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BEST λ-APPROXIMATIONS FOR ANALYTIC FUNCTIONS OF MEDIUM GROWTH ON THE UNIT DISC

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Abstract. In this paper we investigate the asymptotic relation between maximum moduli of a class of functions analytic on the unit disc and their partial sums, i.e. we formulate the problem of best λ-approximations. We also give an application of our results to Karamata's Tauberian Theorem for series.

1. Preliminaries

The problem of maximum moduli of the partial sums of an analytic function defined inside the unit disc is a classical one and has been investigated in many ways. For example, it is well known that the maximum moduli of partial sums of a bounded function need not be bounded, but on the contrary, this is always true (with the same bound) inside the circle $|z| \leqslant 1/2$ (see [6], pp. 236–238).

In general, for a given analytic function $f(x) := \sum_{i=0}^{\infty} a_i z^i$, $|z| < 1$, the moduli of its partial sums $S_n(z) = \sum_{i \leqslant n} a_i z^i$ depend on z and n.

Define, as usual, $M_f(r) := \max_{|z|=r} |f(z)| = |f(re^{i\phi_0})| = |f(z_0)|$; $M_f(r)$ increases with r and we suppose that $M_f(r) \to \infty$, $r \to 1^-$. We want to compare $f(z)$ with the partial sums *at the point* z_0 of maximal growth in the following way:

Determine a real-valued function $n := n(r, \lambda) \to \infty$, $r \to 1^-$; monotone increasing in both variables, such that

$$\frac{S_{n(r,\lambda)}(z_0)}{f(z_0)} = \begin{cases} o(1), & 0 < \lambda < 1 \\ 1 + o(1), & \lambda > 1 \end{cases} \qquad (r \to 1^-). \tag{I}$$

In this sense we are going to find the "shortest" partial sum which is well approximating $f(z_0)$ for r sufficiently close to 1. We call such partial sums best λ-approximating (BLAS). It is evident from (I) that an analogous relation is valid between moduli of BLAS and $M_f(r)$.

Some other questions are related to this one; for a given $n(r, \lambda)$ what can be said about $M_f(r)$ or, how does the ratio $S_{n(r,\lambda)}(z_0)/f(z_0)$ behave when $\lambda \uparrow\downarrow 1$, $r \to 1^-$?

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Apart from self-evident role in numerical calculus, the notion of BLAS appears to be very useful in the theory of Hadamard-type convolutions ([7, 8, 9]).

Here we are going to solve the problem of finding $n(r,\lambda)$ for a class of analytic functions of medium growth inside the unit disc. In this case the form of $n(r,\lambda)$ is very simple but we also show that slight changes of parameters have a drastic influence on it.

A particularly important role in this paper is played by Karamata's regularly varying functions $K_\rho(x)$ which are positive, defined for sufficiently large positive x and can be written in the form $K_\rho(x) := x^\rho L(x)$, $\rho \in \mathbf{R}$. Here, ρ is the index of regular variation and $L(x)$ is a so-called slowly varying function, i.e. positive, measurable and satisfying: $\forall t \in \mathbf{R}$, $\frac{L(tx)}{L(x)} \sim 1$, $x \to \infty$.

Some examples of slowly varying functions are

$$\ln^a x, \quad \ln^b(\ln x), \quad \exp\left(\frac{\ln x}{\ln\ln x}\right), \quad \exp(\ln^c x), \qquad a,b \in \mathbf{R}, \quad 0 < c < 1.$$

The theory of regular variation is very well developed (cf. [4, 5]) but we quote here some facts we are going to use afterwards:

$$K_\rho(\lambda x) \sim \lambda^\rho K_\rho(x), \quad \forall \lambda \in \mathbf{R}; \qquad L(x) = o(x^\varepsilon), \quad \varepsilon > 0;$$
$$\ln L(x) = o(\ln x) \quad (x \to \infty).$$

If $a(x) \sim b(x) \to \infty$ $(x \to \infty)$ then $K_\rho(a(x)) \sim K_\rho(b(x))$ $(x \to \infty)$.

If $L_1(x)$, $L_2(x)$ are slowly varying functions, then $L_1(x)L_2(x)$; $(L_1(x))^a$, $a \in \mathbf{R}$; $L_1 \circ L_2(x)$ $(L_2(x) \to \infty,\ x \to \infty)$, are also slowly varying.

2. Result

Let $f(z)$, $S_n(z)$, $M_f(r)$, $n(r,\lambda)$, $K_\rho(x)$, z_0 be defined as above. Then we have

Proposition 1. *If* $\ln M_f(r) \sim K_\rho(\ln(\frac{1}{1-r}))$, $\rho > 0$, $(r \to 1^-)$ *then we can take*

$$n(r,\lambda) \sim \left(\frac{1}{1-r}\right)^\lambda \quad (r \to 1^-),$$

independent of $K_\rho(\cdot)$.

