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MULTI POINT BOUNDARY VALUE PROBLEMS FOR SECOND ORDER DIFFERENTIAL INCLUSIONS

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Abstract. In this paper we investigate the existence of solutions on a compact interval to a multi-point boundary value problem for a class of second order differential inclusions. We shall rely on a fixed point theorem for condensing maps due to Martelli.

1. Introduction

Let $a_i, b_j \in \mathbf{R}$, with all of the a_i's, (respectively, b_j's), having the same sign, $\xi_i, \zeta_j \in (0,1)$, $i = 1,2,\dots,m-2$, $j = 1,2,\dots,n-2$, $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$, $0 < \zeta_1 < \zeta_2 < \dots < \zeta_{n-2} < 1$. The main purpose of this paper is to get results on the solvability of the following boundary value problems (BVPs for short) for second order differential inclusions of the forms

$$\begin{cases} y''(t) \in F(t, y(t)), \quad t \in (0,1) \\ y(0) = \sum\limits_{i=1}^{m-2} a_i y'(\xi_i), \quad y(1) = \sum\limits_{j=1}^{n-2} b_j y(\zeta_j) \end{cases} \tag{A}$$

$$\begin{cases} y''(t) \in F(t, y(t)), \quad t \in (0,1) \\ y(0) = \sum\limits_{i=1}^{m-2} a_i y'(\xi_i), \quad y'(1) = \sum\limits_{j=1}^{n-2} b_j y'(\zeta_j) \end{cases} \tag{B}$$

$$\begin{cases} y''(t) \in F(t, y(t)), \quad t \in (0,1) \\ y(0) = \sum\limits_{i=1}^{m-2} a_i y(\xi_i), \quad y(1) = \sum\limits_{j=1}^{n-2} b_j y(\zeta_j) \end{cases} \tag{C}$$

and

$$\begin{cases} y''(t) \in F(t, y(t)), \quad t \in (0,1) \\ y(0) = \sum\limits_{i=1}^{m-2} a_i y(\xi_i), \quad y'(1) = \sum\limits_{j=1}^{n-2} b_i y'(\zeta_j) \end{cases} \tag{D}$$

where $F\colon J \times \mathbf{R} \longrightarrow 2^{\mathbf{R}}$ is a multivalued map with compact convex values.

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The study of multi-point boundary value problems for second order ordinary differential equations was initiated by Il'in and Moiseev in [12, 13] motivated by the work of Bitsadze and Samarskii on nonlocal elliptic boundary value problems, [2, 3, 4].

Existence of solutions on compact intervals for multi-point boundary value problems for second order differential equations was given by Gupta in [6], Gupta et al in [7–10]. However, to our knowledge, this type of problems has not been studied for the multivalued case.

It is well known (c.f. [12]) that if a function $y \in C^1$ satisfies one of the boundary conditions stated above and a_i, b_j, $i = 1, 2, \dots, m-2$, $j = 1, 2, \dots, n-2$ are as above, then there exist $\eta \in [\xi_1, \xi_{m-2}]$, $\tau \in [\zeta_1, \zeta_{n-2}]$ such that

$$
\begin{aligned}
y(0) &= \alpha y'(\eta), \quad y(1) = \beta y(\tau) \\
y(0) &= \alpha y'(\eta), \quad y'(1) = \beta y'(\tau) \\
y(0) &= \alpha y(\eta), \quad y(1) = \beta y(\tau) \\
y(0) &= \alpha y(\eta), \quad y'(1) = \beta y'(\tau)
\end{aligned}
$$

respectively with $\alpha = \sum_{i=1}^{m-2} a_i$, $\beta = \sum_{j=1}^{n-2} b_j$. Hence the multi-point BVPs (A)–(D) can be reduced to a corresponding four-point BVP. The method of proof for the existence of a solution for a four-point BVP and for a multi-point BVP (A)–(D) is the same.

In order not to hide the main ideas behind general and technicaly complicated statements, we restrict our discussion to the following four-point BVP

$$y'' \in F(t, y), \quad t \in J = [0, 1] \tag{1.1}$$

$$y(0) = y'(\eta), \quad y(1) = y(\tau) \tag{1.2}$$

where $F \colon J \times \mathbf{R} \longrightarrow 2^{\mathbf{R}}$ is a multivalued map with compact convex values and $\eta, \tau \in (0, 1)$. This is a special case of the BVP (A) when $\alpha = \beta = 1$. All the other four-point BVP and the general multi-point BVP are examined in a similar way, with obvious modifications.

