]^{\langle k(\nu)\rangle}, c_0)$ if and only if $\tilde{V}^a \in ([d^{-1} * c_0, \ell_p]^{\langle k(\nu)\rangle}, c_0)$, where $\tilde{v}^a_{nk} = E_n(a/u)$ for $1 \le k \le n$ and $\tilde{v}^a_{nk} = 0$ for $k > n$ $(n = 1, 2, \ldots)$. Finally, since $z \in [d^{-1} * c_0, \ell_p]^{\langle k(\nu)\rangle}$ if and only if $y \in [c_0, \ell_p]^{\langle k(\nu)\rangle}$ where $y_k = d_\nu z_k$ $(k \in I_\nu; \nu = 0, 1, \ldots)$, we have $\tilde{V}^a \in ([d^{-1} * c_0, \ell_p]^{\langle k(\nu)\rangle}, c_0)$ if and only if $W^a \in ([c_0, \ell_p]^{\langle k(\nu)\rangle}, c_0)$ where $w^a_{nk} = \tilde{v}^a_{nk} 1/d_\nu$ $(k \in I_\nu; \nu = 0, 1, \ldots)$ for all $n = 1, 2, \ldots$. Again, the condition $\lim_{n\to\infty} w_{nk} = 0$ is redundant, and we need

$$\sup_n \sum_{\nu=0}^{\infty} \|(W^a_n)^{\langle \nu \rangle}\|_q < \infty. \tag{5.11}$$

We define the sequence $b^{1,q}(\mu)$ by

$$b^{1,q}_n(\mu) = \begin{cases} \sum\limits_{\nu=0}^{\nu(n)-1} \mu_{k(\nu+1)}(k(\nu+1) - k(\nu))^{1/q} - \mu_{k(\nu(n)+1)}(n+1-k(\nu(n)))^{1/q} & (1 < p < \infty) \\ \sum\limits_{\nu=0}^{\nu(n)} \mu_{k(\nu+1)} & (p = 1). \end{cases}$$

Condition (5.10) is equivalent to

$$\sum_{\nu=0}^{\infty} \mu_{k(\nu+1)} \left(\sum_{k \in I_\nu} \left| \sum_{j=k}^{\infty} \frac{a_j}{\mu_j} \right|^q \right)^{1/q} < \infty \ (1 < p < \infty),$$

$$\sum_{\nu=0}^{\infty} \mu_{k(\nu+1)} \max_{k \in I_\nu} \left| \sum_{j=k} \frac{a_j}{\mu_j} \right| < \infty \ (p = 1),$$

and it is easy to see that condition (5.11) is equivalent to $E(a/\mu) \in (b^{1,q})^{-1} * \ell_\infty$ for $1 < p < \infty$ and $E(a/u) \in (b^{1,\infty}(\mu))^{-1} * \ell_\infty$ for $p = 1$, this condition being redundant, if there are reals s and t with $0 < s \le \mu_{k(\nu)}/\mu_{k(\nu+1)} \le t < 1$ for all ν.

The next result reduces the the characterisation of (X_Δ, Y) to that of (X, Y) and (X, c_0).

THEOREM 5.3. *Let $X \supset \phi$ be a BK space with AK and Y be a subset of ω. Then $A \in (X_\Delta, Y)$ if and only if*

$$E^A \in (X, Y) \text{ where } e^A_{nk} = \sum_{j=k}^{\infty} a_{nj} \text{ for all } n \text{ and } k \tag{5.12}$$

and

$$V^{A_n} \in (X, c_0) \text{ for all } n \tag{5.13}$$

where V^{A_n} is the matrix with the rows $V^{A_n}_m = E_m(A_n) e^{[m]}$ $(m = 1, 2, \ldots)$.

Proof. First we assume $A \in (X_\Delta, Y)$. Then $A_n \in (X_\Delta)^\beta$ for all n, hence condition (5.13) holds and

$$E(A_n) \in X^\beta \text{ for all } n \tag{5.14}$$

by Theorem 5.1. Let $x \in X$ be given. Then $A_n \in (X_\Delta)^\beta$ implies

$$(E^A)_n(x) = A_n(\Sigma(x)) \text{ for all } n \tag{5.15}$$

by (5.2). Since $\Sigma(x) \in X_\Delta$, it follows that $A(\Sigma(x)) \in Y$, hence $E^A(x) \in Y$. Thus (5.12) also holds.

Conversely we assume that conditions (5.12) and (5.13) are satisfied. Then (5.14) holds, and this and (5.13) imply $A_n \in (X_\Delta)^\beta$ for all n by Theorem 5.1. Again (5.15) holds and then $A \in (X_\Delta, Y)$. ∎

Now we give some characterisations of matrix transformations between Z and Z_Δ.

