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ASYMPTOTIC LINEARITY OF MEAN VALUES

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Abstract. The power and some other families of mean values, considered as the functions of parameter, have asymptotically linear distribution. This holds if either all the variables converge to the same value, or under the equal tending to infinity of additive increment of the variables. This limit linearity strengthens asymptotic property of Hoehn and Niven [10] which was investigated in several papers.

Assume one is given a set of positive numbers $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and a decomposition of unity, with variables $q_i \geq 0$, $1 \leq i \leq n$, $\sum_{i=1}^{n} q_i = 1$. The power mean (average) of order p is defined as

$$M_p(\mathbf{x}) = \left(\sum_{i=1}^{n} q_i x_i^p\right)^{1/p}.$$

Mean values necessarily satisfy idempotency $M(\mathbf{a}) = a$, $\mathbf{a} = (a, a, \dots, a)$, which can be replaced by a stronger condition

$$\min_{1\leq i\leq n} x_i \leq M(\mathbf{x}) \leq \max_{1\leq i\leq n} x_i.$$

Special power means are $M_{-\infty} = \min x_i$, $M_{-1} = H$ the harmonic mean, $M_0 = \lim_{p\to 0} M_p = x_1^{q_1} \dots x_n^{q_n} = G$ the geometric mean, $M_1 = A$ the arithmetic mean, $M_2 = Q$ the quadratic mean, and $M_{+\infty} = \max x_i$.

For fixed $\mathbf{x}$ the power mean value $M_p(\mathbf{x})$ is a strictly increasing and logarithmically convex function of the real variable p unless $\mathbf{x} = \mathbf{a}$. All the power mean values lie in the interval $[\min x_i, \max x_i]$. We consider the behaviour of the power and some other means from two perspectives:

$$\begin{array}{lll} \text{asymptotic I:} & M_p(\mathbf{x}), & \mathbf{x} \to \mathbf{a} \\ \text{asymptotic II:} & M_p(\mathbf{x}+\mathbf{a}), & a \to \infty. \end{array}$$

For homogeneous means $M(\lambda\mathbf{x}) = \lambda M(\mathbf{x})$, which are necessarily homogeneous of first degree [8], these two cases are equivalent. In [3] the asymptotic behaviour I was conjectured to be

$$\frac{M_p(\mathbf{x}) - M_q(\mathbf{x})}{M_r(\mathbf{x}) - M_s(\mathbf{x})} \to \frac{p-q}{r-s}, \quad \mathbf{x} \to \mathbf{a}, \tag{1}$$

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a property named as asymptotic linearity. Asymptotic behaviour II is considered in several papers mainly quoted in the references.

Extension of Hoehn and Niven property [10] to all the power means asserts that

$$M_p(\mathbf{x}+\mathbf{a}) - a \to A(\mathbf{x}), \quad a \to \infty. \tag{2}$$

We transform (2) to the equivalent form with two components

$$M_p(\mathbf{x}+\mathbf{a}) - A(\mathbf{x}+\mathbf{a}) \to 0, \quad a \to \infty.$$

The property (2) was firstly proved by L. Hoehn and I. Niven for harmonic, geometric and quadratic means, and later was extended to the other power means and large classes of homogeneous and inhomogeneous means [6,7,8,12]. Observe that the arithmetic mean $A(\mathbf{x}+\mathbf{a})$ keeps the same position with respect to $\min x_i + a$ and $\max x_i + a$ as a varies. The asymptotic linearity claims that the other means $M_p(\mathbf{x}+\mathbf{a})$ as $a \to +\infty$ take positions with distances more and more proportionate to the values of the parameter p. As a consequence of linearity, it straightforwardly follows that the means $M_p(\mathbf{x}+\mathbf{a})$, where $p \in \mathbf{R}$, converge to $A(\mathbf{x}+\mathbf{a})$. Hence, the power mean functions in $\mathbf{R}_+^n$ have asymptotic isometric linear distribution on the diagonal which contains points $\mathbf{a}$, where $a>0$, and along its parallels $\mathbf{x}+\mathbf{a}$, where $a>0$.

THEOREM. *For the power mean the next asymptotic formula is valid:*

$$M_p(\mathbf{x}) = M_1(\mathbf{x}) + (p-1)(M_2(\mathbf{x}) - M_1(\mathbf{x})) + o(M_2(\mathbf{x}) - M_1(\mathbf{x})), \quad \mathbf{x} \to \mathbf{a}, \tag{3}$$

or, equivalently,

$$\begin{aligned} M_p(\mathbf{x}+\mathbf{a}) =& M_1(\mathbf{x}+\mathbf{a}) + (p-1)(M_2(\mathbf{x}+\mathbf{a}) - M_1(\mathbf{x}+\mathbf{a})) \\ &+ o(M_2(\mathbf{x}+\mathbf{a}) - M_1(\mathbf{x}+\mathbf{a})), \quad a \to \infty. \end{aligned} \tag{4}$$

