МАТЕМАТИЧКИ ВЕСНИК
51 (1999), 29–37

UDK 519.213
оригинални научни рад
research paper

EXTREME VALUES OF THE SEQUENCES OF INDEPENDENT RANDOM VARIABLES WITH MIXED DISTRIBUTIONS

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Abstract. In this paper we consider some examples of the sequences of independent random variables with the same mixed distribution. In these cases we determine the type of extreme value distribution and the normalizing constants.

1. Introduction

Let (X_n) be a sequence of independent random variables with the common distribution function F. If for some constants $a_n > 0$ and b_n

$$P\left\{\max_{1\leqslant j\leqslant n} X_j \leqslant \frac{x}{a_n} + b_n\right\} \to_d G(x), \tag{1.1}$$

where G is non-degenerated distribution function, then the function G belongs to one of three classes of *the maximum stable* distributions, and the functions in these classes have the following forms (maybe after linear transformation of the argument):

Type I. $G_1(x) = \exp(-e^{-x})$, $-\infty < x < +\infty$;

Type II. $G_2(x) = \begin{cases} 0, & \text{if } x \leqslant 0, \\ \exp(-x^{-\alpha}), & \text{if } x > 0, \end{cases}$ for some $\alpha > 0$;

Type III. $G_3(x) = \begin{cases} \exp(-(-x)^{\alpha}), & \text{if } x \leqslant 0, \\ 1, & \text{if } x > 0. \end{cases}$ for some $\alpha > 0$.

These three types of distributions are called *the extreme values distributions*. If for some distribution functions F and G the relation (1.1) holds true, then we say that the common distribution function F of the random variables X_1, X_2, X_3, ... belongs to *the domain of attraction* of the function G. We shall use the notation $M_n = \max\{X_1, \dots, X_n\}$. The constants $a_n > 0$ and b_n from the relation (1.1) are called *the normalizing constants*. Note that for $a_n > 0$, the inequality $M_n \leqslant x/a_n + b_n$ is equivalent to $a_n(M_n - b_n) \leqslant x$.

AMS Subject Classification: 60G70.

Keywords and phrases: mixed distributions; extreme value distributions; normal, Cauchy, uniform and truncated exponential distributions.

2. Necessary and sufficient conditions for convergence to the extreme value distributions

If for some distribution function F there exists a domain of attraction, then it is determined by the asymptotic behaviour of the tail $1-F(x)$, as $x \to +\infty$. The following useful theorem 1.5.1 from [1] holds true:

Theorem 1. [1] *Let (X_n) be a sequence of independent random variables with the common distribution function $F(x)$, $-\infty < x < +\infty$, (u_n) a sequence of real numbers, $0 \leqslant \tau \leqslant +\infty$ and $M_n = \max\{X_1, X_2, \ldots, X_n\}$. Then, the equality*

$$\lim_{n\to\infty} P\{M_n \leqslant u_n\} = e^{-\tau},$$

holds true if and only if $\lim_{n\to\infty} n(1-F(u_n)) = \tau$.

Necessary and sufficient conditions for the function F to belong to some domain of attraction can also be formulated as in the following theorem 1.6.2 from [1]:

Theorem 2. [1] *Let (X_n) be a sequence of independent random variables with the common distribution function F, and $x_F = \sup\{x \mid F(x) < 1\}$. Necessary and sufficient conditions for the function F to belong to the domain of attraction of possible types are given by:*

Type I. *There exists a strictly positive function $g(t)$ defined on the set $(-\infty, x_F)$, such that for every real number x the equality $\lim_{t\uparrow x_F} \dfrac{1-F(t+xg(t))}{1-F(t)} = e^{-x}$ holds true.*

Type II. $x_F = +\infty$ *and* $\lim_{t\to\infty} \dfrac{1-F(tx)}{1-F(t)} = x^{-\alpha}$, *for some* $\alpha > 0$ *and all* $x > 0$.