We obtain as an immediate consequence of Theorems 5.3 and 4.4 and of [11, Theorem 3.8, p. 180]

THEOREM 5.4. *Let X be a BK space with AK and Y be any of the spaces ℓ_∞, c_0, ℓ_1, $[\ell_\infty, \ell_1]^{\langle m(\mu)\rangle}$, $[\ell_1, \ell_\infty]^{\langle m(\mu)\rangle}$ or $[c_0, \ell_1]^{\langle m(\mu)\rangle}$.*

(a) Then $A \in (X_\Delta, Y)$ holds if and only if condition (5.13) holds in addition to the respective conditions in Theorem 4.4 with the A replaced by E^A.

(b) Let $C = \Delta A$, that is $c_{nk} = a_{nk} - a_{n-1,k}$ for all n and k. Then $A \in (X_\Delta, Y_\Delta)$ if and only if condition (5.13) with V^{A_n} replaced by V^{C_n} holds in addition to the respective conditions of Theorem 4.4 with A replaced by E^C.

In particular, we have, applying Corollary 4.5

COROLLARY 5.5. *Let $1 \le r < \infty$ and $1 \le p \le \infty$, s and q be the conjugate numbers of r and p, and Y be any of the spaces in Theorem 5.4. Finally, let the sequences $b^{s,q}$ be defined as in Example 5.2(a).*

(a) Then $(A \in ([\ell_r, \ell_p]^{\langle k(\nu)\rangle})_\Delta, Y)$ if and only if $E(A_n) \in (b^{(s,q)})^{-1} * \ell_\infty$ for all n, and the respective conditions in Corollary 4.5 hold with A replaced by E^A. Furthermore, $A \in (([c_0, \ell_p]^{\langle k(\nu)\rangle})_\Delta, Y)$ for $1 \le p < \infty$ if and only if $E(A_n) \in (b^{1,q})^{-1} * \ell_\infty$ for all n, and the respective conditions in Corollary 4.5 hold with A replaced by E^A.

(b) The conditions for $A \in (([\ell_r, \ell_p]^{\langle k(\nu)\rangle})_\Delta, Y_\Delta)$ and $(([c_0, \ell_p]^{\langle k(\nu)\rangle})_\Delta, Y_\Delta)$ are obtained from the respective ones in Part (a) by replacing A by C throughout.

For the next result, we need to know the β-duals of ℓ_∞ and $[\ell_\infty, \ell_1]$ which cannot be determined by Theorem 5.1, since they do not have AK.

LEMMA 5.6. *Let $E = \Sigma^T$. Then*

(a) $a \in ((\ell_\infty)_\Delta)^\beta$ if and only if $a \in (\ell_1 \cap ((n)_{n=1}^\infty)^{-1} * c_0)_E$;

(b) $a \in (([\ell_\infty, \ell_1]^{\langle k(\nu)\rangle})_\Delta)^\beta$ if and only if $a \in ([\ell_1, \ell_\infty]^{\langle k(\nu)\rangle} \cap (b^{1,\infty})^{-1} * c_0)_E$ where the sequence $b^{1,\infty}$ is defined as in Example 5.2(a).

In both parts, if $a \in ((\ell_\infty)_\Delta)^\beta$ or $a \in (([\ell_\infty, \ell_1]^{\langle k(\nu)\rangle})_\Delta)^\beta$ then (5.2) holds.

Proof. (a) This follows from [9, Theorem 2, Corollary 2].

(b) We write $X_\infty = (([\ell_\infty, \ell_1]^{\langle k(\nu)\rangle})_\Delta$ and $X_0 = ([c_0, \ell_1]^{\langle k(\nu)\rangle})_\Delta$, for short.

First $X_0 \subset X_\infty$ implies $X_\infty^\beta \subset X_0^\beta$, hence $X_\infty^\beta \subset ([\ell_1,\ell_\infty]^{\langle k(\nu)\rangle})_E$ by Example 4.3(a). Now we assume $a \in X_\infty^\beta$. Since $e \in X_\infty$, the sequence $E(a)$ is defined. Let $y \in X_\infty$ be given. Then, by (5.6), $a \in X_\infty^\beta$ and $E(a) \in [\ell_1,\ell_\infty]^{\langle k(\nu)\rangle}$ together yield $V^a \in ([\ell_\infty,\ell_1]^{\langle k(\nu)\rangle}, c)$, that is $E(a) \in (X_\infty, c)$ by (5.1). Conversely, if $a \in ([\ell_1,\ell_\infty]^{\langle k(\nu)\rangle} \cap M(X_\infty,c))_E$, then $E(a) \in [\ell_\infty,\ell_1]^{\langle k(\nu)\rangle}$ and $V^a \in (X_\infty, c)$, hence $a \in X_\infty^\beta$ by (5.6). Thus we have shown $X_\infty^\beta = ([\ell_1,\ell_\infty]^{\langle k(\nu)\rangle} \cap M(X_\infty,c))_E$. We will prove

$$M(X_\infty,c) = M(X_\infty,c_0) = (b^{1,\infty})^{-1} * c_0. \tag{5.16}$$

We write $b = b^{1,\infty}$ and observe that $a \in M(X_\infty,c)$ if and only if $B^a \in ([\ell_\infty,\ell_1]^{\langle k(\nu)\rangle}, c)$ by (5.1).

