

## RELATIONSHIPS BETWEEN USUAL AND APPROXIMATE INVERSE SYSTEMS

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**Abstract.** We shall prove that if  $\mathbf{X} = \{X_a, p_{ab}, A\}$  is an approximate inverse system of compact non-metric spaces with surjective bonding mappings  $p_{ab}$  such that each  $X_a$  is a limit of a usual  $\tau$ -directed inverse system  $X(a) = \{X_{(a,\gamma)}, f_{(a,\gamma)(a,\delta)}, \Gamma_a\}$  of metric compact spaces, then there exist: 1) a usual  $\tau$ -directed inverse system  $X_D = \{X_d, F_{de}, D\}$  whose inverse limit  $X_D$  is homeomorphic to  $X = \lim \mathbf{X}$ , 2) every  $X_d$  is a limit of an approximate inverse system  $\{X_{(a,\gamma_a)}, g_{(a,\gamma_a)(b,\gamma_b)}, A\}$  of compact metric spaces  $X_{(a,\gamma_a)}$ , 3) if the mappings  $p_{ab}$  and  $f_{(a,\gamma)(a,\delta)}$  are monotone, then  $g_{(a,\gamma_a)(b,\gamma_b)}$  and  $F_{de}$  are monotone.

### 1. Introduction

In this paper we shall use the notion of *inverse systems*  $\mathbf{X} = \{X_a, p_{ab}, A\}$  and their limits in the usual sense [1, p. 135].

The cardinality of a set  $X$  will be denoted by  $\text{card}(X)$ . The cofinality of a cardinal number  $m$  will be denoted by  $\text{cf}(m)$ .  $\text{Cov}(X)$  is the set of all normal coverings of a topological space  $X$ . If  $\mathcal{U}, \mathcal{V} \in \text{Cov}(X)$  and  $\mathcal{V}$  refines  $\mathcal{U}$ , we write  $\mathcal{V} \leq \mathcal{U}$ . For two mappings  $f, g: Y \rightarrow X$  which are  $\mathcal{U}$ -near (for every  $y \in Y$  there exists a  $U \in \mathcal{U}$  with  $f(y), g(y) \in U$ ), we write  $(f, g) \leq \mathcal{U}$ . A basis of (open) normal coverings of a space  $X$  is a collection  $\mathcal{C}$  of normal coverings such that every normal covering  $\mathcal{U} \in \text{Cov}(X)$  admits a refinement  $\mathcal{V} \in \mathcal{C}$ . We denote by  $\text{cw}(X)$  (*covering weight*) the minimal cardinal of a basis of normal coverings of  $X$  [9, p. 181].

LEMMA 1. [9, Example 2.2] *If  $X$  is a compact Hausdorff space, then  $\text{cw}(X) = w(X)$ .*

The notion of *approximate inverse system*  $\mathbf{X} = \{X_a, p_{ab}, A\}$  will be used in the sense of S. Mardešić [11].

DEFINITION 1. An *approximate inverse system* is a collection  $\mathbf{X} = \{X_a, p_{ab}, A\}$ , where  $(A, \leq)$  is a directed preordered set,  $X_a$ ,  $a \in A$ , is a topological space and  $p_{ab}: X_b \rightarrow X_a$ ,  $a \leq b$ , are mappings such that  $p_{aa} = \text{id}$  and the following condition (A2) is satisfied:

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- (A2) For each  $a \in A$  and each normal cover  $\mathcal{U} \in \text{Cov}(X_a)$  there is an index  $b \geq a$  such that  $(p_{ac}p_{cd}, p_{ad}) \leq \mathcal{U}$ , whenever  $a \leq b \leq c \leq d$ .

An approximate map  $p = \{p_a : a \in A\} : X \rightarrow \mathbf{X}$  into an approximate system  $\mathbf{X} = \{X_a, p_{ab}, A\}$  is a collection of maps  $p_a : X \rightarrow X_a$ ,  $a \in A$ , such that the following condition holds

- (AS) For any  $a \in A$  and any  $U \in \text{Cov}(X_a)$  there is  $b \geq a$  such that  $(p_{ac}p_c, p_a) \leq U$  for each  $c \geq b$ . (See [10]).

Let  $\mathbf{X} = \{X_a, p_{ab}, A\}$  be an approximate system and let  $\mathbf{p} = \{p_a : a \in A\} : X \rightarrow \mathbf{X}$  be an approximate map. We say that  $\mathbf{p}$  is a *limit* of  $\mathbf{X}$  provided it has the following universal property:

- (UL) For any approximate map  $\mathbf{q} = \{q_a : a \in A\} : Y \rightarrow \mathbf{X}$  of a space  $Y$  there exists a unique map  $g : Y \rightarrow X$  such that  $p_a g = q_a$ .

Let  $X = \{X_a, p_{ab}, A\}$  be an approximate system. A point  $x = (x_a) \in \prod \{X_a : a \in A\}$  is called a *thread* of  $X$  provided it satisfies the following condition:

- (L)  $(\forall a \in A)(\forall U \in \text{Cov}(X_a))(\exists b \geq a)(\forall c \geq b) p_{ac}(x_c) \in st(x_a, U)$ .

If  $X_a$  is a  $T_{3,5}$  space, then the sets  $st(x_a, U)$ ,  $U \in \text{Cov}(X_a)$ , form a basis of the topology at the point  $x_a$ . Therefore, for an approximate system of Tychonoff spaces condition (L) is equivalent to the following condition:

- (L)\*  $(\forall a \in A) \lim \{p_{ac}(x_c) : c \geq a\} = x_a$ .

Let  $\tau$  be an infinite cardinal. We say that a partially ordered set  $A$  is  $\tau$ -directed if for each  $B \subseteq A$  with  $\text{card}(B) \leq \tau$  there is an  $a \in A$  such that  $a \geq b$  for each  $b \in B$ . If  $A$  is  $\aleph_0$ -directed, then we will say that  $A$  is  $\sigma$ -directed. An inverse system  $\mathbf{X} = \{X_a, p_{ab}, A\}$  is said to be  $\tau$ -directed if  $A$  is  $\tau$ -directed. An inverse system  $\mathbf{X} = \{X_a, p_{ab}, A\}$  is said to be  $\sigma$ -directed if  $A$  is  $\sigma$ -directed.

The proof of the following theorem is similar to the proof of Theorem 1.1 of [4].

**THEOREM 1.** *Let  $\mathbf{X} = \{X_a, p_{ab}, A\}$  be a  $\sigma$ -directed approximate inverse system of compact spaces with surjective bonding mappings and limit  $X$ . Let  $Y$  be a metric compact space. For each surjective mapping  $f : X \rightarrow Y$  there exists an  $a \in A$  such that for each  $b \geq a$  there exists a mapping  $g_b : X_b \rightarrow Y$  such that  $f = g_b p_b$ .*

**THEOREM 2.** *Let  $X$  be a compact space. There exists a  $\sigma$ -directed inverse system  $\mathbf{X} = \{X_a, p_{ab}, A\}$  of compact metric spaces  $X_a$  and surjective bonding mappings  $p_{ab}$  such that  $X$  is homeomorphic to  $\lim \mathbf{X}$ .*

*Proof.* It is well-known that there exists a usual inverse system  $\mathbf{Y} = \{Y_\alpha, q_{\alpha\beta}, \Sigma\}$  of metric spaces  $Y_\alpha$  and surjective bonding mappings such that  $X$  is homeomorphic to  $\lim \mathbf{Y}$ . By Theorem 9.5 of [12] there exists a  $\sigma$ -directed inverse system  $\mathbf{X} = \{X_a, p_{ab}, A\}$  such that  $\lim \mathbf{X}$  is homeomorphic to  $\lim \mathbf{Y}$  and each  $X_a$  is the limit of a countable inverse subsystem of  $\mathbf{Y}$ . This means that each  $X_a$  is a metric compact space. ■

**THEOREM 3.** [8, p. 163, Theorem 2.] *If  $X$  is a locally connected compact space, then there exists an inverse system  $\mathbf{X} = \{X_a, p_{ab}, A\}$  such that each  $X_a$  is a*

metric locally connected compact space, each  $p_{ab}$  is a monotone surjection and  $X$  is homeomorphic to  $\lim \mathbf{X}$ . Conversely, the inverse limit of such system is always a locally connected compact space.