The method we are going to use is to reduce the existence of solutions to problem (1.1)–(1.2) to the search for fixed points of a suitable multivalued map on the Banach space $C(J, \mathbf{R})$. In order to prove the existence of fixed points, we shall rely on a fixed point theorem for condensing maps due to Martelli [15].

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper.

Let $(X, \|\cdot\|)$ be a Banach space. A multivalued map $G \colon X \longrightarrow 2^X$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \bigcup_{x \in B} G(x)$ is bounded in X for any bounded set B of X (i.e. $\sup_{x \in B}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$).

G is called upper semicontinuous (u.s.c.) on X if for each $x_* \in X$ the set $G(x_*)$ is a nonempty, closed subset of X, and if for each open set B of X containing $G(x_*)$, there exists an open neighbourhood V of x_* such that $G(V) \subseteq B$.

G is said to be completely continuous if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \longrightarrow x_*$, $y_n \longrightarrow y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$).

G has a fixed point if there is $x \in X$ such that $x \in G(x)$.

In the following $CC(X)$ denotes the set of all nonempty compact and convex subsets of X.

A multivalued map $G\colon J \longrightarrow CC(E)$ is said to be measurable if for each $x \in E$ the function $Y\colon J \longrightarrow \mathbf{R}$ defined by

$$Y(t) = d(x, G(t)) = \inf\{|x - z| : z \in G(t)\}$$

is measurable.

Definition 2.1. A multivalued map $F\colon J \times \mathbf{R} \longrightarrow 2^{\mathbf{R}}$ is said to be an L^1-Carathéodory map if

(i) $t \longmapsto F(t, y)$ is measurable for each $y \in \mathbf{R}$;

(ii) $y \longmapsto F(t, y)$ is upper semicontinuous for almost all $t \in J$;

(iii) for each $k > 0$, there exists $h_k \in L^1(J, \mathbf{R}_+)$ such that

$$\|F(t, y)\| = \sup\{|v| : v \in F(t, y)\} \leq h_k(t)$$
$$\text{for all } |y| \leq k \text{ and for almost all } t \in J.$$

An upper semi-continuous map $G\colon X \longrightarrow 2^X$ is said to be condensing if for any subset $B \subseteq X$ with $\alpha(B) \neq 0$, we have $\alpha(G(B)) < \alpha(B)$, where α denotes the Kuratowski measure of noncompacteness. For properties of the Kuratowski measure, we refer to Banas and Goebel [1].

We remark that a completely continuous multivalued map is the easiest example of a condensing map. For more details on multivalued maps see the books of Deimling [5] and Hu and Papageorgiou [11].

We will need the following hypotheses:

(H1) $F\colon J \times \mathbf{R} \longrightarrow CC(\mathbf{R})$ is an L^1-Carathéodory multivalued map.

(H2) There exists a function $H \in L^1(J, \mathbf{R}_+)$ such that

$$\|F(t, y)\| := \sup\{|v| : v \in F(t, y)\} \leq H(t) \text{ for almost all } t \in J \text{ and all } y \in \mathbf{R}.$$

Definition 2.2. A function $y\colon J \longrightarrow \mathbf{R}$ is called a solution for the BVP (1.1)–(1.2) if y and its first derivative are absolutely continuous and y'' (which exists almost everywhere) satisfies the differential inclusion (1.1) a.e. on J and the condition (1.2).

Our considerations are based on the following lemmas.

Lemma 2.3. [14] *Let I be a compact real interval and X be a Banach space. If F is a multivalued map satisfying (H1) and Γ is a linear continuous mapping from $L^1(I,X)$ to $C(I,X)$, then the operator*

$$\Gamma \circ S_F : C(I,X) \longrightarrow CC(C(I,X)),\; y \longmapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F,y})$$

is a closed graph operator in $C(I,X) \times C(I,X)$.

Lemma 2.4. [15] *Let X be a Banach space and $N\colon X \longrightarrow CC(X)$ be a u.s.c. condensing map. If the set*

$$\Omega := \{y \in X : \lambda y \in Ny \;\; \text{for some} \;\; \lambda > 1\}$$

is bounded, then N has a fixed point.

3. Main Result

Now, we are able to state and prove our main theorem.