First we assume $B^a \in ([\ell_\infty,c]^{\langle k(\nu)\rangle}, c)$. Then, by [7, Satz 4.8],

$$\sum_{\nu=0}^{\infty} \max_{k \in I_\nu} |b_{nk}^a| \text{ converges uniformly in } n \tag{5.17}$$

and $\lim_{n\to\infty} b_{nk}^a = \alpha_k$ exists for each k. Since $[c_0,\ell_1]^{\langle k(\nu)\rangle} \subset [\ell_\infty,\ell_1]^{\langle k(\nu)\rangle}$ implies $([\ell_\infty,\ell_1]^{\langle k(\nu)\rangle}, c) \subset ([c_0,\ell_1]^{\langle k(\nu)\rangle}, c)$, we have $\sup_n \|B_n^a\|_{1,\infty} < \infty$ by Corollary 4.5(**2.**), and this is equivalent to $a \in b^{-1} * \ell_\infty$, by Example 5.2(a). Thus there is a constant K such that $\sup_n |a_n| b_n \leq K$, whence $|a_n| \leq K/b_n \to 0$ $(n \to \infty)$, that is $a \in c_0$. By (5.17), given $\varepsilon > 0$ there is $\nu_0 \in \mathbb{N}_0$ such that

$$\sum_{\nu=\nu_0}^{\infty} \max_{k \in I_\nu} |b_{nk}^a| \leq |a_n| \left(b_n - b_{k(\nu_0)-1}\right) < \varepsilon/2 \text{ for all } n.$$

Furthermore, since $a \in c_0$, we can choose $n_0 \in \mathbb{N}$ such that $|a_n| b_{k(\nu_0)-1} < \varepsilon/2$ for all $n \geq n_0$. Then $|a_n| b_n < \varepsilon$ for all $n \geq n_0$, that is $a \in b^{-1} * c_0$.

Conversely we assume $a \in b^{-1} * c_0$. Then obviously $a \in c_0$. Furthermore $a \in b^{-1} * c_0$ implies

$$\|B_n^a\|_{(1,\infty)} = \sum_{\nu=0}^{\infty} \max_{k \in I_\nu} |b_{nk}^a| \to 0 \ (n \to \infty) \text{ and } \sup \|B_n^a\|_{(1,\infty)} < \infty$$

By [6, Lemma, p. 168], these two conditions together imply (5.17). From this and $\lim_{n\to\infty} b_{nk}^a = \lim_{n\to\infty} a_n = 0$, we conclude $B^a \in ([\ell_\infty,\ell_1]^{\langle k(\nu)\rangle}, c_0)$ by [7, Satz 4.8], hence $a \in M(X_\infty, c_0)$. ∎

We obtain as an immediate consequence of Theorems 5.3, 4.6, Example 5.2(a) and Lemma 5.6

THEOREM 5.7. *Let W be a BK space with AK and $Y = W^\beta$ and X be any of the spaces ℓ_∞, c_0, ℓ_1, $[\ell_1,\ell_\infty]^{\langle k(\nu)\rangle}$, $[\ell_\infty,\ell_1]^{\langle k(\nu)\rangle}$ or $[c_0,\ell_1]^{\langle k(\nu)\rangle}$.*

(a) Then $A \in (X_\Delta, Y)$ if and only if the respective conditions in Theorem 4.6 hold with A replaced by E^A and, in addition for all m, $E(A_m) \in ((n)_{n=1}^\infty)^{-1} * c_0$ when $X = \ell_\infty$, $E(A_m) \in ((n)_{n=1}^\infty)^{-1} * \ell_\infty$ when $X = c_0$, $E(A_m) \in (b^{1,\infty})^{-1} * c_0$ when $X = [\ell_\infty,\ell_1]^{\langle k(\nu)\rangle}$, $E(A_m) \in (b^{\infty,1})^{-1} * \ell_\infty$ when $X = [\ell_1,\ell_\infty]^{\langle k(\nu)\rangle}$, and $E(A_m) \in (b^{1,\infty})^{-1} * \ell_\infty$ when $X = [c_0,\ell_1]^{\langle k(\nu)\rangle}$; no additional condition is needed when $X = \ell_1$ by Example 5.2(a).

(b) Then $A \in (X_\Delta, Y_\Delta)$ if and only if the respective conditions in Part (a) hold with A replaced by $C = \Delta A$.

We obtain from Corollary 4.7

COROLLARY 5.8. *Let $1 < r < \infty$ and $1 < p < \infty$ ad X be any of the spaces in Theorem 5.7.*

(a) Then $A \in (X_\Delta, [\ell_r, \ell_p]^{\langle m(\mu)\rangle})$ if and only if the conditions in Corollary 4.7 with A replaced by E^A and the additional conditions of Theorem 5.7(a) hold.

(b) Then $A \in (X_\Delta, ([\ell_r, \ell_p]^{\langle m(\mu)\rangle})_\Delta)$ if and only if the conditions of Part (a) hold with A replaced by $C = \Delta A$.

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