REMARK 1. We may assume that  $\mathbf{X} = \{X_a, p_{ab}, A\}$  in Theorem 3 is  $\sigma$ -directed [12, Theorem 9.5].

THEOREM 4. [13, Corollary 2.9] *If  $X$  is a hereditarily locally connected continuum, then there exists a  $\sigma$ -directed inverse system  $\mathbf{X} = \{X_a, p_{ab}, A\}$  such that each  $X_a$  is a metrizable hereditarily locally connected continuum, each  $p_{ab}$  is a monotone surjection and  $X$  is homeomorphic to  $\lim \mathbf{X}$ .*

THEOREM 5. [3, Corollary 3] *Let  $\mathbf{X} = \{X_a, p_{ab}, A\}$  be a  $\sigma$ -directed inverse system of hereditarily locally connected continua  $X_a$ . Then  $X = \lim \mathbf{X}$  is hereditarily locally connected.*

The following theorem is Theorem 1.7 from [5].

THEOREM 6. *Let  $\mathbf{X} = \{X_a, p_{ab}, A\}$  be a  $\sigma$ -directed inverse system of compact metrizable spaces and surjective bonding mappings. Then  $X = \lim \mathbf{X}$  is metrizable if and only if there exists an  $a \in A$  such that  $p_b: X \rightarrow X_b$  is a homeomorphism for each  $b \geq a$ .*

## 2. Approximate subsystems

In this Section we investigate the approximate subsystem of an approximate system  $\mathbf{X} = \{X_a, p_{ab}, A\}$ . We start with the following definition.

DEFINITION 2. Let  $\mathbf{X} = \{X_a, p_{ab}, A\}$  be an approximate inverse system and let  $B$  be a directed subset of  $A$  such that  $\{X_b, p_{bc}, B\}$  is an approximate inverse system. We say that  $\{X_b, p_{bc}, B\}$  is an *approximate subsystem* of  $\mathbf{X} = \{X_a, p_{ab}, A\}$  if there exists a mapping  $q: \lim \mathbf{X} \rightarrow \lim \{X_b, p_{bc}, B\}$  such that

$$p_b q = P_b, \quad b \in B,$$

where  $p_b: \lim \{X_b, p_{bc}, B\} \rightarrow X_b$  and  $P_b: \lim \mathbf{X} \rightarrow X_b$ ,  $b \in B$ , are natural projections.

We say that an approximate system  $\mathbf{X} = \{X_a, p_{ab}, A\}$  is *irreducible* if for each  $B \subset A$  with  $\text{card}(B) < \text{card}(A)$  it follows that  $B$  is not cofinal in  $A$ .

LEMMA 2. *Let  $\mathbf{X} = \{X_a, p_{ab}, A\}$  be an approximate inverse system. There exists a cofinal subset  $B$  of  $A$  such  $\mathbf{X} = \{X_a, p_{ab}, B\}$  is irreducible.*

*Proof.* Consider the family  $\mathcal{B}$  of all cofinal subsets of  $B$  of  $A$ . The set  $\{\text{card}(B) : B \in \mathcal{B}\}$  has the minimal element  $b$  since each  $\text{card}(B)$  is some initial ordinal number. Let  $B \in \mathcal{B}$  be such that  $\text{card}(B) = b$ . It is clear that  $\{X_a, p_{ab}, B\}$  is irreducible. ■

In the sequel we will assume that  $\mathbf{X} = \{X_a, p_{ab}, A\}$  is irreducible.

LEMMA 3. *Let  $\mathbf{X} = \{X_a, p_{ab}, A\}$  be an approximate inverse system of compact spaces such that  $\text{card}(A) = \aleph_0$ . Then there exists a countable well-ordered subset  $B$*

of  $A$  such that the collection  $\{X_b, p_{bc}, B\}$  is an approximate inverse sequence and  $\lim \mathbf{X}$  is homeomorphic to  $\lim \{X_b, p_{bc}, B\}$ .

*Proof.* Let  $\nu$  be any finite subset of  $A$ . There exists a  $\delta(\nu) \in A$  such that  $\delta \leq \delta(\nu)$  for each  $\delta \in \nu$ . Since  $A$  is infinite, there exists a sequence  $\{\nu_n : n \in \mathbb{N}\}$  such that  $\nu_1 \subseteq \dots \subseteq \nu_n \subseteq \dots$  and  $A = \bigcup \{\nu_n : n \in \mathbb{N}\}$ . Recursively, we define the sets  $A_1, \dots, A_n, \dots$  by

$$A_1 = \nu_1 \cup \{\delta(\nu_1)\},$$

and

$$A_{n+1} = A_n \cup \nu_{n+1} \cup \{\delta(A_n \cup \nu_{n+1})\}.$$

It follows that there exists a sequence

$$A_1 \subseteq A_2 \subseteq \dots \subseteq A_n \subseteq \dots$$

of finite sets  $A_n$  such that  $A = \bigcup \{A_n : n \in \mathbb{N}\}$ . Let  $b_1 = \delta(A_1)$  and  $b_n \geq \delta(A_n)$ ,  $b_{n-1}$  if  $n \geq 2$ . We obtain a sequence  $B = \{b_n : n \in \mathbb{N}\}$  such that  $B$  is cofinal in  $A$ . By virtue of [10, Theorem (1.19)] it follows that  $\lim \mathbf{X}$  is homeomorphic to  $\lim \{X_b, p_{bc}, B\}$ . ■

Now we consider irreducible approximate inverse systems  $\mathbf{X} = \{X_a, p_{ab}, A\}$  with  $\text{card}(A) \geq \aleph_1$ .

LEMMA 4. *Let  $A$  be a directed set. For each subset  $B$  of  $A$  there exists a directed set  $F_\infty(B)$  such that  $\text{card}(F_\infty(B)) = \text{card}(B)$ .*

*Proof.* For each  $B \subseteq A$  there exists a set  $F_1(B) = B \cup \{\delta(\nu) : \nu \in B\}$ , where  $\nu$  is a finite subset of  $B$  and  $\delta(\nu)$  is defined as in the proof of Lemma 3. Put

$$F_{n+1} = F_1(F_n(B)),$$

and

$$F_\infty(B) = \bigcup \{F_n(B) : n \in \mathbb{N}\}.$$

It is clear that

$$F_1(B) \subseteq F_2(B) \subseteq \dots \subseteq F_n(B) \subseteq \dots$$

The set  $F_\infty(B)$  is directed since each finite subset  $\nu$  of  $F_\infty(B)$  is contained in some  $F_n(B)$  and, consequently,  $\delta(\nu)$  is contained in  $F_\infty(B)$ .

If  $B$  is finite, then  $\text{card}(F_\infty(B)) = \aleph_0$ . If  $\text{card}(B) \geq \aleph_0$ , then we have  $\text{card}(\{\delta(\nu) : \nu \in B\}) \leq \text{card}(B)\aleph_0$ . We infer that  $\text{card}(F_1(B)) \leq \text{card}(B)\aleph_0$ . Similarly,  $\text{card}(F_n(B)) \leq \text{card}(B)\aleph_0$ . This means that  $\text{card}(F_\infty(B)) \leq \text{card}(B)\aleph_0$ . Thus

$$\text{card}(F_\infty(B)) \leq \text{card}(B)\aleph_0, \quad \text{if} \quad \text{card}(B) < \text{card}(A).$$

The proof is completed. ■

LEMMA 5. *Let  $\{X_a, p_{ab}, A\}$  be an approximate inverse system such that  $\text{cw}(X_a) < \text{card}(A)$ ,  $a \in A$ . For each subset  $B$  of  $A$  with  $\text{card}(B) < \text{card}(A)$ , there exists a directed set  $G_\infty(B) \supseteq B$  such that the collection  $\{X_a, p_{ab}, G_\infty(B)\}$  is an approximate system and  $\text{card}(G_\infty(B)) = \text{card}(B)$ .*

*Proof.* Let  $\mathcal{B}_a$  be a base of normal coverings of  $X_a$ . Let  $\mathcal{U}_a \in \mathcal{B}_a$ . By virtue of (A2) there exists an  $a(\mathcal{U}_a) \in A$  such that  $(p_{ad}, p_{ac}p_{cd}) \leq \mathcal{U}_a$ ,  $a \leq a(\mathcal{U}_a) \leq c \leq d$ . For each subset  $B$  of  $A$  we define  $G_\infty(B)$  by induction as follows:

a) Let  $G_1(B) = F_\infty(B)$ . From Lemma 4 it follows that  $\text{card}(G_1(B)) = \text{card}(F_\infty(B)) = \text{card}(B)$ .

b) For each  $n > 1$  we define  $G_n(B)$  as follows:

1) If  $n$  is odd then  $G_n(B) = F_\infty(G_{n-1}(B))$ ,

2) If  $n$  is even, then  $G_n(B) = G_{n-1}(B) \cup \{a(\mathcal{U}_a) : \mathcal{U}_a \in \mathcal{B}_a, a \in G_{n-1}(B)\}$ .