Proof. Let $\mathbf{x} = \mathbf{a} + \mathbf{h}t$, and let $M_{\mathbf{h}}^{(k)}$ be the derivative of order k with respect to t in direction $\mathbf{h} = (h_1, h_2, \ldots, h_n)$

$$M_{\mathbf{h}}^{(k)}(\mathbf{a}) = \left(\sum_{i=1}^{n} h_i \frac{\partial}{\partial x_i}\right)^k M(\mathbf{a}).$$

Differentiating the both sides of the equality $M(\mathbf{x})^p = \sum_{i=1}^n q_i x_i^p$ gives

$$pM(\mathbf{x})^{p-1} M'_{\mathbf{h}}(\mathbf{x}) = \sum_{i=1}^{n} p q_i h_i x_i^{p-1}, \qquad M'_{\mathbf{h}}(\mathbf{a}) = \sum_{i=1}^{n} q_i h_i.$$

Further,

$$(p-1)M(\mathbf{x})^{p-2} M'_{\mathbf{h}}(\mathbf{x})^2 + M(\mathbf{x})^{p-1} M''_{\mathbf{h}}(\mathbf{x}) = (p-1)\sum_{i=1}^{n} q_i h_i^2 x_i^{p-2},$$

for $\mathbf{x} = \mathbf{a}$ obtains

$$M''_{\mathbf{h}}(\mathbf{a}) = \frac{p-1}{a}\left[\sum_{i=1}^{n} q_i h_i^2 - \left(\sum_{i=1}^{n} q_i h_i\right)^2\right] = \frac{p-1}{a}\left[Q^2(\mathbf{h}) - A^2(\mathbf{h})\right].$$

By L'Hopital's theorem for $\mathbf{h} \neq \mathbf{1}$

$$\lim_{t\to 0} \frac{M_p(\mathbf{x}) - M_q(\mathbf{x})}{M_r(\mathbf{x}) - M_s(\mathbf{x})} = \lim_{t\to 0} \frac{M_p'(\mathbf{x}) - M_q'(\mathbf{x})}{M_r'(\mathbf{x}) - M_s'(\mathbf{x})} = \lim_{t\to 0} \frac{M_p''(\mathbf{x}) - M_q''(\mathbf{x})}{M_r''(\mathbf{x}) - M_s''(\mathbf{x})}$$
$$= \frac{\left(\frac{p-1}{a} - \frac{q-1}{a}\right)\left(Q^2(\mathbf{h}) - A^2(\mathbf{h})\right)}{\left(\frac{r-1}{a} - \frac{s-1}{a}\right)\left(Q^2(\mathbf{h}) - A^2(\mathbf{h})\right)} = \frac{p-q}{r-s}.$$

Equations (3) and (4) are immediate consequences of this result. ∎

For the homogeneous means the asymptotic linearity for equal values of variables is equivalent to the asymptotic linearity at infinity. Asymptotic linearity states that the mean values M_p, $-\infty \leq p \leq +\infty$, on the infinitesimal segment $[\min x_i, \max x_i]$, $\mathbf{x} \to \mathbf{a}$, as well as on the remote segment $[\min x_i + a, \max x_i + a]$, $a \to +\infty$, tend to the linear distribution in accordance with the order p (linear asymptotic mean loci). The equivalent form of (3) and (4) is given by the next extension of divided differences of means

$$\frac{M_p(\mathbf{x}) - M_1(\mathbf{x})}{M_2(\mathbf{x}) - M_1(\mathbf{x})} \to p - 1, \quad \mathbf{x} \to \mathbf{a}, \tag{3'}$$

and

$$\frac{M_p(\mathbf{x}+\mathbf{a}) - M_1(\mathbf{x}+\mathbf{a})}{M_2(\mathbf{x}+\mathbf{a}) - M_1(\mathbf{x}+\mathbf{a})} \to p - 1, \quad a \to \infty. \tag{4'}$$

The continuous extension (3') is infinitely differentiable; a slightly more general assertion is that the extension of divided differences of means (1) is a continuous infinitely differentiable extension to the points with equal coordinates.

Some other means are also asymptotic(ally) linear functions of parameters. Let μ be a probability measure on the simplex of nonnegative normalized weights $u_1, u_2, \ldots, u_n$, and let $\mathbf{ux} = \sum_{i=1}^n u_i x_i$. A one-parameter family of means is defined by integration with respect to μ over the simplex

$$\left(\int (\mathbf{ux})^p \, d\mu(u)\right)^{1/p}, \quad p \neq 0; \qquad \exp \int \log(\mathbf{ux}) \, d\mu(u), \quad p = 0.$$