Type III. $x_F < +\infty$ *and* $\lim_{h\downarrow 0} \dfrac{1-F(x_F-hx)}{1-F(x_F-h)} = x^{\alpha}$, *for some* $\alpha > 0$ *and all* $x > 0$.

3. Mixed distributions

Let X_1 and X_2 be random variables with distribution functions $F_1(x)$ and $F_2(x)$, respectively, and

$$X = \begin{cases} X_1 & \text{with probability } p, \\ X_2 & \text{with probability } q, \end{cases}$$

where $p+q=1$. The distribution function of the random variable X is given by

$$\begin{aligned} F(x) = P\{X \leqslant x\} &= pP\{X_1 \leqslant x\} + qP\{X_2 \leqslant x\} \\ &= pF_1(x) + qF_2(x). \end{aligned}$$

The distribution of probability determined by the distribution function F is called the mixture of the distributions determined by the functions F_1 and F_2.

We shall consider some examples of sequences of independent random variables with common mixed distribution. In these cases we are going to determine the type of extreme value distribution and the normalizing constants.

3.1. *Mixture of normal distributions.* Let (X_n) be a sequence of independent random variables with normal $\mathcal{N}(0,1)$ distribution and $M_n = \max\{X_1, \dots, X_n\}$. As is well known, the limiting distribution of the maximum M_n is given by

$$P\{a_n(M_n - b_n) \leqslant x\} \to e^{-e^{-x}}, \quad n \to \infty, \tag{3.1}$$

where the normalizing constants a_n and b_n are

$$a_n = \sqrt{2\ln n}, \quad b_n = \sqrt{2\ln n} - \frac{1}{2}\frac{\ln\ln n + \ln 4\pi}{\sqrt{2\ln n}}. \tag{3.2}$$

Corollary. Let (Y_n) be a sequence of independent random variables with normal $\mathcal{N}(m, \sigma^2)$ distribution. Then, we have $X_n = (Y_n - m)/\sigma \in \mathcal{N}(0,1)$. If $M_n = \max\{X_1, \dots, X_n\}$ and $\widetilde{M}_n = \max\{Y_1, \dots, Y_n\}$, then

$$\begin{aligned} P\{a_n(M_n - b_n) \leqslant x\} &= P\left\{a_n\left(\frac{\widetilde{M}_n - m}{\sigma} - b_n\right) \leqslant x\right\} \\ &= P\{\widetilde{a}_n(\widetilde{M}_n - \widetilde{b}_n) \leqslant x\} \to \exp(-e^{-x}), \quad n \to \infty, \end{aligned}$$

where the constants $\widetilde{a}_n$ and $\widetilde{b}_n$ are given by:

$$\begin{aligned} \widetilde{a}_n &= \frac{a_n}{\sigma} = \frac{\sqrt{2\ln n}}{\sigma}, \\ \widetilde{b}_n &= m + \sigma b_n = m + \sigma\sqrt{2\ln n} - \sigma\frac{\ln\ln n + \ln 4\pi}{2\sqrt{2\ln n}}. \end{aligned}$$

Theorem 3. *Let (Z_n) be a sequence of independent random variables such that*

$$Z_n \in \begin{cases} \mathcal{N}(m_1, \sigma_1^2), & \text{with probability } p, \\ \mathcal{N}(m_2, \sigma_2^2), & \text{with probability } q, \end{cases} \quad \text{for all } n,$$

where $p + q = 1$. Let us denote $M_n^* = \max\{Z_1, \dots, Z_n\}$. If

(a) $\sigma_1 > \sigma_2$, $m_1, m_2 \in R$ *or* (b) $\sigma_1 = \sigma_2$ *and* $m_1 > m_2$,

then for every real number x the equality

$$\lim_{n\to\infty} P\{a_n^*(M_n^* - b_n^*) \leqslant x\} = e^{-e^{-x}} \tag{3.3}$$

holds true, where the constants a_n^* and b_n^* are given by

$$a_n^* = \frac{\sqrt{2\ln n}}{\sigma_1}, \quad b_n^* = m_1 + \sigma_1\sqrt{2\ln n} - \frac{\sigma_1}{2\sqrt{2\ln n}}\left(\ln\ln n + \ln\frac{4\pi}{p^2}\right). \tag{3.4}$$

Remark. Since M_n^* in distribution is the same as $\widetilde{M}_{np}$, then all normalizing constants with $a_n^*/\widetilde{a}_{np} \to 1$ and $a_n^*(\widetilde{b}_{np} - b_n^*) \to 0$ will work. The maximum M_n^* asymptotically comes from F_1.