Since  $\text{card}(\mathcal{B}_a) < \text{card}(A)$  the set  $G_n(B)$  has the cardinality  $< \text{card}(A)$ . Now we define  $G_\infty(B) = \bigcup\{G_n(B) : n \in \mathbb{N}\}$ . It is obvious that  $\text{card}(G_\infty(B)) < \text{card}(A)$ .

*The set  $G_\infty(B)$  is directed.* Let  $a, b$  be a pair of elements of  $G_\infty(B)$ . There exists an  $n \in \mathbb{N}$  such that  $a, b \in G_n(B)$ . We may assume that  $n$  is odd. Then  $a, b \in F_\infty(G_{n-1}(B))$ . Thus there exists a  $c \in F_\infty(G_{n-1}(B))$  such that  $c \geq a, b$ . It is clear that  $c \in G_\infty(B)$ . The proof of directedness of  $G_\infty(B)$  is completed.

*The collection  $\{X_a, p_{ab}, G_\infty(B)\}$  is an approximate system.* It suffices to prove that the condition (A2) is satisfied. Let  $a$  be any member of  $G_\infty(B)$ . There exists an  $n \in \mathbb{N}$  such that  $a \in G_n(B)$ . We have two cases.

1) If  $n$  is odd then  $G_n(B) = F_\infty(G_{n-1}(B))$ . This means that  $a \in F_\infty(G_{n-1}(B))$ . By definition of  $F_\infty(G_{n-1}(B))$  we infer that  $a(\mathcal{U}_a) \in F_\infty(G_{n-1}(B))$ . Thus (A2) is satisfied.

2) If  $n$  is even, then  $G_n(B) = G_{n-1}(B) \cup \{a(\mathcal{U}_a) : \mathcal{U}_a \in \text{Cov}(X_a), a \in G_{n-1}(B)\}$ . In this case  $a \in G_{n+1}(B) \subseteq G_\infty(B)$ . Arguing as in the case 1, we infer that (A2) is satisfied. ■

**THEOREM 7.** *Let  $\mathbf{X} = \{X_a, p_{ab}, A\}$  be an approximate inverse system of compact spaces. If  $\lambda \leq w(X_a) \leq \tau < \text{card}(A)$  for each  $a \in A$ , then  $\lim \mathbf{X}$  is homeomorphic to a limit of a  $\lambda$ -directed usual inverse system  $\{X_\alpha, q_{\alpha\beta}, T\}$ , where each  $X_\alpha$  is a limit of an approximate inverse subsystem  $\{X_\gamma, p_{\alpha\beta}, \Phi\}$ ,  $\text{card}(\Phi) = \lambda$ .*

*Proof.* The proof consists of several steps.

**Step 1.** Let  $\mathcal{B} = \{B_\mu : \mu \in M\}$  be a family of all subsets of  $A$  with  $\text{card}(B_\mu) = \lambda$ . Put  $A_\mu = G_\infty(B_\mu)$  (Lemma 5) and let  $\Delta = \{A_\mu : \mu \in M\}$  be ordered by inclusion  $\subseteq$ .

**Step 2.** *If  $\Phi$  and  $\Psi$  are in  $\Delta$  such that  $\Phi \subset \Psi$ , then there exists a mapping  $q_{\Phi\Psi} : \lim\{X_\alpha, p_{\alpha\beta}, \Psi\} \rightarrow \lim\{X_\gamma, p_{\alpha\beta}, \Phi\}$ .*

Namely, if  $x = (x_\alpha, \alpha \in \Psi) \in \lim\{X_\alpha, p_{\alpha\beta}, \Psi\}$ , then by definition of the threads of  $\{X_\alpha, p_{\alpha\beta}, \Psi\}$  the condition (L) is satisfied. If (L) is satisfied for  $x = (x_\alpha, \alpha \in \Psi) \in \lim\{X_\alpha, p_{\alpha\beta}, \Psi\}$ , then it is satisfied for  $(x_\gamma, \gamma \in \Phi)$  since the required  $a'$  in (L) lies—by definition of the set  $\Phi$ —in the set  $\Phi$ . This means that  $(x_\gamma, \gamma \in \Phi) \in \lim\{X_\gamma, p_{\alpha\beta}, \Phi\}$ . Now we define  $q_{\Phi\Psi}(x) = (x_\gamma, \gamma \in \Phi)$ .

**Step 3.** *The collection  $\{X_\Phi, q_{\Phi\Psi}, \Delta\}$  is a usual inverse system.* It suffices to prove transitivity, i.e., if  $\Phi \subseteq \Psi \subseteq \Omega$ , then  $q_{\Phi\Psi}q_{\Psi\Omega} = q_{\Phi\Omega}$ . This easily follows from the definition of  $q_{\Phi\Psi}$ .

**Step 4.** *The space  $\lim \mathbf{X}$  is homeomorphic to  $\lim\{X_\Psi, q_{\Phi\Psi}, \Delta\}$ , where  $X_\Phi = \lim\{X_\gamma, p_{\alpha\beta}, \Phi\}$ . We shall define a homeomorphism  $H: \lim \mathbf{X} \rightarrow \lim\{X_\Psi, q_{\Phi\Psi}, \Delta\}$ . Let  $x = (x_a : a \in A)$  be any point of  $\lim \mathbf{X}$ . Each collection  $\{x_a : a \in \Phi \in \Delta\}$  is a point  $x_\Phi$  of  $X_\Phi$  since  $X_\Phi = \lim\{X_a, p_{ab}, \Phi\}$ . Moreover, from the definition of  $q_{\Phi\Psi}$  (Step 2) it follows that  $q_{\Phi\Psi}(x_\Psi) = x_\Phi$ ,  $\Psi \supseteq \Phi$ . Thus, the collection  $\{x_\Phi : \Phi \in \Delta\}$  is a point of  $\lim\{X_\Psi, q_{\Phi\Psi}, \Delta\}$ . Let  $H(x) = \{x_\Phi, \Phi \in \Delta\}$ . Thus,  $H$  is a continuous mapping of  $\lim \mathbf{X}$  to  $\lim\{X_\Psi, q_{\Phi\Psi}, \Delta\}$ . In order to complete the proof it suffices to prove that  $H$  is 1-1 and onto. Let us prove that  $H$  is 1-1. Let  $x = (x_a : a \in A)$  and  $y = (y_a : a \in A)$  be a pair of points of  $\lim \mathbf{X}$ . This means that there exists an  $a \in A$  such that  $y_a \neq x_a$ . There exists a  $\Phi \in \Delta$  such that  $a \in \Phi$ . Thus, the collections  $\{x_a : a \in \Phi\}$  and  $\{y_a : a \in \Phi\}$  are different. From this we conclude that  $x_\Phi \neq y_\Phi$ ,  $x_\Phi, y_\Phi \in X_\Phi = \lim\{X_a, p_{ab}, \Phi\}$ . Hence  $H$  is 1-1. Let us prove that  $H$  is onto. Let  $y = (y_\Phi : \Phi \in \Delta)$  be any point of  $\lim\{X_\Psi, q_{\Phi\Psi}, \Delta\}$ . Each  $y_\Phi$  is a collection  $\{x_a : a \in \Phi\}$  and if  $\Psi \supseteq \Phi$ , then the collection  $\{x_a : a \in \Phi\}$  is the restriction of the collection  $\{x_a : a \in \Psi\}$  on  $\Phi$ . Let  $x$  be the collection which is the union of all collections  $\{x_a : a \in \Phi\}$ ,  $\Phi \in \Delta$ . Hence  $x$  is a collection  $(x_a : a \in A)$  which is a point of  $\lim \mathbf{X}$  and  $H(x) = y$ .*