Theorem 3.1. *Assume that Hypotheses $(H1)$–$(H2)$ hold. Then the BVP (1.1)–(1.2) has at least one solution on J.*

Proof. Let $C(J,\mathbf{R})$ be the Banach space provided with the norm

$$\|y\|_\infty := \sup\{\,|y(t)| : t \in J\,\},\ \text{for}\ y \in C(J,\mathbf{R}).$$

Transform the problem (1.1)–(1.2) into a fixed point problem. Consider the multivalued map, $N\colon C(J,\mathbf{R}) \longrightarrow 2^{C(J,\mathbf{R})}$ defined by:

$$\begin{aligned} Ny = \Big\{\, h \in C(J,\mathbf{R})\colon h(t) &= \int_0^t (t-s)g(s)\,ds + \int_0^\eta g(s)\,ds \\ &\quad + \frac{1+t}{1-\tau}\Big[\int_0^\tau (\tau-s)g(s)\,ds - \int_0^1 (1-s)g(s)\,ds\Big]\Big\} \end{aligned}$$

where

$$g \in S_{F,y} = \Big\{\, g \in L^1(J,\mathbf{R}) : g(t) \in F(t,y(t)) \;\; \text{for a.e. in}\ J\,\Big\}.$$

Remark 3.2. (i) It is clear that the fixed points of N are solutions to (1.1)–(1.2).

(ii) For each $y \in C(J,\mathbf{R})$ the set $S_{F,y}$ is nonempty (see Lasota and Opial [14]).

We shall show that N satisfies the assumptions of Lemma 2.4. The proof will be given in several steps.

Step 1. Ny is convex for each $y \in C(J,\mathbf{R})$.

Indeed, if $h_1,\ h_2$ belong to Ny, then there exist $g_1, g_2 \in S_{F,y}$ such that for each $t \in J$ we have

$$\begin{aligned} h(t) &= \int_0^t (t-s)g_i(s)\,ds + \int_0^\eta g_i(s)\,ds \\ &\quad + \frac{1+t}{1-\tau}\Big[\int_0^\tau (\tau-s)g_i(s)\,ds - \int_0^1 (1-s)g_i(s)\,ds\Big], \quad i = 1,2. \end{aligned}$$

Let $0 \leq \alpha \leq 1$. Then for each $t \in J$ we have

$$
\begin{aligned}
(\alpha h_1 + (1-\alpha)h_2)(t) = & \int_0^t (t-s)\{\alpha g_1(s) + (1-\alpha)g_2(s)\}\, ds \\
& + \int_0^\eta \{\alpha g_1(s) + (1-\alpha)g_2(s)\}\, ds \\
& + \frac{1+t}{1-\tau}\Big[\int_0^\tau (\tau-s)\{\alpha g_1(s) + (1-\alpha)g_2(s)\}\, ds \\
& - \int_0^1 (1-s)\{\alpha g_1(s) + (1-\alpha)g_2(s)\} ds\Big].
\end{aligned}
$$

Since $S_{F,y}$ is convex (because F has convex values) then

$$\alpha h_1 + (1-\alpha)h_2 \in Ny.$$

Step 2. N is bounded on bounded sets of $C(J, \mathbf{R})$.

Indeed, it is enough to show that there exists a positive constant c such that for each $h \in Ny$, $y \in B_r = \{y \in C(J, \mathbf{R}) : |y\|_\infty \leq r\}$ one has $\|h\|_\infty \leq c$.

If $h \in Ny$, then there exists $g \in S_{F,y}$ such that for each $t \in J$ we have

$$
\begin{aligned}
h(t) = & \int_0^t (t-s)g(s)\, ds + \int_0^\eta g(s)\, ds \\
& + \frac{1+t}{1-\tau}\Big[\int_0^\tau (\tau-s)g(s)\, ds - \int_0^1 (1-s)g(s)\, ds\Big], \quad t \in J.
\end{aligned}
$$

By (H1) we have for each $t \in J$ that

$$|h(t)| \leq \int_0^t h_r(s)\, ds + \int_0^\eta h_r(s)\, ds + \frac{2}{1-\tau}\Big[\int_0^\tau (\tau-s)h_r(s)\, ds + \int_0^1 (1-s)h_r(s)\, ds\Big].$$

Then

$$\|h\|_\infty \leq \int_0^1 h_r(s)\, ds + \int_0^\eta h_r(s)\, ds + \frac{2}{1-\tau}\Big[\int_0^\tau (\tau-s)h_r(s)\, ds + \int_0^1 (1-s)h_r(s)\, ds\Big] = c.$$

Step 3. N sends bounded sets of $C(J, \mathbf{R})$ into equicontinuous sets.

