

A MULTIVALUED FIXED POINT THEOREM IN ULTRAMETRIC SPACES

Ljiljana Gajić

Abstract. The purpose of this paper is to prove that a class of generalized contractive multivalued mappings on a spherically complete ultrametric space has a fixed point.

Let (X, d) be a metric space. If the metric d satisfies strong triangle inequality: for all $x, y, z \in X$

$$d(x, y) \leq \max\{d(x, z), d(z, y)\},$$

it is called *ultrametric* on X [4]. Pair (X, d) is now an *ultrametric space*.

REMARK. If $X \neq \emptyset$, then the so called discrete metric d defined on X by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$$

is an ultrametric.

EXAMPLE. For $a \in \mathbb{R}$ let $[a]$ be the entire part of a . By

$$d(x, y) = \inf\{2^{-n} : n \in \mathbb{Z}, [2^n(x - e)] = [2^n(y - e)]\}$$

(here e is any irrational number) an ultrametric d on \mathbb{Q} is defined which determines the usual topology on \mathbb{Q} [4].

An ultrametric space (X, d) is said to be *spherically complete* if every shrinking collection of balls in X has a nonempty intersection.

In [3] authors proved a fixed point theorem for contractive function on spherically complete ultrametric space X . Let us recall: $T: X \rightarrow X$ is said to be contractive if for every $x, y \in X$, $x \neq y$, $d(Tx, Ty) < d(x, y)$. This result is generalized in [2] for multivalued mappings $T: X \rightarrow 2_c^X$ (2_c^X is the space of all nonempty compact subsets in X with Hausdorff metric H).

AMS Subject Classification: 47H10

Keywords and phrases: Ultrametric space, spherically complete, fixed point, multivalued mappings.

Communicated at the 5th International Symposium on Mathematical Analysis and its Applications, Niška banja, Yugoslavia, October, 2–6, 2002.

On the other side, the result from [3] is generalized for a class of functions $T: X \rightarrow X$ such that for every $x, y \in X$, $x \neq y$

$$d(Tx, Ty) < \max\{d(x, y), d(x, Tx), d(y, Ty)\}.$$

Now we are going to prove the related result for multivalued mappings.

THEOREM. *Let (X, d) be a spherically complete ultrametric space. If $T: X \rightarrow 2_c^X$ is such that for any $x, y \in X$, $x \neq y$,*

$$H(Tx, Ty) < \max\{d(x, y), d(x, Tx), d(y, Ty)\}, \quad (1)$$

then T has a fixed point (i.e., there exists $x \in X$ such that $x \in Tx$).

Proof. Let $B_a = B[a; d(a, Ta)]$ denote the closed ball centered at a with radius $d(a, Ta) = \inf_{z \in Ta} d(a, z)$, and let \mathcal{A} be the collection of these spheres for all $a \in X$. The relation

$$B_a \leq B_b \quad \text{iff} \quad B_b \subseteq B_a$$

is a partial order on \mathcal{A} . Let \mathcal{A}_1 be a totally ordered subfamily of \mathcal{A} . Since X is spherically complete, $\bigcup_{B_a \in \mathcal{A}_1} B_a = B \neq \emptyset$. Let $b \in B$ and $B_a \in \mathcal{A}_1$. Obviously, $b \in B_a$, so $d(b, a) \leq d(a, Ta)$.

Take $u \in T(a)$ such that $d(a, u) = d(a, Ta)$ (it is possible since Ta is a nonempty compact set). Then

$$\begin{aligned} d(b, Tb) &\leq \inf_{c \in Tb} d(b, c) \leq \max\{d(b, a), d(a, u), \inf_{c \in Tb} d(u, c)\} \\ &\leq \max\{d(a, Ta), H(Ta, Tb)\} < \max\{d(a, Ta), d(a, b), d(b, Tb)\} \\ &= \max\{d(a, Ta), d(b, Tb)\} \end{aligned}$$

which is possible only for $d(b, Tb) < d(a, Ta)$. Now, for any $x \in B_b$,

$$\begin{aligned} d(x, b) &\leq d(b, Tb) < d(a, Ta), \\ d(x, a) &\leq \max\{d(x, b), d(b, a)\} < d(a, Ta), \end{aligned}$$

so $x \in B_a$. We have just proved that $B_b \subseteq B_a$ for any $B_a \in \mathcal{A}_1$. Thus B_b is an upper bound in \mathcal{A} for the family \mathcal{A}_1 . By Zorn's lemma there is a maximal element in \mathcal{A} , say B_z . We shall prove that $z \in Tz$.

In opposite case, $z \notin Tz$, there exists $\bar{z} \in Tz$, $\bar{z} \neq z$, such that $d(z, \bar{z}) = d(z, Tz)$. Let us prove that $B_{\bar{z}} \subseteq B_z$.

$$\begin{aligned} d(\bar{z}, T\bar{z}) &\leq H(Tz, T\bar{z}) < \max\{d(z, \bar{z}), d(z, Tz), d(\bar{z}, T\bar{z})\} \\ &= \max\{d(z, Tz), d(\bar{z}, T\bar{z})\}, \end{aligned}$$

which is possible only for $d(\bar{z}, T\bar{z}) < d(z, Tz)$. Now, for any $y \in B_{\bar{z}}$,

$$\begin{aligned} d(y, \bar{z}) &\leq d(\bar{z}, T\bar{z}) < d(z, Tz), \\ d(y, z) &\leq \max\{d(y, \bar{z}), d(\bar{z}, z)\} \leq d(z, Tz), \end{aligned}$$

which means that $y \in B_z$, so $B_{\bar{z}} \subseteq B_z$. But $d(z, \bar{z}) = d(z, Tz) > d(\bar{z}, T\bar{z})$, hence $z \notin B_{\bar{z}}$, so $B_{\bar{z}} \subsetneq B_z$. This fact contradicts the maximality of B_z . So we have proved that T has a fixed point. ■

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(received 31.12.2002)

Deaprtment of Mathematics and Informatics, 21000 Novi Sad, trg D. Obradovića 4, Yugoslavia