**Step 5.** *Inverse system  $\{X_\Phi, q_{\Phi\Psi}, \Delta\}$  is a  $\lambda$ -directed inverse system. Let  $\{\{X_\gamma, p_{\alpha\beta}, \Phi_\kappa\} : \kappa \leq \lambda\}$  be a collection of approximate subsystems  $\{X_\gamma, p_{\alpha\beta}, \Phi_\kappa\}$ . The set  $\Phi = \bigcup\{\Phi_\kappa : \kappa \leq \lambda\}$  has the cardinality  $\leq \lambda$  since  $\text{card}(\Phi_\kappa) \leq \lambda$ . By virtue of Steps 1-4 there exists an approximate subsystem  $\{X_\gamma, p_{\alpha\beta}, \Phi\}$ ,  $\text{card}(\Phi) = \lambda$ . This means that  $\{X_\Phi, q_{\Phi\Psi}, \Delta\}$  is a  $\lambda$ -directed inverse system. ■*

**COROLLARY 1.** *Let  $\mathbf{X} = \{X_a, p_{ab}, A\}$  be an approximate inverse system of compact metric spaces. Then  $\lim \mathbf{X}$  is homeomorphic to the limit of a  $\sigma$ -directed usual inverse system  $\{X_\alpha, q_{\alpha\beta}, \Delta\}$ , where each  $X_\alpha$  is a limit of an approximate inverse subsystem  $\{X_\gamma, p_{\alpha\beta}, \Phi\}$ ,  $\text{card}(\Phi) = \aleph_0$ .*

**LEMMA 6.** *Let  $\mathbf{X} = \{X_a, p_{ab}, A\}$  be an approximate system such that  $X_a$ ,  $a \in A$ , are compact locally connected spaces and  $p_{ab}$  are monotone surjections. If  $\mathbf{Y} = \{X_b, p_{cd}, B\}$  is an approximate subsystem of  $\mathbf{X}$ , then the mapping  $q_{AB}: \lim \mathbf{X} \rightarrow \lim \mathbf{Y}$  (defined in Step 2 of the proof of Theorem 7) is a monotone surjection.*

*Proof.* Let  $P_a: \lim \mathbf{X} \rightarrow X_a$ ,  $a \in A$ , be the natural projection. Similarly, let  $p_a: \lim \mathbf{Y} \rightarrow X_a$ ,  $a \in B$ , be the natural projection. From the definition of  $q_{AB}$  (Step 2 of the proof of Theorem 7) it follows that  $p_a q_{AB} = P_a$  for each  $a \in B$ . By virtue of [10, Corollary 4.5] and [7, Corollary 5.6] it follows that  $P_a$  and  $p_a$  are monotone surjections. Let us prove that  $q_{AB}$  is a surjection. Let  $y = (y_a : a \in B) \in \lim \mathbf{Y}$ . The sets  $P_a^{-1}(y_a)$ ,  $a \in B$ , are non-empty since  $P_a$  is surjective for each  $a \in A$ . From the compactness of  $\lim \mathbf{X}$  it follows that a limit superior  $Z = \text{Ls}\{P_a^{-1}(y_a), a \in B\}$  is a non-empty subset of  $\lim \mathbf{X}$ . We shall prove that for each  $z = (z_a : a \in A) \in Z$ ,  $P_a(z) = y_a$ . Suppose that  $P_a(z) \neq y_a$ . There exists a pair  $U, V$  of open disjoint subsets of  $X_a$  such that  $y_a \in U$  and  $P_a(z) \in V$ . For sufficiently large  $b \in B$ ,  $P_a(P_b^{-1}(b))$  is in  $U$  because of (AS). This means that  $P_a^{-1}(V) \cap P_b^{-1}(y_b) = \emptyset$  for sufficiently large  $b \in B$ . This contradicts the assumption  $z \in \text{Ls}\{P_a^{-1}(y_a), a \in B\}$ . Hence  $q_{AB}$  is a surjection. In order to complete the proof it suffices to prove that

$q_{AB}$  is monotone. Take a point  $y \in \lim \mathbf{Y}$  and suppose that  $q_{AB}^{-1}(y)$  is disconnected. There exists a pair  $U, V$  of disjoint open sets in  $\lim \mathbf{X}$  such that  $q_{AB}^{-1}(y) \subseteq U \cup V$ . From the compactness of  $\lim \mathbf{X}$  it follows that  $q_{AB}$  is closed. This means that there exists an open neighborhood  $W$  of  $y$  such that  $q_{AB}^{-1}(y) \subseteq q_{AB}^{-1}(W) \subseteq U \cup V$ . From the definition of the basis in  $\lim \mathbf{Y}$  it follows that there exists an open set  $W_a$  in some  $X_a$ ,  $a \in B$  such that  $y \in p_a^{-1}(W_a) \subseteq W$ . Moreover, we may assume that  $W_a$  is connected since  $X_a$  is locally connected. Then  $P_a^{-1}(W_a)$  is connected since  $P_a$  is monotone [7, Corollary 5.6]. Moreover,  $q_{AB}^{-1}(y) \subseteq P_a^{-1}(W_a)$  and  $P_a^{-1}(W_a) \subseteq U \cup V$  since  $P_a = p_a q_{AB}$ . This is impossible since  $U$  and  $V$  are disjoint open sets and  $P_a^{-1}(W_a)$  is connected. ■

**THEOREM 8.** *Let  $\mathbf{X} = \{X_a, p_{ab}, A\}$  be an approximate inverse system of compact spaces. If  $\lambda \leq w(X_a) < \text{card}(A)$  for each  $a \in A$  and  $\text{cf}(\text{card}(A)) \neq \lambda$ , then  $X = \lim \mathbf{X}$  is homeomorphic to a limit of a  $\lambda$ -directed usual inverse system  $\{X_\alpha, q_{\alpha\beta}, T\}$ , where each  $X_\alpha$  is a limit of an approximate inverse subsystem  $\{X_\gamma, p_{\alpha\beta}, \Phi\}$ ,  $\text{card}(\Phi) = \lambda$ . Moreover, if  $\text{card}(A)$  is a regular cardinal, then  $X = \lim \mathbf{X}$  is homeomorphic to a limit of a  $\lambda$ -directed usual inverse system  $\{X_\alpha, q_{\alpha\beta}, T\}$ , where each  $X_\alpha$  is a limit of an approximate inverse subsystem  $\{X_\gamma, p_{\alpha\beta}, \Phi\}$ ,  $\text{card}(\Phi) = \lambda$ .*

A directed preordered set  $(A, \leq)$  is said to be *cofinite* provided each  $a \in A$  has only finitely many predecessors. If  $a \in A$  has exactly  $n$  predecessors, we shall write  $p(a) = n + 1$ . Hence,  $a \in A$  is the first element of  $(A, \leq)$  if and only if  $p(a) = 1$ .

**LEMMA 7.** *If  $(A, \leq)$  is cofinite, then it satisfies the following principle of induction:*

*Let  $B \subset A$  be a set such that:*

- (i)  *$B$  contains all the first elements of  $A$ ,*
- (ii) *if  $B$  contains all the predecessors of  $a \in A$ , then  $a \in B$ .*

*Then  $B = A$ .*

**LEMMA 8.** [15, Lemma 1] *Let  $q = (q_a): Y \rightarrow \mathcal{Y} = \{Y_b, \mathcal{V}_b, q_{ab'}, B\}$  be an approximate map (approximate resolution) of a space  $Y$ . Then there exists an approximate map (approximate resolution)  $q = (q_a): Y \rightarrow \mathcal{Y} = \{Y'_c, \mathcal{V}'_c, q_{cc'}, C\}$  of the space  $Y$  and an increasing surjection  $t: C \rightarrow B$  satisfying the following conditions:*

- (i)  *$C$  is directed, unbounded, antisymmetric and cofinite set;*
- (ii)  *$(\forall c \in C)(\forall b \in B)(\exists c' > c) t(c') > b$ ;*
- (iii)  *$(\forall c \in C) Y'_c = Y_{t(c)}$ ,  $\mathcal{V}'_c = \mathcal{V}_{t(c)}$ ,  $q'_c = q_{t(c)}$  and  $q'_{cc'} = q_{t(c)t(c')}$ , whenever  $c < c'$ .*

**COROLLARY 2.** *Let  $\mathbf{X} = \{X_a, p_{ab}, A\}$  be an approximate inverse system of compact spaces. Then there exists a cofinite approximate inverse system  $\mathbf{Y} = \{Y_c, p_{cc'}, C\}$  such that each  $Y_c$  is some  $X_a$ , each  $p_{cc'}$  is some  $p_{ab}$  and  $\lim \mathbf{X}$  is homeomorphic to  $\lim \mathbf{Y}$ .*

