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SOME MEAN VALUE THEOREMS IN TERMS OF AN INFINITESIMAL FUNCTION

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Abstract. In this paper we survey some mean value theorems in terms of an infinitesimal function.

1. Introduction

Let $f\colon [a,b] \to \mathbf{R}$ be a function which is differentiable in a segment $[a,b]$ and differentiable arbitrary number of times in a right neighbourhood of the point $x = a$. According to the following relation:

$$\Delta y = f(x) - f(a) = f'(a)(x-a) + \alpha(x)(x-a),$$

where $\lim_{x\to a} \alpha(x) = \lim_{x\to a+} \alpha(x) = 0$, let us introduce the function α_1 with:

$$\alpha_1(x) \stackrel{\text{def}}{=} \begin{cases} \frac{f(x)-f(a)}{x-a} - f'(a), & x \in (a,b] \\ 0, & x = a \end{cases} \tag{1}$$

It is true that:

$$\alpha_1(x) = \frac{f(x)-f(a)}{x-a} - f'(a) = \frac{f(x)-f(a)-f'(a)(x-a)}{x-a} = \frac{R_2^f(a,x)}{x-a} \qquad (x \neq a), \tag{2}$$

where $R_2^f(a,x)$ is the remainder of the second order in Taylor's formula. Then,

$$\begin{aligned} \alpha_1'(a) &= \lim_{x\to a} \frac{\alpha_1(x)-\alpha_1(a)}{x-a} = \lim_{x\to a} \frac{f(x)-f(a)-f'(a)(x-a)}{(x-a)^2} \\ &= \lim_{x\to a} \frac{f'(x)-f'(a)}{2(x-a)} = \frac{1}{2} f''(a). \end{aligned}$$

We generalize this by induction as:

$$\alpha_1^{(n)}(a) = \frac{1}{n+1}\, f^{(n+1)}(a), \tag{3}$$

for an arbitrary $n \in \mathbf{N}$. The given formula is also a corollary of [3].

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Let us introduce analogously:

$$\alpha_2(x) \stackrel{\text{def}}{=} \begin{cases} \frac{\alpha_1(x)-\alpha_1(a)}{x-a} - \alpha_1{}'(a), & x \in (a,b] \\ 0, & x = a. \end{cases} \tag{4}$$

It is true that

$$\begin{aligned} \alpha_2(x) &= \frac{\alpha_1(x) - \alpha_1(a)}{x-a} - \alpha_1'(a) \\ &= \frac{f(x) - f(a) - f'(a)(x-a) - \frac{1}{2}f''(a)(x-a)^2}{(x-a)^2} = \frac{R_3^f(a,x)}{(x-a)^2} \quad (x \neq a), \end{aligned}$$

where $R_3^f(a,x)$ is the remainder of the third order in Taylor's formula. According to the formula (3) the following formula for n^{th} derivative is true:

$$\alpha_2^{(n)}(a) = \frac{1}{n+1}\alpha_1^{(n+1)}(a) = \frac{1}{(n+1)(n+2)} f^{(n+2)}(a).$$

Next, let us introduce a sequence of functions:

$$\alpha_{k+1}(x) \stackrel{\text{def}}{=} \begin{cases} \frac{\alpha_k(x)-\alpha_k(a)}{x-a} - \alpha_k'(a), & x \in (a,b] \\ 0, & x = a; \end{cases} \qquad (k = 1, 2, \ldots). \tag{5}$$

For the function α_k let us assume:

$$\alpha_k^{(n)}(a) = \frac{1}{(n+1)(n+2)\cdots(n+k)} f^{(n+k)}(a) \qquad (n \in \mathbf{N})$$

and $\alpha_k(x) = R_{k+1}^f(a,x)/(x-a)^k$ $(x \neq a)$, where $R_{k+1}^f(a,x)$ is the remainder of $(k{+}1)^{\text{th}}$ order in Taylor's formula. Then, according to the formula (3) and inductive assumption for k, we have:

$$\alpha_{k+1}^{(n)}(a) = \frac{1}{n+1}\alpha_k^{(n+1)}(a) = \frac{1}{(n+1)(n+2)\cdots(n+k+1)} f^{(n+k+1)}(a) \tag{6}$$

