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A GENERALIZATION OF DARBOUX THEOREM

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Abstract. We show a generalization of the fundamental Darboux theorem that states intermediate property for the derivative function of a real differentiable function. We extend this result for pairs of differentiable functions, i.e., for flat differentiable arcs.

1. Introduction

The well-known theorem of Darboux asserts that the set of values of the first derivative of a real differentiable function $x(t)$, $t \in [a,b]$ is an interval (connected set on the real line). For a pair $(x(t), y(t))$ of differentiable functions on an interval $[a,b]$, the set $\{\,(x'(t), y'(t)) \mid t \in [a,b]\,\}$ need not be connected as the following counterexample shows (see [1]).

Example. Let $(x(t), y(t)) = (t^2 \cos \frac{1}{t}, t^2 \sin \frac{1}{t})$ for $t \in (0,1]$ and $(x(0), y(0)) = (0,0)$. Then the set $\{\,(x'(t), y'(t)) \mid t \in [0,1]\,\}$ is disconnected.

In spite of this, there is still an extension of the Darboux theorem for pairs of differentiable functions. The aim of the paper is to show such a generalization.

2. Result

Theorem. *Let $(x(t), y(t))$, $t \in [a,b]$ be a differentiable arc, such that the tangent vectors $(x'_+(a), y'_+(a))$ and $(x'_-(b), y'_-(b))$ are not collinear. Then, for every direction d in the interior of the angle between these vectors, there exists a point $c \in (a,b)$ such that the tangent vector $(x'(c), y'(c))$ is collinear with d.*

Proof. Denote $T = \{\,(u,v) \mid a \leqslant u < v \leqslant b\,\}$. Let $F\colon T \to \mathbf{R}^2$ be a function defined by
$$F(u,v) = \left(\frac{x(v) - x(u)}{v - u}, \frac{y(v) - y(u)}{v - u} \right).$$
Function F is obviously continuous and maps the connected triangle T onto a connected set $F(T)$. Since $F(a,v) \to (x'_+(a), y'_+(a))$, when $v \to a+0$ and $F(u,b) \to$

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$(x'_-(b), y'_-(b))$, when $u \to b - 0$, the points $(x'_+(a), y'_+(a))$ and $(x'_-(b), y'_-(b))$ are accumulation points of $F(T)$. Since d lies in the interior of the angle between the radius vectors of these points, the line $p = \{\, td \mid t \in \mathbf{R}\}$ strongly separates these points, so that the former point belongs to the open half-plane $H(a)$ and the latter one to the open half-plane $H(b)$, both having line p as its border. The intersection $F(T) \cap p$ is not empty. Otherwise, $F(T)$, being the union of non-empty sets $F(T) \cap H(a)$ and $F(T) \cap H(b)$, would not be connected. Let $(\alpha, \beta) \in F(T) \cap p$. Then

$$(x(\beta) - x(\alpha), y(\beta) - y(\alpha)) = t_0(\beta - \alpha)d \tag{1}$$

for some $t_0 \in \mathbf{R}$. Since $\alpha < \beta$, it follows from (1) that the points $(x(a), y(a))$ and $(x(b), y(b))$ are distinct. Using Cauchy's mean value theorem for functions $x(t)$ and $y(t)$ on the interval $[\alpha, \beta]$, we conclude that there exists a $c \in (\alpha, \beta)$ such that

$$(x'(c), y'(c)) = s(x(\beta) - x(\alpha), y(\beta) - y(\alpha)) \tag{2}$$

for some $s \in \mathbf{R}$. From (1) and (2) it follows that the vectors $(x'(c), y'(c))$ and d are collinear. ∎

The Theorem we have just proved extends the Theorem of Darboux. In order to prove that, consider the differentiable arc $(x(t), t)$, $t \in [a, b]$. Notice that if $x'_+(a) \neq x'_-(b)$, then the tangent vectors $(x'_+(a), 1)$ and $(x'_-(b), 1)$ span a nonzero angle and the direction $(d, 1)$ lies in its interior (it is a convex combination of these vectors). Applying our Theorem, we can find a $c \in (a, b)$ such that $(x'(c), 1)$ is collinear with $(d, 1)$. Hence, $x'(c) = d$ is a convex combination of $x'_+(a)$ and $x'_-(b)$.

REFERENCES

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