*Proof.* By virtue of Theorem (4.2) of [10] an approximate map  $p: X \rightarrow \mathbf{X}$  is an approximate resolution if and only if it is a limit of  $\mathbf{X} = \{X_a, p_{ab}, A\}$ . Apply Lemma 8. ■

**THEOREM 9.** *Let  $\mathbf{X} = \{X_a, p_{ab}, A\}$  be an approximate inverse system of compact non-metric spaces with surjective bonding mappings  $p_{ab}$ . If each  $X_a$  is a limit of a usual  $\sigma$ -directed inverse system  $\mathbf{X}(a) = \{X_{(a,\gamma)}, f_{(a,\gamma)(a,\delta)}, \Gamma_a\}$  of metric compact spaces, then:*

1. *there exists a usual  $\sigma$ -directed inverse system  $\mathbf{X}_D = \{X_d, F_{de}, D\}$  whose inverse limit  $X_D$  is homeomorphic to  $X = \lim \mathbf{X}$ ,*
2. *every  $X_d$  is a limit of an approximate inverse system  $\{X_{(a,\gamma_a)}, g_{(a,\gamma_a)(b,\gamma_b)}, A\}$  of compact metric spaces  $X_{(a,\gamma_a)}$ ,*
3. *if the mappings  $p_{ab}$  and  $f_{(a,\gamma)(a,\delta)}$  are monotone, then  $g_{(a,\gamma_a)(b,\gamma_b)}$  and  $F_{de}$  are monotone.*

*Proof.* The proof consists of several steps. In the Steps 0.–11. we shall define a usual inverse system  $\mathbf{X}_D = \{X_d, F_{de}, D\}$  whose inverse limit  $X_D$  is homeomorphic to  $X = \lim \mathbf{X}$ .

**Step 0.** From Corollary 2 it follows that we may assume that  $A$  is cofinite.

**Step 1.** For each  $X_a$  there exists a  $\sigma$ -directed inverse system

$$\mathbf{X}(a) = \{X_{(a,\gamma)}, f_{(a,\gamma)(a,\delta)}, \Gamma_a\} \quad (1)$$

such that each  $X_{(a,\gamma)}$  is a metric compact space, each  $f_{(a,\gamma)(a,\delta)}$  is monotone and surjective and  $X_a$  is homeomorphic to  $\lim \mathbf{X}(a)$ . Now we have the following diagram

$$\begin{array}{ccccc} X_a & \xleftarrow{p_{ab}} & X_b & \xleftarrow{p_{bc}} & X_c & \xleftarrow{p_d} & X \\ \downarrow f_{(a,\gamma_a)} & & \downarrow f_{(b,\gamma_b)} & & \downarrow f_{(c,\gamma_c)} & & \\ X_{(a,\gamma_a)} & & X_{(b,\gamma_b)} & & X_{(c,\gamma_c)} & & \\ \downarrow f_{(a,\gamma_a)(a,\delta_a)} & & \downarrow f_{(b,\gamma_b)(b,\delta_b)} & & \downarrow f_{(c,\gamma_c)(c,\delta_c)} & & \\ X_{(a,\delta_a)} & & X_{(b,\delta_b)} & & X_{(c,\delta_c)} & & \\ \downarrow & & \downarrow & & \downarrow & & \end{array} \quad (2)$$

**Step 2.** Put  $B = \{(a, \gamma_a) : a \in A, \gamma_a \in \Gamma_a\}$  and put  $C$  to be the set of all subsets  $c$  of  $B$  of the form

$$c = \{(a, \gamma_a) : a \in A\}, \quad (3)$$

where every  $\gamma_a$  is the fixed element of  $\Gamma_a$ .

**Step 3.** Let  $D$  be a subset of  $C$  containing all  $c \in C$  for which there exist the mappings

$$g_{(a,\gamma_a)(b,\gamma_b)} : X_{(b,\gamma_b)} \rightarrow X_{(a,\gamma_a)}, \quad b \geq a, \quad (4)$$

such that

$$\{X_{(a,\gamma_a)}, g_{(a,\gamma_a)(b,\gamma_b)}, A\} \quad (5)$$



is an approximate inverse system and each diagram

$$\begin{array}{ccc}
X_a & \xleftarrow{p_{ab}} & X_b \\
f_{(a,\gamma_a)} \downarrow & & \downarrow f_{(b,\gamma_b)} \\
X_{(a,\gamma_a)} & \xleftarrow{g_{(a,\gamma_a)(b,\gamma_b)}} & X_{(b,\gamma_b)}
\end{array} \quad (6)$$

commutes, where  $f_{(a,\gamma_a)}: X_a \rightarrow X_{(a,\gamma_a)}$  is the canonical projection. It is clear that the mapping  $g_{(a,\gamma_a)(b,\gamma_b)}$  is unique since  $f_{(b,\gamma_b)}$  is a surjection.

**Step 4.** *The set  $D$  is non-empty. Moreover, for each subset  $S_a \subset \Gamma_a$ ,  $a \in A$ ,  $\text{card}(S_a) \leq \aleph_0$ , there exists a  $d \in D$  such that  $d = \{(a, \gamma_a) : a \in A\}$ ,  $\gamma_a \geq \gamma$  for every  $\gamma \in S_a$ . Let  $a \in A$  be some first element of  $A$  and let  $\gamma_a \in \Gamma_a$  such that  $\gamma_a \geq \gamma$  for every  $\gamma \in S_a$ . The space  $X_{(a,\gamma_a)}$  is a metric compact space and there exist mappings  $f_{(a,\gamma_a)}p_{ab}: X_b \rightarrow X_{(a,\gamma_a)}$ ,  $b \geq a$ . By virtue of Theorem 1 for each  $b \geq a$  there exists a  $\gamma_b^1 \in \Gamma_b$  such that for each  $\gamma_b \geq \gamma_b^1, \gamma$ , where  $\gamma \in S_b$ , there exists a monotone surjective mapping  $g_{(a,\gamma_a)(b,\gamma_b)}: X_{(b,\gamma_b)} \rightarrow X_{(a,\gamma_a)}$  with  $f_{(a,\gamma_a)}p_{ab} = g_{(a,\gamma_a)(b,\gamma_b)}f_{(b,\gamma_b)}$ , i.e., the diagram*

$$\begin{array}{ccc}
X_a & \xleftarrow{p_{ab}} & X_b \\
f_{(a,\gamma_a)} \downarrow & & \downarrow f_{(b,\gamma_b)} \\
X_{(a,\gamma_a)} & \xleftarrow{g_{(a,\gamma_a)(b,\gamma_b)}} & X_{(b,\gamma_b)}
\end{array} \quad (7)$$

commutes. Suppose that  $(a, \gamma_b^1), (a, \gamma_b^2), \dots, (a, \gamma_b^{n-1})$  are defined for each  $a \in A$  with  $p(a) \leq n-1$  such that the each diagram (6) commutes. Let  $a \in A$  be a member of  $A$  with  $p(a) = n$ . This means that  $(a, \gamma_b^1), (a, \gamma_b^2), \dots, (a, \gamma_b^{n-1})$  are defined. From the cofiniteness of  $A$  it follows that the set of  $\gamma_a^j$  which are defined in  $\Gamma_a$  is finite. Hence there exists  $\gamma_a^n \geq \gamma_a^{n-1}, \dots, \gamma_a^1$ . We define  $\gamma_b^n \in \Gamma_b$  considering the space  $X_{(a,\gamma_a^n)}$  and the mappings  $f_{(a,\gamma_a^n)}p_{ab}: X_b \rightarrow X_{(a,\gamma_a^n)}$ . Again, by Theorem 1 for each  $b \geq a$  there exists a  $\gamma_b^n \in \Gamma_b$  such that for each  $\gamma_b \geq \gamma_b^n, \gamma_b^{n-1}, \dots, \gamma_b^1$  and there is a mapping  $g_{(a,\gamma_a^n)(b,\gamma_b)}: X_{(b,\gamma_b)} \rightarrow X_{(a,\gamma_a^n)}$  with  $f_{(a,\gamma_a^n)}p_{ab} = g_{(a,\gamma_a^n)(b,\gamma_b)}f_{(b,\gamma_b)}$ , i.e., the diagram