and

$$\begin{aligned} \alpha_{k+1}(x) &= \frac{\alpha_k(x) - \alpha_k(a)}{x-a} - \alpha_k'(a) \\ &= \frac{R_{k+1}^f(a,x) - 0 - \frac{1}{(k+1)!} f^{(k+1)}(a)\,(x-a)^{k+1}}{(x-a)^{k+1}} = \frac{R_{k+2}^f(a,x)}{(x-a)^{k+1}} \quad (x \neq a), \end{aligned} \tag{7}$$

where R_{k+2}^f is the remainder of $(k+2)^{\text{th}}$ order in Taylor's formula.

2. Mean value theorems

Flett's theorem. We determine the conditions which are sufficient for the existence of the point $\xi \in (a,b)$ in which the function α_1 has an extreme value and the equality $\alpha_1'(\xi) = 0$ is true. The reflection of the previous equality on the function f is Flett's equality:

$$f'(\xi) = \frac{f(\xi) - f(a)}{\xi - a}. \tag{8}$$

Namely, Flett [1] has proved that the condition:

$$f'(a) = f'(b) \tag{9}$$

is sufficient for the existence of the point $\xi \in (a, b)$ so that (8) is true. Besides the previous condition, it is shown that the conclusion of the Flett's Theorem is true under the other conditions, as well. The following statement is true.

THEOREM. *If one of the following two conditions is true:*

$$\alpha_1'(b) \cdot \alpha_1(b) < 0, \qquad \mathrm{T}_1\text{-}\textit{condition;} \tag{10}$$

and

$$\alpha_1'(a) \cdot \alpha_1(b) < 0 \qquad \mathrm{M}_1\text{-}\textit{condition;} \tag{11}$$

then there exists a point $\xi \in (a, b)$*, such that the following holds:*

$$f'(\xi) = \frac{f(\xi) - f(a)}{\xi - a}. \tag{12}$$

If both of the conditions (10) and (11) are fulfilled, then there exists at least two points $\xi_i \in (a, b)$*, such that the following holds:*

$$f'(\xi_i) = \frac{f(\xi_i) - f(a)}{\xi_i - a} \qquad (i = 1, 2). \tag{13}$$

Proof. From the condition of the corollary 3 of the theorem 2 [4]:

$$\alpha_1'(a)[\alpha_1(b) - \alpha_1(a)] < 0, \quad \text{and} \quad \alpha_1'(b)[\alpha_1(b) - \alpha_1(a)] < 0, \tag{14}$$

we obtain the T_1 and M_1 conditions:

$$\alpha_1'(a) \cdot \alpha(b) < 0, \quad \text{and} \quad \alpha_1'(b) \cdot \alpha(b) < 0, \tag{15}$$

respectively. According to the second part of the stated corollary for $r = a$ the point $\xi_2 \in (a, b)$ exists, for which $\alpha_1'(\xi_2) = 0$, i.e. $f'(\xi_2) = \frac{f(\xi_2) - f(a)}{\xi_2 - a}$; while for $r = b$ the point $\xi_1 \in (a, b)$ exists, for which $\alpha_1'(\xi_1) = 0$, i.e. $f'(\xi_1) = \frac{f(\xi_1) - f(a)}{\xi_1 - a}$. If both of the stated conditions (10) and (11) are fulfilled (T_1-M_1-condition), according to the first part of the stated corollary, then two points $\xi_i \in (a, b)$ exist, for which $\alpha_1'(\xi_i) = 0$, i.e. $f'(\xi_i) = \frac{f(\xi_i) - f(a)}{\xi_i - a}$ $(i = 1, 2)$. ∎

We give extensions of the previous conditions on function α_n, with the suitable conclusions.