Proof of Theorem 3. Let $\varphi(x) = (2\pi)^{-1/2}e^{-x^2/2}$ and $\Phi(x) = \int_{-\infty}^{x} \varphi(t)\,dt$. We shall use the following asymptotic relation

$$1 - \Phi(x) \sim x^{-1}\varphi(x), \quad x \to \infty. \tag{3.5}$$

If $X_i \in \mathcal{N}(m_i, \sigma_i^2)$, $i = 1, 2$, then distribution function of the random variable X_i can be represented in the form

$$F_i(x) = P\{X_i \leqslant x\} = P\left\{\frac{X_1 - m_1}{\sigma_1} \leqslant \frac{x - m_1}{\sigma_1}\right\} = \Phi\left(\frac{x - m_1}{\sigma_1}\right), \quad i = 1, 2.$$

Distribution function of the random variable Z_n has the form $F(x) = pF_1(x) + qF_2(x)$. Using this representation of $F(x)$ we obtain

$$\begin{aligned}
1 - F(t) &= p\left[1 - \Phi\left(\frac{t - m_1}{\sigma_1}\right)\right] + q\left[1 - \Phi\left(\frac{t - m_2}{\sigma_2}\right)\right] \\
&\sim \frac{p\sigma_1}{t - m_1}\varphi\left(\frac{t - m_1}{\sigma_1}\right) + \frac{q\sigma_2}{t - m_2}\varphi\left(\frac{t - m_2}{\sigma_2}\right) \\
&= \frac{1}{\sqrt{2\pi}}\left\{\frac{p\sigma_1}{t - m_1}\exp\left[-\frac{1}{2}\left(\frac{t - m_1}{\sigma_1}\right)^2\right] + \frac{q\sigma_2}{t - m_2}\exp\left[-\frac{1}{2}\left(\frac{t - m_2}{\sigma_2}\right)^2\right]\right\} \\
&= \frac{1}{\sqrt{2\pi}}\frac{p\sigma_1}{t - m_1}\exp\left[-\frac{1}{2}\left(\frac{t - m_1}{\sigma_1}\right)^2\right](1 + o(1)), \quad t \to \infty;
\end{aligned}$$

$$\begin{aligned}
\frac{1 - F(t + xg(t))}{1 - F(t)} &= \exp\left\{-\frac{(t - m_1 + xg(t))^2 - (t - m_1)^2}{2\sigma_1^2}\right\}\frac{(t - m_1)(1 + o(1))}{t - m_1 + xg(t)} \\
&= \exp\left(-\frac{xg(t)(t - m_1)}{\sigma_1^2}\right)\exp\left(-\frac{x^2g^2(t)}{2\sigma_1^2}\right)\frac{(t - m_1)(1 + o(1))}{t - m_1 + xg(t)}.
\end{aligned}$$

For $g(t) = \sigma_1/(t - m_1)$, we get

$$\begin{aligned}
\frac{1 - F(t + xg(t))}{1 - F(t)} &= e^{-x}\exp\left(-\frac{x^2\sigma_1^2}{2(t - m_1)^2}\right)\frac{1}{1 + x\sigma_1^2(t - m_1)^{-2}}(1 + o(1)) \\
&\to e^{-x}, \quad \text{as } t \to +\infty.
\end{aligned}$$