$$\begin{array}{ccc}
X_a & \xleftarrow{p_{ab}} & X_b \\
f_{(a,\gamma_a^n)} \downarrow & & \downarrow f_{(b,\gamma_b)} \\
X_{(a,\gamma_a^n)} & \xleftarrow{g_{(a,\gamma_a^n)(b,\gamma_b)}} & X_{(b,\gamma_b)}
\end{array} \quad (8)$$

commutes. By induction on  $A$  (Lemma 7) the set  $D$  is defined. It remains to prove that  $\{X_{(a,\gamma_a)}, g_{(a,\gamma_a)(b,\gamma_b)}, A\}$  is an approximate inverse system. Let  $\mathcal{U}$  be a normal cover of  $X_{(a,\gamma_a)}$ . Then  $\mathcal{V} = f_{(a,\gamma_a)}^{-1}(\mathcal{U})$  is a normal cover of  $X_a$ . By virtue of (A2) there exists a  $b \geq a$  such that for each  $c \geq d \geq b$  we have  $(p_{ad}, p_{ca}p_{cd}) \leq \mathcal{V}$ . By virtue of the commutativity of the diagrams of the form (8) it follows that

$$(g_{(a,\gamma_a)(d,\gamma_d)}, g_{(a,\gamma_a)(c,\gamma_c)}g_{(c,\gamma_c)(d,\gamma_d)}) \leq \mathcal{V}.$$

Thus,  $\{X_{(a,\gamma_a)}, g_{(a,\gamma_a)(b,\gamma_b)}, A\}$  is an approximate inverse system.

**Step 5.** We define a partial order on  $D$  as follows. Let  $d_1, d_2$  be a pair of members of  $D$  such that  $d_1 = \{(a, \gamma_a) : a \in A, \gamma_a \in \Gamma_a\}$  and  $d_2 = \{(a, \delta_a) : a \in A, \delta_a \in \Gamma_a\}$ . We write  $d_2 \leq d_1$  if and only if  $\delta_a \leq \gamma_a$  for each  $a \in A$ . From Step 4. it follows that  $(D, \leq)$  is  $\tau$ -directed. Moreover,  $X_D$  is a usual inverse system.

**Step 6.** For each  $d \in D$  the limit space  $X_d$  of the inverse system (5) is a compact space. Moreover, there exists a mapping  $F_d: X \rightarrow X_d$ . The existence of  $F_d$  follows from the commutativity of the diagram (6). The following diagram illustrates the construction of  $d \in D$  and the space  $X_d$ .

$$\begin{array}{ccccc}
X_a & \xleftarrow{p_{ab}} & X_b & \xleftarrow{p_{bc}} & X_c & \xleftarrow{p_d} & X \\
\downarrow f_{(a, \delta_a)} & & \downarrow f_{(b, \delta_b)} & & \downarrow f_{(c, \delta_c)} & & \\
X_{(a, \delta_a)} & & X_{(b, \delta_b)} & & X_{(c, \delta_c)} & & \\
\downarrow f_{(a, \gamma_a)(a, \delta_a)} & & \downarrow f_{(b, \gamma_b)(b, \delta_b)} & & \downarrow f_{(c, \gamma_c)(c, \delta_c)} & & \\
X_{(a, \gamma_a)} & \xleftarrow{g_{(a, \gamma_a)(b, \gamma_b)}} & X_{(b, \gamma_b)} & \xleftarrow{g_{(b, \gamma_b)(c, \gamma_c)}} & X_{(c, \gamma_c)} & \xleftarrow{g_{(c, \gamma_c)}} & X_d \\
\downarrow & & \downarrow & & \downarrow & & 
\end{array} \tag{9}$$

**Step 7.** If  $d_1, d_2$  is a pair of members of  $D$  such that  $d_1 = \{(a, \gamma_a) : a \in A, \gamma_a \in \Gamma_a\}$ ,  $d_2 = \{(a, \delta_a) : a \in A, \delta_a \in \Gamma_a\}$  and  $d_2 \geq d_1$ , then for each  $a \in A$  commutes the diagram

$$\begin{array}{ccc}
X_{(a, \delta_a)} & \xleftarrow{g_{(a, \delta_a)(b, \delta_b)}} & X_{(b, \delta_b)} \\
f_{(a, \gamma_a)(a, \delta_a)} \downarrow & & \downarrow f_{(b, \gamma_b)(b, \delta_b)} \\
X_{(a, \gamma_a)} & \xleftarrow{g_{(a, \gamma_a)(b, \gamma_b)}} & X_{(b, \gamma_b)}
\end{array} \tag{10}$$

This follows from the surjectivity of the mappings  $f_{(b, \gamma_b)}$  and from the commutativity of the diagrams of the form (6) for  $d_1$  and  $d_2$ , i.e., from the commutativity of the diagrams

$$\begin{array}{ccc}
X_a & \xleftarrow{p_{ab}} & X_b \\
f_{(a, \gamma_a)} \downarrow & & \downarrow f_{(b, \gamma_b)} \\
X_{(a, \gamma_a)} & \xleftarrow{g_{(a, \gamma_a)(b, \gamma_b)}} & X_{(b, \gamma_b)}
\end{array} \tag{11}$$

and

$$\begin{array}{ccc}
X_a & \xleftarrow{p_{ab}} & X_b \\
f_{(a, \delta_a)} \downarrow & & \downarrow f_{(b, \delta_b)} \\
X_{(a, \delta_a)} & \xleftarrow{g_{(a, \delta_a)(b, \delta_b)}} & X_{(b, \delta_b)}
\end{array} \tag{12}$$

**Step 8.** From Step 7. it follows that for  $d_1, d_2 \in D$  with  $d_2 \geq d_1$  there exists a mapping  $F_{d_1 d_2}: X_{d_2} \rightarrow X_{d_1}$  (see [1, p. 138]) such that  $F_{d_1} = F_{d_1 d_2} F_{d_2}$ .

*Proof of Step 8.* Let  $d_1, d_2, d_3 \in D$  and let  $d_1 \leq d_2 \leq d_3$ . Then  $F_{d_1 d_3} = F_{d_1 d_2} F_{d_2 d_3}$ . This follows from Step 7. and the commutativity condition in each inverse system  $\mathbf{X}(a) = \{X_{(a,\gamma)}, f_{(a,\gamma)(a,\delta)}, \Gamma_a\}$  (see (1) of Step 1.).

**Step 9.** *The collection  $\{X_d, F_{de}, D\}$  is a usual inverse system of compact spaces.*

Apply Steps 1.–8.

**Step 10.** *There is a mapping  $F: X \rightarrow X_D$  which is 1–1.*

By Step 6. and Step 8. for each  $d \in D$  there is a mapping  $F_d: X \rightarrow X_d$  such that  $F_{d_1} = F_{d_1 d_2} F_{d_2}$  for  $d_2 \geq d_1$ . This means that there exists a mapping  $F: X \rightarrow X_D$  [1, p. 138]. Let us prove that  $F$  is 1–1. Take a pair  $x, y$  of distinct points of  $X$ . There exists an  $a \in A$  such that  $x_a = p_a(x)$  and  $y_a = p_a(y)$  are distinct points of  $X_a$ . Now, there exists an  $(a, \gamma_a)$  such that  $f_{(a,\gamma_a)}(x_a)$  and  $f_{(a,\gamma_a)}(y_a)$  are distinct points of  $X_{(a,\gamma_a)}$ . From Step 4. it follows that there is a  $d \in D$  such that  $F_d(x)$  and  $F_d(y)$  are distinct points of  $X_d$ . Thus,  $F$  is 1–1.