Proof. We start with a simple implementation of the Cauchy integral formula:

$$\frac{1}{2\pi i}\int_C f(w)\frac{\left(\frac{z_0}{w}\right)^{n+1}}{w - z_0}\,dw = \begin{cases} -S_n(z_0), & z_0 \notin \operatorname{int} C; \\ f(z_0) - S_n(z_0), & z_0 \in \operatorname{int} C. \end{cases} \tag{1}$$

Let the contour C be a circle $w = Re^{i\phi}$, where $R := R(r,\lambda) = 1 - (1-r)^\lambda$. Since $|z_0| = r > R$, $0 < \lambda < 1$; $r < R$, $\lambda > 1$, from (1) we obtain

$$I := \frac{1}{2\pi}\int_0^{2\pi} \frac{f(Re^{i\phi})(\frac{r}{R}e^{i(\phi_0 - \phi)})^n}{f(z_0)(\frac{R}{r}e^{i(\phi-\phi_0)} - 1)}\,d\phi = \begin{cases} -\frac{S_n(z_0)}{f(z_0)}, & 0 < \lambda < 1, \\ 1 - \frac{S_n(z_0)}{f(z_0)}, & \lambda > 1. \end{cases} \tag{2}$$

Since $|f(z_0)| = M_f(r)$, by estimating the integral on the left-hand side of (2), we get

$$|I| = O(1)\frac{M_f(R)}{M_f(r)}\exp(n\ln(r/R))\int_0^{2\pi}\frac{1}{|\frac{R}{r}e^{i(\phi-\phi_0)}-1|}\,d\phi. \tag{3}$$

But

$$\begin{aligned}
M_f(R) &= \exp\left[K_\rho\left(\ln\frac{1}{1-R}\right)(1+o(1))\right] = \exp\left[K_\rho\left(\ln\left(\frac{1}{1-r}\right)^\lambda\right)(1+o(1))\right]\\
&= \exp\left[K_\rho\left(\lambda\ln\frac{1}{1-r}\right)(1+o(1))\right] = \exp\left[\lambda^\rho K_\rho\left(\ln\frac{1}{1-r}\right)(1+o(1))\right],\\
\ln\frac{r}{R} &= ((1-r)^\lambda-(1-r))(1+o(1)) \quad (r\to 1^-);\\
\int_0^{2\pi}&\frac{1}{|\frac{R}{r}e^{i(\phi-\phi_0)}-1|}\,d\phi = O(1)\ln\frac{1}{|\frac{R}{r}-1|}; \quad (r\to 1^-).
\end{aligned}$$

Putting this in (3) with $n = n(r,\lambda) = (\frac{1}{1-r})^\lambda(1+o(1))$ $(r\to 1^-)$, we get

$$\begin{aligned}
|I| = O(1)\exp&\left[(\lambda^\rho-1)K_\rho\left(\ln\left(\frac{1}{1-r}\right)\right)-(1-r)^{1-\lambda}+\ln\ln\left(\frac{1}{1-r}\right)+1\right]\times\\
&\times(1+o(1)) \quad (r\to 1^-).
\end{aligned} \tag{4}$$

Since $K_\rho(\ln\frac{1}{1-r}) = o(\ln(\frac{1}{1-r}))^{\rho+\varepsilon}$, $\varepsilon>0$, it follows from (4) that

$$|I| = \begin{cases} O(1)e^{-(1-\lambda^\rho)K_\rho(\ln\frac{1}{1-r})(1+o(1))}, & 0<\lambda<1,\\ O(1)e^{-(\frac{1}{1-r})^{\lambda-1}(1+o(1))}, & \lambda>1,\end{cases} \qquad (r\to 1^-)$$

i.e., according to (2),

$$\frac{S_{n(r,\lambda)}(z_0)}{f(z_0)} = \begin{cases} O(1)e^{-(1-\lambda^\rho)K_\rho(\ln\frac{1}{1-r})(1+o(1))}, & 0<\lambda<1,\\ 1+O(1)e^{-(\frac{1}{1-r})^{\lambda-1}(1+o(1))}, & \lambda>1\end{cases} \qquad (r\to 1^-). \tag{5}$$

Thus we have proved our Proposition 1 with good approximation of the o's from (I). ■

3. Comments

Especially interesting applications can be found if we suppose that the coefficients of $f(z)$ are non-negative, since in that case

$$|f(z)| = \left|\sum_{n=0}^\infty a_nz^n\right| \leqslant \sum_{n=0}^\infty a_n|z|^n = f(|z|);$$

i.e. for $z = re^{i\phi}$, $z_0 = r$, $M_f(r) = f(r)$.