Using theorem 1 we conclude that the distribution function $F(x)$ belongs to the domain of attraction of type I, i.e. there exist constants a_n^* and b_n^*, such that the following equality holds true:

$$\lim_{n\to\infty} P\left\{M_n^* \leqslant \frac{x}{a_n^*} + b_n^*\right\} = e^{-e^{-x}}.$$

The constants a_n^* and b_n^* can be determined as follows: let us first determine the constant u_n, such that $1 - F(u_n) \sim \frac{1}{n}e^{-x}$ as $n \to \infty$ i.e.

$$1 - pF_1(u_n) - qF_2(u_n) \sim \frac{1}{n}e^{-x}, \quad n \to \infty.$$

Using the equalities $F_1(u_n) = \Phi((u_n - m_1)/\sigma_1)$ and $F_2(u_n) = \Phi((u_n - m_2)/\sigma_2)$ and asymptotic relation (3.5), we obtain

$$\begin{aligned} 1 - pF_1(u_n) - qF_2(u_n) &= p\left[1 - \Phi(\frac{u_n - m_1}{\sigma_1})\right] + q\left[1 - \Phi(\frac{u_n - m_2}{\sigma_2})\right] \\ &\sim \frac{p\sigma_1}{u_n - m_1}\varphi\left(\frac{u_n - m_1}{\sigma_1}\right) + \frac{q\sigma_2}{u_n - m_2}\varphi\left(\frac{u_n - m_2}{\sigma_2}\right) \\ &\sim \frac{1}{n}e^{-x}, \quad \text{as } n \to \infty. \end{aligned}$$

Let us denote: $v_n = (u_n - m_1)/\sigma_1$ and $w_n = (u_n - m_2)/\sigma_2$. For large values of n the inequality $v_n < w_n$ holds true, and

$$\begin{aligned} p\frac{\varphi(v_n)}{v_n} + q\frac{\varphi(w_n)}{w_n} &= \frac{1}{\sqrt{2\pi}}\left(\frac{p}{v_n}e^{-v_n^2/2} + \frac{q}{w_n}e^{-w_n^2/2}\right) \\ &= \frac{1}{\sqrt{2\pi}}\frac{p}{v_n}e^{-v_n^2/2}\left(1 + \frac{v_n}{p}\frac{q}{w_n}e^{-(w_n^2 - v_n^2)/2}\right). \end{aligned}$$

Let $\Delta_n = w_n^2 - v_n^2 = (u_n - m_1)^2/\sigma_1^2 - (u_n - m_2)^2/\sigma_2^2$. If $\sigma_1 > \sigma_2 > 0$, then

$$\Delta_n = \left(\frac{1}{\sigma_2^2} - \frac{1}{\sigma_1^2}\right)u_n^2 + Au_n + B \to +\infty, \quad \text{as } n \to \infty.$$

If $\sigma_1 = \sigma_2 = \sigma$ and $m_1 > m_2$, then $\Delta_n = [2u_n(m_1 - m_2) + m_2^2 - m_1^2]/\sigma^2 \to +\infty$, as $n \to \infty$. In both cases we have the following asymptotic equality

$$p\frac{\varphi(v_n)}{v_n} + q\frac{\varphi(w_n)}{w_n} = \frac{1}{\sqrt{2\pi}}\frac{p}{v_n}e^{-v_n^2/2}(1 + o(1)), \quad \text{as } n \to \infty.$$

Hence, the constant u_n should be determined from the conditions $u_n = \sigma_1 v_n + m_1$ and

$$p\frac{\varphi(v_n)}{v_n} \sim \frac{1}{n}e^{-x}, \quad \text{as } n \to \infty. \tag{3.6}$$

Asymptotic relation (3.6) can be transformed in the following way:

$$\begin{gathered} \frac{1}{pn}e^{-x} \cdot \frac{v_n}{\varphi(v_n)} \to 1, \\ -\ln n - \ln p - x + \ln v_n - \ln\varphi(v_n) \to 0, \\ -\ln n - \ln p - x + \ln v_n + \frac{1}{2}\ln 2\pi + \frac{v_n^2}{2} \to 0. \end{gathered} \tag{3.7}$$