**Step 11.** *The mapping  $F$  is a homeomorphism onto  $X_D$ .* Let  $y$  be a point of  $X_D$ . Let us prove that there exists a point  $x \in X$  such that  $F(x) = y$ . For each  $d \in D$  we have a point  $y_d = F_d(y)$ . Now, we have the points  $g_{(a,\gamma_a)} F_d(y)$  in  $X_{(a,\gamma_a)}$  and the subsets  $Y_a = f_{(a,\gamma_a)}^{-1}(g_{(a,\gamma_a)} F_d(y))$  of  $X_a$ . Let  $U$  be an open neighborhood  $Y_a$ . There exists an open neighborhood  $V$  of  $g_{(a,\gamma_a)} F_d(y)$  such that  $f_{(a,\gamma_a)}^{-1}(V) \subseteq U$ . We infer that  $\text{Ls}\{g_{(b,\gamma_b)}(Y_b) : b \geq a\} \subseteq Y_a$  since  $g_{(a,\gamma_a)} F_d(y) = \lim\{g_{(a,\gamma_a)(b,\gamma_b)} g_{(b,\gamma_b)} F_d(y) : b \geq a\}$  and the diagrams (6) commute. By virtue of [6, Lemma 2.1] it follows that there exists a non-empty closed subset  $C_d$  of  $\lim \mathbf{X}$  such that  $p_b(C_d) \subseteq Y_b$ . The family  $\{C_d : d \in D\}$  has the finite intersection property. This means that  $X' = \bigcap \{C_d : d \in D\}$  is non-empty. For each  $x \in X'$  we have  $F_d(x) = F_d(y)$ ,  $d \in D$ . Thus,  $F(y) = x$ . The proof is completed. ■

By the similar method of proof we obtain the following theorem.

**THEOREM 10.** *Let  $\mathbf{X} = \{X_a, p_{ab}, A\}$  be an approximate inverse system of compact non-metric spaces with surjective bonding mappings  $p_{ab}$ . If each  $X_a$  is a limit of a usual  $\tau$ -directed inverse system  $\mathbf{X}(a) = \{X_{(a,\gamma)}, f_{(a,\gamma)(a,\delta)}, \Gamma_a\}$  of compact spaces with  $w(X_{(a,\gamma)}) \leq \tau$ , then:*

1. *there exists a usual  $\tau$ -directed inverse system  $\mathbf{X}_D = \{X_d, F_{de}, D\}$  whose inverse limit  $X_D$  is homeomorphic to  $X = \lim \mathbf{X}$ ,*
2. *every  $X_d$  is a limit of an approximate inverse system  $\{X_{(a,\gamma_a)}, g_{(a,\gamma_a)(b,\gamma_b)}, A\}$  of compact spaces  $X_{(a,\gamma_a)}$ ,*
3. *if the mappings  $p_{ab}$  and  $f_{(a,\gamma)(a,\delta)}$  are monotone, then  $g_{(a,\gamma_a)(b,\gamma_b)}$  and  $F_{de}$  are monotone.*

**COROLLARY 3.** *Let  $\mathbf{X} = \{X_n, p_{nm}, \mathbb{N}\}$  be an approximate inverse sequence of compact non-metric spaces with surjective bonding mappings  $p_{nm}$ . If each  $X_n$  is a limit of a usual  $\sigma$ -directed inverse system  $\mathbf{X}(n) = \{X_{(n,\gamma)}, f_{(n,\gamma)(n,\delta)}, \Gamma_n\}$  of metric compact spaces, then:*

1. there exists a usual  $\sigma$ -directed inverse system  $\mathbf{X}_D = \{X_d, F_{de}, D\}$  whose inverse limit  $X_D$  is homeomorphic to  $X = \lim \mathbf{X}$ ,
2. every  $X_d$  is a limit of an approximate inverse sequence  $\{X_{(n,\gamma_n)}, g_{(n,\gamma_n)(m,\gamma_m)}, \mathbb{N}\}$  of compact metric spaces  $X_{(n,\gamma_n)}$ ,
3. if the mappings  $p_{nm}$  and  $f_{(n,\gamma)(m,\delta)}$  are monotone, then  $g_{(n,\gamma_n)(m,\gamma_m)}$  and  $F_{de}$  are monotone.

Let  $\mathcal{P}$  be a topological property of spaces.

**THEOREM 11.** *Let  $\mathbf{X} = \{X_a, p_{ab}, A\}$  be an approximate inverse system of compact non-metric spaces with surjective bonding mappings  $p_{ab}$  and let  $\mathcal{P}$  be a topological property of spaces such that:*

1. each  $X_a$  is a limit of a usual  $\sigma$ -directed inverse system  $\mathbf{X}(\mathbf{a}) = \{X_{(a,\gamma)}, f_{(a,\gamma)(a,\delta)}, \Gamma_a\}$  of metric compact spaces with property  $\mathcal{P}$ ,
2. if  $X_d$  is a limit of an approximate inverse system  $\{X_{(a,\gamma_a)}, g_{(a,\gamma_a)(b,\gamma_b)}, A\}$  of compact metric spaces  $X_{(a,\gamma_a)}$  with property  $\mathcal{P}$ , then  $X_d$  has  $\mathcal{P}$ ,
3. if  $Y$  is a limit of  $\sigma$ -directed usual inverse system of compact spaces with property  $\mathcal{P}$ , then  $Y$  has  $\mathcal{P}$ .

Then  $X = \lim \mathbf{X}$  has the property  $\mathcal{P}$ .

### 3. Applications

**LEMMA 9.** *Let  $\mathbf{X} = \{X_n, p_{nm}, \mathbb{N}\}$  be an approximate inverse sequence of locally connected metric continua. If the bonding mappings are monotone and surjective, then  $X = \lim \mathbf{X}$  is locally connected.*

*Proof.* There exists a usual inverse sequence  $\mathbf{Y} = \{Y_i, q_{ij}, M\}$  such that  $Y_i = X_{n_i}$ ,  $q_{ij} = p_{n_i n_{i+1}} p_{n_{i+1} n_{i+2}} \cdots p_{n_{j-1} n_j}$  for each  $i, j \in \mathbb{N}$  and a homeomorphism  $H: \lim \mathbf{X} \rightarrow \lim \mathbf{Y}$  [2, Proposition 8]. Each mapping  $q_{ij}$  as a composition of the monotone mappings is monotone. This means that  $\mathbf{Y}$  is a usual inverse sequence of locally connected continua with monotone bonding mappings  $q_{ij}$ . Hence  $\lim \mathbf{Y}$  is locally connected. We infer that  $X = \lim \mathbf{X}$  is locally connected since there exists a homeomorphism  $H: \lim \mathbf{X} \rightarrow \lim \mathbf{Y}$ . ■

**LEMMA 10.** *Let  $\mathbf{X} = \{X_a, p_{ab}, A\}$  be an approximate inverse system of locally connected continua such that  $\text{card}(A) = \aleph_0$ . Then  $X = \lim \mathbf{X}$  is locally connected.*

*Proof.* By virtue of Lemma 3 there exists a countable well-ordered subset  $B$  of  $A$  such that the collection  $\{X_b, p_{bc}, B\}$  is an approximate inverse sequence and  $\lim \mathbf{X}$  is homeomorphic to  $\lim \{X_b, p_{bc}, B\}$ . From Lemma 9 it follows that  $\lim \{X_b, p_{bc}, B\}$  is locally connected. Hence  $X = \lim \mathbf{X}$  is locally connected. ■

**LEMMA 11.** *Let  $\mathbf{X} = \{X_a, p_{ab}, A\}$  be an approximate inverse system of locally connected metric continua and monotone bonding mappings. Then  $X = \lim \mathbf{X}$  is locally connected.*

*Proof.* If  $\text{card}(A) = \aleph_0$  then we apply Lemma 10. If  $\text{card}(A) \geq \aleph_1$  then from Corollary 1 it follows that  $X = \lim \mathbf{X}$  is homeomorphic to the limit of a  $\sigma$ -directed

usual inverse system  $\{X_\alpha, q_{\alpha\beta}, \Delta\}$ , where each  $X_\alpha$  is a limit of an approximate inverse subsystem  $\{X_\gamma, p_{\alpha\beta}, \Phi\}$ ,  $\text{card}(\Phi) = \aleph_0$ . From Lemma 10 it follows that each  $X_a$  is locally connected. By Theorem 3 we infer that the limit of  $\{X_\alpha, q_{\alpha\beta}, \Delta\}$  is locally connected. Hence,  $X$  is locally connected since  $X$  is homeomorphic to  $\lim\{X_\alpha, q_{\alpha\beta}, \Delta\}$ . ■

**THEOREM 12.** *Let  $\mathbf{X} = \{X_a, p_{ab}, A\}$  be an approximate inverse system of locally connected continua and monotone bonding mappings. Then  $X = \lim \mathbf{X}$  is a locally connected continuum.*

*Proof.* By virtue of Theorem 3 and Remark 1 every  $X_a$  is a limit of a usual  $\sigma$ -directed inverse system  $X(a) = \{X_{(a,\gamma)}, f_{(a,\gamma)(a,\delta)}, \Gamma_a\}$  of metric locally connected continua with monotone bonding mappings  $f_{(a,\gamma)(a,\delta)}$ . From Theorem 9 it follows that there exist : 1) a usual  $\sigma$ -directed inverse system  $\mathbf{X}_D = \{X_d, F_{de}, D\}$  whose inverse limit  $X_D$  is homeomorphic to  $X = \lim \mathbf{X}$ ; 2) every  $X_d$  is a limit of an approximate inverse system  $\{X_{(a,\gamma_a)}, g_{(a,\gamma_a)(b,\gamma_b)}, A\}$  of compact metric spaces  $X_{(a,\gamma_a)}$  and 3) if the mappings  $p_{ab}$  and  $f_{(a,\gamma)(a,\delta)}$  are monotone, then  $g_{(a,\gamma_a)(b,\gamma_b)}$  and  $F_{de}$  are monotone. Now, every  $X_d$  as the limit of approximate inverse system  $\{X_{(a,\gamma_a)}, g_{(a,\gamma_a)(b,\gamma_b)}, A\}$  is locally connected because of Lemma 11. Finally,  $X$  is locally connected since  $X$  is homeomorphic to  $X_D = \lim \mathbf{X}_D$  and  $X_D$  is locally connected (Theorem 3). ■

We shall say that a non-empty compact space is *perfect* if it has no isolated points.