Now, denoting by $U(x) := \sum_{n \leqslant x} a_n$, we find that $U(x)$ is non-decreasing and its Laplace-Stieltjes transform is $\widehat{U(s)} := s\int_0^\infty e^{-st}U(t)\,dt = f(e^{-s})$, $s > 0$. In our case, we have

$$\ln \widehat{U(s)} = \ln f(e^{-s}) = \ln M_f(e^{-s}) \sim K_\rho\left(\ln\frac{1}{1-e^{-s}}\right)$$
$$\sim \ln^\rho\left(\frac{1}{s}\right) L\left(\ln\left(\frac{1}{s}\right)\right) \quad (s \to 0^+).$$

The function on the right $L_1(1/s) := \ln^\rho(1/s)L(\ln(1/s))$ is clearly slowly varying at 0^+ (see Prelimimaries). Therefore, a variant of the Tauberian theorem for Kohlbacker transforms (cf. [3]) gives $\ln U(x) \sim 1/L_1^*(x)$ $(x \to \infty)$, where L_1^* is the so-called De Bruijn's conjugate of L_1.

A slowly varying function l^* is called De Bruijn's conjugare of l if it satisfies

$$l(x)l^*(xl(x)) \to 1, \quad l^*(x)(xl^*(x)) \to 1, \quad l^{**}(x) \sim l(x) \qquad (x \to \infty).$$

It always exists and it is asymptotically unique (cf. [4], p. 29).

Since, in our case,

$$\frac{L_1(xL_1(x))}{L_1(x)} = \left(\frac{\ln x(1+o(1))}{\ln x}\right)^\rho \frac{L(\ln x(1+o(1)))}{L(\ln x)} \to 1 \quad (x \to \infty),$$

we have

$$L_1^*(xL_1(x)) \sim 1/L_1(x) \sim 1/L_1(xL_1(x)), \quad (x \to \infty).$$

Hence, because of the asymptotic uniqueness, $L_1^*(x) \sim 1/L_1(x)$ and we finally get

$$\ln U(x) = \ln \sum_{n \leqslant x} a_n \sim \ln^\rho(x)L(\ln x) \quad (x \to \infty).$$

We can get much better information in the case $0 < \rho \leqslant 1$, using a variant of Karamata's Tauberian theorem for power series (cf. [2]), i.e.,

If $a_n \geqslant 0$ and the power series $f(r) := \sum_{n=0}^\infty a_n r^n$ converges for $r \in [0,1)$, then for $c > 0$, l slowly varying and a_n ultimately monotone,

$$f(r) \sim l(1/(1-r))/(1-r)^c \qquad (r \to 1^-),$$

is equivalent to either of asymptotic relations

$$a_n \sim n^{c-1}l(n)/\Gamma(c); \qquad \sum_{k=0}^n a_k \sim n^c l(n)/\Gamma(1+c), \quad (n \to \infty).$$

For l slowly varying and $a_n \geqslant 0$, we put in $K_\rho(\cdot)$:

$$L(\log(1/(1-r))) = c + \left(\frac{\log l(1/(1-r))}{\log(1/(1-r))}\right)^\rho, \quad c > 0, 0 < \rho \leqslant 1,$$

and obtain

$$f(r) \sim \begin{cases} l(1/(1-r))/(1-r)^c, & \rho = 1,\\ \exp(c\log^\rho(1/(1-r)))\exp(\log^\rho l(1/(1-r))), & 0 < \rho < 1. \end{cases}$$

Therefore, combining in the first case (5) and Karamata's theorem we get

PROPOSITION 2. *Under the conditions of Karamata's theorem*

$$\frac{\sum\limits_{n\leqslant(1-r)^{-\lambda}} n^{c-1}l(n)r^n}{\Gamma(c)f(r)} = \\ = \begin{cases} O(1)(f(r))^{-(1-\lambda)(1+o(1))}, & 0<\lambda<1; \\ 1+O(1)\exp(-(1-r)^{1-\lambda}(1+o(1))), & \lambda>1 \end{cases} \qquad (r\to 1^-).$$

Or, in a weaker but simpler form

PROPOSITION 2'. *Under the conditions of Karamata's Tauberian theorem for power series*

$$\frac{(1-r)^c}{l(1/(1-r))}\sum_{n\leqslant(1-r)^{-\lambda}} n^{c-1}l(n)r^n \to \begin{cases} 0, & 0<\lambda<1, \\ \Gamma(c), & \lambda>1, \end{cases} \qquad c>0, \quad (r\to 1^-).$$

The second case $0<\rho<1$ can be treated similarly because in that case $f(r)$ is a product of two slowly varying functions.

Although the case $c=0$ is included in the full version of Karamata's theorem, the form of $n(r,\lambda)$ is drastically changed here, as the next example shows.

It is not difficult to check that for

$$\log\left(\frac{1}{1-r}\right) = \sum_{k=1}^{\infty}\frac{r^k}{k}, \qquad r\in[0,1),$$

the considered function $n(r,\lambda)$ is

$$n(r,\lambda) = \exp\left[\log(1/(1-r))e^{-(\log\log(1/(1-r)))^{1-\lambda}}\right].$$

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