Let $t_1, t_2 \in J$, $t_1 < t_2$ and B_r be a bounded set of $C(J, \mathbf{R})$. For each $y \in B_r$ and $h \in Ny$, there exists $g \in S_{F,y}$ such that

$$
\begin{aligned}
h(t) = & \int_0^t (t-s)g(s)\, ds + \int_0^\eta g(s)\, ds \\
& + \frac{1+t}{1-\tau}\Big[\int_0^\tau (\tau-s)g(s)\, ds - \int_0^1 (1-s)g(s)\, ds\Big], \quad t \in J.
\end{aligned}
$$

Thus we obtain

$$\begin{aligned}|h(t_2)-h(t_1)| &\le \int_0^{t_2}(t_2-s)\|g(s)\|\,ds+\int_{t_1}^{t_2}(t_1-s)\|g(s)\|\,ds\\ &\quad+\frac{t_2-t_1}{1-\tau}\Big[\int_0^{\tau}(\tau-s)\|g(s)\|\,ds+\int_0^1(1-s)\|g(s)\|\,ds\Big]\\ &\le \int_0^{t_2}(t_2-s)h_r(s)\,ds+\int_{t_1}^{t_2}(t_1-s)h_r(s)\,ds\\ &\quad+\frac{t_2-t_1}{1-\tau}\Big[\int_0^{\tau}(\tau-s)h_r(s)\,ds+\int_0^1(1-s)h_r(s)\,ds\Big].\end{aligned}$$

As $t_2 \longrightarrow t_1$ the right-hand side of the above inequality tends to zero.

As a consequence of Step 2, Step 3 together with the Arzela-Ascoli theorem we can conclude that N is completely continuous.

Step 4. N has a closed graph.

Let $y_n \longrightarrow y_*$, $h_n \in Ny_n$, and $h_n \longrightarrow h_*$. We shall prove that $h_* \in Ny_*$.

$h_n \in Ny_n$ means that there exists $g_n \in S_{F,y_n}$ such that

$$\begin{aligned}h_n(t) &= \int_0^t (t-s)g_n(s)\,ds+\int_0^{\eta} g_n(s)ds\\ &\quad+\frac{1+t}{1-\tau}\Big[\int_0^{\tau}(\tau-s)g_n(s)\,ds-\int_0^1(1-s)g_n(s)\,ds\Big], \quad t\in J.\end{aligned}$$

We must prove that there exists $g_* \in S_{F,y_*}$ such that

$$\begin{aligned}h_*(t) &= \int_0^t (t-s)g_*(s)\,ds+\int_0^{\eta} g_*(s)\,ds\\ &\quad+\frac{1+t}{1-\tau}\Big[\int_0^{\tau}(\tau-s)g_*(s)\,ds-\int_0^1(1-s)g_*(s)\,ds\Big], \quad t\in J.\end{aligned}$$

Now, we consider the linear continuous operator

$$\Gamma\colon L^1(J,\mathbf{R}) \longrightarrow C(J,\mathbf{R})$$

$$\begin{aligned}g \longmapsto \Gamma(g)(t) &= \int_0^t (t-s)g(s)\,ds+\int_0^{\eta} g(s)\,ds\\ &\quad+\frac{1+t}{1-\tau}\Big[\int_0^{\tau}(\tau-s)g(s)\,ds-\int_0^1(1-s)g(s)\,ds\Big], \quad t\in J.\end{aligned}$$

From Lemma 2.3, it follows that $\Gamma \circ S_F$ is a closed graph operator.

Moreover from the definition of Γ we have

$$h_n(t) \in \Gamma(S_{F,y_n}).$$

Since $y_n \longrightarrow y_*$, it follows from Lemma 2.3 that

$$\begin{aligned}h_*(t) &= \int_0^t (t-s)g_*(s)\,ds+\int_0^{\eta} g_*(s)\,ds\\ &\quad+\frac{1+t}{1-\tau}\Big[\int_0^{\tau}(\tau-s)g_*(s)\,ds-\int_0^1(1-s)g_*(s)\,ds\Big], \quad t\in J\end{aligned}$$

for some $g_* \in S_{F,y_*}$.