A continuum is said to be *totally regular* [12, p. 47] if for each  $x \neq y$  in  $X$  there is a positive integer  $n$  and perfect subsets  $A_1, \dots, A_n$  of  $X$  such that  $x_i \in A_i$  for  $i = 1, \dots, n$  implies that  $\{x_1, \dots, x_n\}$  separates  $x$  from  $y$  in  $X$ .

**LEMMA 12.** [12, Proposition 7.4] *Each totally regular continuum is hereditarily locally connected and rim-finite.*

The following theorem is a part of [12, Theorem 7.15].

**THEOREM 13.** *If  $X$  is a continuum then the following conditions are equivalent:*

1.  $X$  is totally regular,
2.  $X$  is homeomorphic to  $\lim\{X_a, f_{ab}, \Gamma\}$  such that each  $X_a$  is a totally regular continuum and each  $f_{ab}$  is a monotone surjection.

**THEOREM 14.** [12, Theorem 7.7] *Let  $\mathbf{X} = \{X_a, p_{ab}, A\}$  be an inverse system of totally regular continua  $X_a$  and monotone surjective mappings  $p_{ab}$ . Then  $X = \lim \mathbf{X}$  is totally regular.*

**THEOREM 15.** *Let  $X$  be a non-metric totally regular continuum. There exists a  $\sigma$ -directed inverse system  $\mathbf{X} = \{X_a, p_{ab}, A\}$  such that each  $X_a$  is totally regular, each  $f_{ab}$  is a monotone surjection and  $X$  is homeomorphic to  $\lim \mathbf{X}$ .*

*Proof.* Apply [12, Theorem 9.4], Theorem 14 and Lemma 3.5 of [14]. ■

Now we shall prove the following theorem.

**THEOREM 16.** *Let  $\mathbf{X} = \{X_n, p_{nm}, \mathbb{N}\}$  be an approximate inverse sequence of totally regular metric continua. If the bonding mappings are monotone and surjective, then  $X = \lim \mathbf{X}$  is totally regular.*

*Proof.* There exists a usual inverse sequence  $\mathbf{Y} = \{Y_i, q_{ij}, M\}$  such that  $Y_i = X_{n_i}$ ,  $q_{ij} = p_{n_i n_{i+1}} p_{n_{i+1} n_{i+2}} \cdots p_{n_{j-1} n_j}$  for each  $i, j \in \mathbb{N}$  and a homeomorphism  $H: \lim \mathbf{X} \rightarrow \lim \mathbf{Y}$  [2, Proposition 8]. Each mapping  $q_{ij}$  as a composition of the monotone mappings is monotone. This means that  $\mathbf{Y}$  is a usual inverse sequence of totally regular continua with monotone bonding mappings  $q_{ij}$ . By virtue of Theorem 14  $\lim \mathbf{Y}$  is totally regular. We infer that  $X = \lim \mathbf{X}$  is totally regular since there exists a homeomorphism  $H: \lim \mathbf{X} \rightarrow \lim \mathbf{Y}$ . ■

**THEOREM 17.** *Let  $\mathbf{X} = \{X_a, p_{ab}, A\}$  be an approximate inverse system of totally regular continua such that  $\text{card}(A) = \aleph_0$ . Then  $X = \lim \mathbf{X}$  is totally regular.*

*Proof.* By virtue of Lemma 3 there exists a countable well-ordered subset  $B$  of  $A$  such that the collection  $\{X_b, p_{bc}, B\}$  is an approximate inverse sequence and  $\lim \mathbf{X}$  is homeomorphic to  $\lim \{X_b, p_{bc}, B\}$ . From Theorem 16 it follows that  $\lim \{X_b, p_{bc}, B\}$  is totally regular. Hence  $X = \lim \mathbf{X}$  is totally regular. ■

**THEOREM 18.** *Let  $\mathbf{X} = \{X_a, p_{ab}, A\}$  be an approximate inverse system of totally regular continua and monotone bonding mappings. If  $w(X_a) < \tau < \text{card}(A)$  for each  $a \in A$ , then  $X = \lim \mathbf{X}$  is a totally regular continuum.*

*Proof.* By virtue of Theorem 7 (for  $\lambda = \aleph_0$ ) there exists a  $\sigma$ -directed inverse system  $\{X_\alpha, q_{\alpha\beta}, T\}$ , where each  $X_\alpha$  is a limit of an approximate inverse subsystem  $\{X_\gamma, p_{\alpha\beta}, \Phi\}$ ,  $\text{card}(\Phi) = \aleph_0$ . From Theorem 17 it follows that every  $X_\alpha$  is totally regular. Theorem 14 completes the proof. ■

**THEOREM 19.** *Let  $\mathbf{X} = \{X_a, p_{ab}, A\}$  be an approximate inverse system of totally regular metric continua and monotone bonding mappings. Then  $X = \lim \mathbf{X}$  is totally regular continuum.*

*Proof.* If  $\text{card}(A) = \aleph_0$  then we apply Theorem 17. If  $\text{card}(A) \geq \aleph_1$  then from Theorem 18 it follows that  $X$  is totally regular. ■

**THEOREM 20.** *Let  $\mathbf{X} = \{X_a, p_{ab}, A\}$  be an approximate inverse system of totally regular non-metric continua with surjective monotone bonding mappings  $p_{ab}$ . Then  $X = \lim \mathbf{X}$  is totally regular.*

*Proof.* By virtue of Theorem 15 every  $X_a$  is a limit of a usual  $\sigma$ -directed inverse system  $X(a) = \{X_{(a,\gamma)}, f_{(a,\gamma)(a,\delta)}, \Gamma_a\}$  of metric totally regular continua with monotone bonding mappings  $f_{(a,\gamma)(a,\delta)}$ . From Theorem 9 it follows that there exist: 1) a usual  $\sigma$ -directed inverse system  $\mathbf{X}_D = \{X_d, F_{de}, D\}$  whose inverse limit  $X_D$  is homeomorphic to  $X = \lim \mathbf{X}$ , 2) every  $X_d$  is a limit of an approximate inverse system  $\{X_{(a,\gamma_a)}, g_{(a,\gamma_a)(b,\gamma_b)}, A\}$  of compact metric spaces  $X_{(a,\gamma_a)}$  and 3) if the mappings  $p_{ab}$  and  $f_{(a,\gamma)(a,\delta)}$  are monotone, then  $g_{(a,\gamma_a)(b,\gamma_b)}$  and  $F_{de}$  are monotone. Now, every  $X_d$  as the limit of approximate inverse system  $\{X_{(a,\gamma_a)}, g_{(a,\gamma_a)(b,\gamma_b)}, A\}$

is totally regular because of Theorem 19. Finally,  $X$  is totally regular since  $X$  is homeomorphic to  $X_D = \lim \mathbf{X}_D$  and  $X_D$  is totally regular (Theorem 14). ■

**THEOREM 21.** *Let  $\mathbf{X} = \{X_a, p_{ab}, A\}$  be an approximate inverse system of totally regular continua with surjective monotone bonding mappings  $p_{ab}$ . Then  $X = \lim \mathbf{X}$  is totally regular.*

*Proof.* Apply Theorems 19 and 20. ■

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