It follows from (3.7) that $v_n^2/(2\ln n) \to 1$ as $n \to \infty$, and

$$\ln v_n = \frac{1}{2}(\ln 2 + \ln\ln n) + o(1). \tag{3.8}$$

The relation (3.7) can also be writen in the form

$$\frac{v_n^2}{2} = x + \ln n + \ln p - \frac{1}{2}\ln 2\pi - \ln v_n + o(1). \tag{3.9}$$

Now let us substitute the value of $\ln v_n$ from (3.8) into (3.9). We obtain

$$\begin{aligned}\frac{v_n^2}{2} &= x + \ln n + \ln p - \frac{1}{2}\ln 2\pi - \frac{1}{2}\ln 2 - \frac{1}{2}\ln\ln n + o(1)\\ &= x + \ln n + \ln p - \frac{1}{2}\ln 4\pi - \frac{1}{2}\ln\ln n + o(1),\\ v_n^2 &= 2\ln n\biggl\{1 + \frac{x + \ln p - \frac{1}{2}\ln 4\pi - \frac{1}{2}\ln\ln n}{\ln n} + o\Bigl(\frac{1}{\ln n}\Bigr)\biggr\}.\end{aligned}$$

Note that $\ln p - \frac{1}{2}\ln 4\pi = -\frac{1}{2}\ln\frac{4\pi}{p^2}$. Using the formula $\sqrt{1+x} = 1 + \frac{1}{2}x + o(x)$ as $x \to 0$, we get

$$\begin{aligned}v_n &= \sqrt{2\ln n}\biggl\{1 + \frac{x + \ln p - \frac{1}{2}\ln 4\pi - \frac{1}{2}\ln\ln n}{2\ln n} + o\Bigl(\frac{1}{\ln n}\Bigr)\biggr\}\\ &= \sqrt{2\ln n}\biggl\{1 + \frac{1}{2\ln n}\Bigl(x - \frac{1}{2}\ln\frac{4\pi}{p^2} - \frac{1}{2}\ln\ln n\Bigr) + o\Bigl(\frac{1}{\ln n}\Bigr)\biggr\}.\end{aligned}$$

Since $u_n = m_1 + \sigma_1 v_n$, it follows that

$$u_n = \frac{\sigma_1 x}{\sqrt{2\ln n}} + m_1 + \sigma_1\sqrt{2\ln n} - \frac{\sigma_1}{2\sqrt{2\ln n}}\Bigl(\ln\ln n + \ln\frac{4\pi}{p^2}\Bigr) + o\Bigl(\frac{1}{\sqrt{\ln n}}\Bigr).$$

On the other hand $u_n \sim \dfrac{x}{a_n^*} + b_n^*$, as $n \to \infty$. Consequently it is easy to obtain the normalizing constants a_n^* and b_n^* in the form (3.4). ∎

3.2. *Mixture of Cauchy distributions.* Let (X_n) be a sequence of independent random variables with the Cauchy distribution $K(1,0)$, determined by the distribution function

$$F(x) = \frac{1}{2} + \frac{1}{\pi}\operatorname{arctg} x.$$

Let us denote $M_n = \max\{X_1, \ldots, X_n\}$. For $x > 0$ we have

$$\frac{1 - F(tx)}{1 - F(t)} = \frac{\frac{\pi}{2} - \operatorname{arctg}(tx)}{\frac{\pi}{2} - \operatorname{arctg} t} \to \frac{1}{x}, \quad t \to \infty. \tag{3.10}$$

Indeed, on substituting $\operatorname{arctg} t = \pi/2 - \vartheta$, we obtain $t = \operatorname{tg}(\pi/2 - \vartheta) = \operatorname{ctg}\vartheta$,

$$\lim_{t\to\infty}\Bigl(\frac{\pi}{2} - \operatorname{arctg} t\Bigr)\cdot t = \lim_{\vartheta\to 0}\vartheta\operatorname{ctg}\vartheta = 1.$$