The logarithmic means L_p, $p \in \mathbf{R}$, are such integral power means [12]. Since $L_{-n} = G$ and $L_1 = A$ we have

$$\begin{aligned} L_p(\mathbf{x}) &= L_1(\mathbf{x}) + \frac{p-1}{n+1}\big(L_1(\mathbf{x}) - L_{-n}(\mathbf{x})\big) + o\big(L_1(\mathbf{x}) - L_{-n}(\mathbf{x})\big) \\ &= A(\mathbf{x}) + \frac{p-1}{n+1}\big(A(\mathbf{x}) - G(\mathbf{x})\big) + o\big(A(\mathbf{x}) - G(\mathbf{x})\big) \\ &= G(\mathbf{x}) + \frac{p+n}{n+1}\big(A(\mathbf{x}) - G(\mathbf{x})\big) + o\big(A(\mathbf{x}) - G(\mathbf{x})\big), \quad \mathbf{x} \to \mathbf{a}. \end{aligned}$$

Thus we obtain an asymptotic linear correspondence between two parameterized families of means

$$L_p(\mathbf{x}) = M_q(\mathbf{x}) + o\big(Q(\mathbf{x}) - A(\mathbf{x})\big), \qquad q = \frac{p+n}{n+1}, \qquad \mathbf{x} \to \mathbf{a}.$$

The ratio $(M_p(\mathbf{x})-M_q(\mathbf{x}))/(L_r(\mathbf{x})-L_s(\mathbf{x}))$ has an extension to $(n+1)(p-q)/(r-s)$ as $\mathbf{x}\to\mathbf{a}$.

Dresher means [8]

$$\left(\sum_{i=1}^n q_i x_i^s \Big/ \sum_{j=1}^n q_j x_j^t\right)^{1/(s-t)}, \qquad s\neq t$$
$$\exp\left(\sum_{i=1}^n q_i x_i^t \log x_i \Big/ \sum_{j=1}^n q_j x_j^t\right), \qquad s=t$$

are increasing in each parameter. The asymptotic representation for both cases has one form which can be derived from Taylor's expansion of elementary functions by a straightforward calculation

$$\begin{aligned} D_{s,t}(\mathbf{x}) &= A(\mathbf{x}) + \frac{s+t-1}{2a}(Q^2(\mathbf{x})-A^2(\mathbf{x})) + o(Q^2(\mathbf{x})-A^2(\mathbf{x})) \\ &= G(\mathbf{x}) + (s+t)(A(\mathbf{x})-G(\mathbf{x})) + o\big(A(\mathbf{x})-G(\mathbf{x})\big), \qquad \mathbf{x}\to\mathbf{a}. \end{aligned}$$

Tobey means are another two-parameter family

$$T_{s,t}(\mathbf{x}) = \left(\int (\mathbf{ux}^s)^t \, d\mu\right)^{1/st}, \qquad \mathbf{ux}^s = \sum_{i=1}^n u_i x_i^s, \quad st\neq 0.$$

In limiting cases $st=0$ the expression is similar to the limit power and Dresher means. The asymptotic representation obtains similarly

$$\begin{aligned} T_{s,t}(\mathbf{x}) = A(\mathbf{x}) &+ (st-1)(Q(\mathbf{x})-A(\mathbf{x})) + (s-1)(K(\mathbf{x})-Q(\mathbf{x})) \\ &+ o\big(Q(\mathbf{x})-A(\mathbf{x})\big) + o\big(K(\mathbf{x})-Q(\mathbf{x})\big), \quad \mathbf{x}\to\mathbf{a}, \end{aligned}$$

where $K=T_{2,1}$ is a mean appearing in the other asymptotics also. If μ is a $\mathbf{q}$-measure, then the Tobey mean is the power mean of order st and $K=Q$.

The generalization of Muirhead means [8] is

$$\left(\frac{1}{n!}\sum x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}\right)^{1/r},$$

where the sum is over the $n!$ permutations of x_i, $1\le i\le n$; and $r=\sum\alpha_i$, with $r\neq 0$. For Muirhead means $r=1$. In particular, if $\alpha_i(1-\alpha_i)=0$, $1\le i\le n$, one obtains the symmetric mean S_r of degree r, $r=1,2,\ldots,n$. In particular, $S_1=A$ and $S_n=G$, A and G are means with weights $\frac{1}{n}$. The generalization of Muirhead means is an integral mean with the geometric nucleus $(\int G^r \, d\mu_\alpha)^{1/r}$ and measure μ_α which at any of $n!$ points, obtained by permutations of the coordinates $\frac{1}{r}(\alpha_1,\ldots,\alpha_n)$, $\alpha_i\ge 0$, has point measure $\frac{1}{n!}$. Therefore, Muirhead means are also asymptotically linear.

Some comprehensive classes of means have nonlinear asymptotic behaviour; e.g., the quasi-arithmetic integral means

$$M_\mu(\mathbf{x}) = \varphi^{-1}\left(\int \varphi(\mathbf{ux}) \, d\mu\right)$$

have characteristic function $\phi = \varphi''/\varphi'$, which should be included into the limit ratio (1). Note that the power means of functions are also asymptotically linear.

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