Step 5. The set

$$\Omega := \{\, y \in C(J, \mathbf{R}) : \lambda y \in Ny \quad \textit{for some} \quad \lambda > 1 \,\}$$

is bounded.

Let $y \in \Omega$. Then $\lambda y \in Ny$ for some $\lambda > 1$. Thus there exists $g \in S_{F,y}$ such that

$$\begin{aligned} y(t) &= \lambda^{-1} \int_0^t (t-s) g(s)\, ds + \lambda^{-1} \int_0^\eta g(s)\, ds \\ &\quad + \lambda^{-1} \frac{1+t}{1-\tau} \Big[\int_0^\tau (\tau - s) g(s)\, ds - \int_0^1 (1-s) g(s)\, ds \Big], \quad t \in J. \end{aligned}$$

This implies by (H2) that for each $t \in J$ we have

$$|y(t)| \le \int_0^t (t-s) H(s)\, ds + \int_0^\eta H(s) ds + \frac{2}{1-\tau} \Big[\int_0^\tau (\tau - s) H(s)\, ds + \int_0^1 (1-s) H(s)\, ds \Big].$$

Thus

$$\begin{aligned} \|y\|_\infty &\le \int_0^1 (1-s) H(s)\, ds + \int_0^\eta H(s)\, ds \\ &\quad + \frac{2}{1-\tau} \Big[\int_0^\tau (\tau - s) H(s)\, ds + \int_0^1 (1-s) H(s)\, ds \Big] = K. \end{aligned}$$

This shows that Ω is bounded.

Set $X := C(J, \mathbf{R})$. As a consequence of Lemma 2.4 we deduce that N has a fixed point which is a solution of (1.1)–(1.2) on J. ■

REFERENCES

[**1**] J. Banas and K. Goebel, *Measures of Noncompactness in Banach Spaces*, Marcel-Dekker, New York, 1980.

[**2**] A. V. Bitsadze, *On the theory of nonlocal boundary value problem*, Soviet. Math. Dokl. **30** (1) (1984), 8–10.

[**3**] A. V. Bitsadze, *On a class of conditionally solvable nonlocal boundary value problems for harmonic functions*, Soviet. Math. Dokl. **31** (1) (1985), 91–94.

[**4**] A. V. Bitsadze and A. A. Samarskii, *On some simple generalizations of linear elliptic boundary value problem*, Soviet. Math. Dokl. **10** (2) (1969), 398–400.

[**5**] K. Deimling, *Multivalued Differential Equations*, Walter de Gruyter, Berlin-New York, 1992.

[**6**] C. P. Gupta, *A generalized multi-point boundary value problem for a second order ordinary differential equation*, Appl. Math. Comput. **89** (1998), 133–146.

[**7**] C.P. Gupta, S.K. Ntouyas and P.Ch. Tsamatos, *On an m-point boundary value problem for second order ordinary differential equations*, Nonlinear Anal. **23** (1994), 1427–1436.

[**8**] C.P. Gupta, S.K. Ntouyas and P.Ch. Tsamatos, *Existence results for m-point boundary value problems*, Differ. Equ. Dyn. Syst. **2** (1994), 289–298.

[**9**] C.P. Gupta, S.K. Ntouyas and P.Ch. Tsamatos, *On the solvability of some m-point boundary value problems*, Appl. Math. **41** (1996), 1–17.

[**10**] C.P. Gupta, S.K. Ntouyas and P.Ch. Tsamatos, *Existence results for multi-point boundary value problem for second order ordinary differential equations*, Bull. Greek Math. Soc., to appear.

[11] Sh. Hu and N. Papageorgiou, *Handbook of Multivalued Analysis, Volume I: Theory*, Kluwer Academic Publishers, Dordrecht, 1997.

[12] V.A. Il'in and E.I. Moiseev, *Nonlocal boundary value problem of the first kind for a Sturm-Liouville operator in its differential and finite difference aspects*, Differential Equations **23** (7) (1987), 803–810.

[13] V.A. Il'in and E.I. Moiseev, *Nonlocal boundary value problem of the second kind for a Sturm-Liouville operator*, Differential Equations **23** (8) (1987), 979–987.

[14] A. Lasota and Z. Opial, *An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. **13** (1965), 781–786.

[15] M. Martelli, *A Rothe's type theorem for noncompact acyclic-valued map*, Boll. Un. Math. Ital. **11** (1975), 3, suppl., 70–76.

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