Similarly, $\lim\limits_{t\to\infty}\Bigl(\dfrac{\pi}{2} - \operatorname{arctg}(tx)\Bigr)tx = 1$, and (3.10) follows easily. In this case, the distribution function F belongs to the domain of attraction of the function $G_2(x)$, and we have the type II of limiting distribution. The normalizing constants are $a_n = 1/\gamma_n$ and $b_n = 0$, where the constant γ_n can be determined from the equality

$$1 - F(\gamma_n) = \frac{1}{2} - \frac{1}{\pi}\operatorname{arctg}\gamma_n = \frac{1}{n}.$$

Hence, $\gamma_n = \operatorname{tg}\left(\frac{\pi}{2} - \frac{\pi}{n}\right) = \operatorname{ctg}\frac{\pi}{n}$. For every $x > 0$ we have:

$$\lim_{n\to\infty} P\left\{ M_n \cdot \operatorname{tg}\frac{\pi}{n} \leqslant x \right\} = e^{-x^{-1}}.$$

Corollary. Let (Y_n) be a sequence of independent random variables with the Cauchy distribution $K(\lambda, 0)$, which is determined by the distribution function $F(x) = \dfrac{1}{2} + \dfrac{1}{\pi}\operatorname{arctg}\dfrac{x}{\lambda}$. Let us denote $\widetilde{M}_n = \max\{Y_1, \dots, Y_n\}$ and $Y_n/\lambda = X_n$. Then, $\widetilde{M_n} = \lambda M_n$. It is easy to see that the random variables Y_n/λ has $K(1,0)$ distribution. Since the inequality $M_n \operatorname{tg}\frac{\pi}{n} \leqslant x$ is equivalent to $\frac{1}{\lambda}\widetilde{M}_n \operatorname{tg}\frac{\pi}{n} \leqslant x$, it follows that the normalizing constants in this case are given by $\widetilde{a}_n = \frac{1}{\lambda}\operatorname{tg}\frac{\pi}{n}$ and $\widetilde{b}_n = 0$. For these values of normalizing constants and every $x > 0$ we have

$$\lim_{n\to\infty} P\left\{ \frac{1}{\lambda}\widetilde{M}_n \cdot \operatorname{tg}\frac{\pi}{n} \leqslant x \right\} = e^{-x^{-1}}.$$

Theorem 4. *Let $K(\lambda_i, 0)$ be the class of random variables with the distribution function $F_i(x) = \frac{1}{2} + \frac{1}{\pi}\operatorname{arctg}\frac{x}{\lambda_i}$, $i = 1, 2$. Let (Z_n) be a sequence of independent random variables such that for every n,*

$$Z_n \in \begin{cases} K(\lambda_1, 0), & \text{with probability } p, \\ K(\lambda_2, 0), & \text{with probability } q, \end{cases}$$

where $p + q = 1$, and $M_n^* = \max\{Z_1, \dots, Z_n\}$. Then, for all $x > 0$ we have

$$\lim_{n\to\infty} P\left\{ \frac{\pi}{n(p\lambda_1 + q\lambda_2)} M_n^* \leqslant x \right\} = e^{-x^{-1}}. \tag{3.11}$$

Remark. For Cauchy variables with different scale parameters, M_n^* will come from either of the two parent distributions, which explains why the extremal distribution is also a mixture. More specifically, for pure Cauchy variables, $\pi M_n / n\lambda \to U$ with $F_U(x) = e^{-x^{-1}}$. In a sample of size n from the mixture of two Cauchy distributions one observes approximately np and nq variables of the two types, and hence $M_n^* \approx \max(M_{np}^{(1)}, M_{nq}^{(2)})$ in distribution. Since F_U is a max-stable distribution, this explains the result.

Proof of Theorem 4. The distribution function of the random variable Z_n is

$$F(x) = \frac{1}{2} + \frac{p}{\pi}\operatorname{arctg}\frac{x}{\lambda_1} + \frac{q}{\pi}\operatorname{arctg}\frac{x}{\lambda_2}.$$

It is easy to prove that $\lim\limits_{t\to\infty} t\left(\dfrac{\pi}{2} - \operatorname{arctg}(at)\right) = \dfrac{1}{a}$ for $a > 0$. Consequently, for every $x > 0$ we obtain the following relations:

$$\frac{1-F(tx)}{1-F(t)} = \frac{\dfrac{1}{2} - \dfrac{p}{\pi}\operatorname{arctg}\dfrac{tx}{\lambda_1} - \dfrac{q}{\pi}\operatorname{arctg}\dfrac{tx}{\lambda_2}}{\dfrac{1}{2} - \dfrac{p}{\pi}\operatorname{arctg}\dfrac{t}{\lambda_1} - \dfrac{q}{\pi}\operatorname{arctg}\dfrac{t}{\lambda_2}}$$

$$= \frac{pt\left(\dfrac{\pi}{2} - \operatorname{arctg}\dfrac{tx}{\lambda_1}\right) + qt\left(\dfrac{\pi}{2} - \operatorname{arctg}\dfrac{tx}{\lambda_2}\right)}{pt\left(\dfrac{\pi}{2} - \operatorname{arctg}\dfrac{t}{\lambda_1}\right) + qt\left(\dfrac{\pi}{2} - \operatorname{arctg}\dfrac{t}{\lambda_2}\right)}$$

$$\to \frac{p\lambda_1/x + q\lambda_2/x}{p\lambda_1 + q\lambda_2} = \frac{1}{x}, \quad \text{as } t \to \infty.$$

It follows from theorem 2 that the function F belongs to the domain of attraction of the function G_2, i.e. in this case we have the type II of extreme value distribution. The normalizing constants are $a_n^* = 1/\gamma_n$ and $b_n^* = 0$, where $1 - F(\gamma_n) = 1/n$. This equation can be transformed equivalently as follows:

$$\frac{1}{2} - \frac{p}{\pi}\operatorname{arctg}\frac{\gamma_n}{\lambda_1} - \frac{q}{\pi}\operatorname{arctg}\frac{\gamma_n}{\lambda_2} = \frac{1}{n},$$

$$\gamma_n p\left(\frac{\pi}{2} - \operatorname{arctg}\frac{\gamma_n}{\lambda_1}\right) + \gamma_n q\left(\frac{\pi}{2} - \operatorname{arctg}\frac{\gamma_n}{\lambda_2}\right) = \frac{\pi}{n}\gamma_n.$$

From the last equality we obtain $p\lambda_1 + q\lambda_2 - \dfrac{\pi}{n}\gamma_n \to 0$, as $\gamma_n \to \infty$. Consequently, we have $\gamma_n \sim \dfrac{n(p\lambda_1 + q\lambda_2)}{\pi}$ and $a_n^* = \dfrac{1}{\gamma_n} \sim \dfrac{\pi}{n(p\lambda_1 + q\lambda_2)}$. Now it is easy to derive the equality (3.11). ■

3.3. *Mixture of uniform and truncated ecponential distribution.* Let (X_n) be a sequence of independent random variables with the uniform $U[0, c]$ distribution. The distribution function of the random variable X_n is given by $F_1(x) = x/c$, $0 \leqslant x \leqslant c$. For $n \geqslant \tau > 0$ it follows from the equality $1 - F_1(u_n) = \tau/n$ that $u_n = c(1 - \tau/n)$. Using theorem 1 we obtain $P\{M_n \leqslant c(1 - \tau/n)\} \to e^{-\tau}$ as $n \to \infty$. For $x < 0$ and $\tau = -x$ we obtain $P\{\frac{n}{c}(M_n - c) \leqslant x\} \to e^x$, as $n \to \infty$. Hence, in this case we have the extreme value distribution of type III, where $\alpha = 1$. The normalizing constants are $a_n = \frac{n}{c}$ and $b_n = c$.

Let (Y_n) be a sequence of independent random variables with the same truncated exponential distribution $\mathcal{E}(\lambda, c)$. This distribution is determined by the distribution function F_2 which is given by $F_2(x) = 0$ for $x \leqslant 0$, $F_2(x) = 1$ for $x \geqslant c$ and

$$F_2(x) = \frac{1 - e^{-\lambda x}}{1 - e^{-\lambda c}}, \quad \text{if } 0 \leqslant x \leqslant c.$$

For $x < 0$ and $\tau = -x$, it follows from theorem 1 that

$$\lim_{n\to\infty} P\left\{M_n \leqslant c + \frac{x(e^{\lambda c} - 1)}{\lambda n} + o\left(\frac{1}{n}\right)\right\} = e^x.$$

In this case we also obtain the limiting distribution of type III. The normalizing constants are given by $\widetilde{a}_n = \lambda n(e^{\lambda c} - 1)^{-1}$ and $b_n = c$.

THEOREM 5. *Let $U[0,c]$ be the class of random variables with the uniform distribution on the interval $[0,c]$ and $\mathcal{E}(\lambda,c)$ the class of random variables with the truncated exponential distribution. Let (Z_n) be a sequence of independent random variables such that for every n,*

$$Z_n \in \begin{cases} U[0,c], & \text{with probability } p, \\ \mathcal{E}(\lambda,c), & \text{with probability } q, \end{cases}$$

where $p+q=1$. If $M_n^* = \max\{Z_1,\dots,Z_n\}$, then for every $x<0$ the following equality holds true:

$$\lim_{n\to\infty} P\left\{ M_n^* \leqslant c + \frac{x}{n}\left(\frac{p}{c} + \frac{\lambda q}{e^{\lambda c}-1}\right)^{-1}\right\} = e^x. \tag{3.12}$$

Proof. The distribution function of the random variable Z_n is given by

$$F(x) = \frac{px}{c} + q\frac{1-e^{-\lambda x}}{1-e^{-\lambda c}}, \quad \text{if } 0 \leqslant x \leqslant c.$$

It is easy to prove that for every $x>0$ the equality

$$\lim_{h\downarrow 0} \frac{1-F(c-hx)}{1-F(c-x)} = x,$$

holds true. Using theorem 2 we conclude that the function F belongs to the domain of attraction of the function G_3. The normalizing constants are $b_n^* = c$ and $a_n^* = n/k$, where k should be determined. Using theorem 1 we obtain that for $x<0$ the condition $P\{M_n \leqslant c + xk/n\} \to e^x$ as $n\to\infty$, can be writen in the form:

$$1 - F\left(c + \frac{kx}{n}\right) \sim -\frac{x}{n}, \quad n\to\infty,$$

$$1 - \frac{p}{c}\left(c + \frac{kx}{n}\right) - \frac{q(1-e^{-\lambda c}e^{-\lambda kx/n})}{1-e^{-\lambda c}} \sim -\frac{x}{n}, \quad n\to\infty,$$

$$1 - p - \frac{pkx}{cn} - \frac{q}{1-e^{-\lambda c}}\left\{1 - e^{-\lambda c} + e^{-\lambda c}\frac{\lambda kx}{n} + o\left(\frac{1}{n}\right)\right\} \sim -\frac{x}{n}, \quad n\to\infty.$$

Now, we obtain that $-\dfrac{pkx}{cn} - \dfrac{e^{-\lambda c}q\lambda kx}{n(1-e^{-\lambda c})} \sim -\dfrac{x}{n}$ as $n\to\infty$. Consequently, the constant k is given by $k = \left(\dfrac{p}{c} + \dfrac{\lambda q}{e^{\lambda c}-1}\right)^{-1}$, and the equality (3.12) holds true. ■

ACKNOWLEDGMENT. I would like to thank two anonymous referees for helpful comments.

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(received 14.10.1998)

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