Corollary 1 (T_n-condition). By applying T_1-condition on function α_n we get T_n-condition. Thus, the following implication is true:

$$\alpha'_n(b)\cdot\alpha_n(b)<0 \implies (\exists\xi\in(a,b))\,\alpha'_n(\xi)=0. \tag{16}$$

The reflection of the previous relation on the original function f is given by implication:

$$\begin{gathered}\Big(\sum_{j=2}^{n-1}\tfrac{n-j}{n}\tfrac{f^{(j)}(a)}{j!}(b-a)^{j-1}+\tfrac{f'(b)+(n-1)f'(a)}{n}-\tfrac{f(b)-f(a)}{b-a}\Big)\cdot R^f_{n+1}(a,b)<0\\ \implies \big(\exists\xi\in(a,b)\big)\,\tfrac{f(\xi)-f(a)}{\xi-a}=\sum_{j=2}^{n-1}(1-\tfrac{j}{n})\tfrac{f^{(j)}(a)}{j!}(\xi-a)^{j-1}+\tfrac{f'(\xi)+(n-1)f'(a)}{n}.\end{gathered} \tag{17}$$

For $n=1,2,3$ we get the corresponding implications:

$$\big(f'(a)-\tfrac{f(b)-f(a)}{b-a}\big)\cdot\big(f'(b)-\tfrac{f(b)-f(a)}{b-a}\big)>0 \implies (\exists\xi\in(a,b))\,\tfrac{f(\xi)-f(a)}{\xi-a}=f'(\xi), \tag{18}$$

$$\big(\tfrac{f'(b)+f'(a)}{2}-\tfrac{f(b)-f(a)}{b-a}\big)\cdot R^f_3(a,b)<0 \implies (\exists\xi\in(a,b))\,\tfrac{f(\xi)-f(a)}{\xi-a}=\tfrac{f'(\xi)+f'(a)}{2} \tag{19}$$

and

$$\begin{gathered}\big(\tfrac{f''(a)}{6}(b-a)+\tfrac{f'(b)+2f'(a)}{3}-\tfrac{f(b)-f(a)}{b-a}\big)\cdot R^f_4(a,b)<0\\ \implies (\exists\xi\in(a,b))\,\tfrac{f(\xi)-f(a)}{\xi-a}=\tfrac{f'(\xi)+2f'(a)}{3}+\tfrac{f''(a)}{6}(\xi-a).\quad\blacksquare\end{gathered} \tag{20}$$

Implication (18) is actually Trahan's Theorem [2].

Corollary 2 (M_n-condition). By applying M_1-condition on function α_n we get M_n-condition. Thus, the following implication is true:

$$\alpha'_n(a)\cdot\alpha_n(b)<0 \implies (\exists\xi\in(a,b))\,\alpha'_n(\xi)=0. \tag{21}$$

The reflection of the previous relation on the original function f is given by implication:

$$\begin{gathered}f^{(n+1)}(a)\cdot R^f_{n+1}(a,b)<0 \implies\\ \big(\exists\xi\in(a,b)\big)\,\tfrac{f(\xi)-f(a)}{\xi-a}=\sum_{j=2}^{n-1}(1-\tfrac{j}{n})\tfrac{f^{(j)}(a)}{j!}(\xi-a)^{j-1}+\tfrac{f'(\xi)+(n-1)f'(a)}{n}.\end{gathered} \tag{22}$$

For $n=1,2,3$ we get the corresponding implications:

$$f''(a)\cdot R^f_2(a,b)<0 \implies (\exists\xi\in(a,b))\,\frac{f(\xi)-f(a)}{\xi-a}=f'(\xi), \tag{23}$$

$$f'''(a)\cdot R^f_3(a,b)<0 \implies (\exists\xi\in(a,b))\,\frac{f(\xi)-f(a)}{\xi-a}=\frac{f'(\xi)+f'(a)}{2} \tag{24}$$

and

$$f^{IV}(a)\cdot R^f_4(a,b)<0 \implies (\exists\xi\in(a,b))\,\tfrac{f(\xi)-f(a)}{\xi-a}=\tfrac{f'(\xi)+2f'(a)}{3}+\tfrac{f''(a)}{6}(\xi-a).\quad\blacksquare \tag{25}$$

Implication (23) provides one more sufficient condition that the conclusion of the Flett's Theorem is true.

Taylor's theorem. On the basis of Taylor's expansion of the function f with the remainder in Peano's form:

$$f(a+\Delta x)-f(a)=\sum_{k=1}^{m}\frac{f^{(k)}(a)}{k!}(\Delta x)^k+o\big((\Delta x)^m\big),$$

using the equality $\frac{f^{(k)}(a)}{k!}=\alpha'_{k-1}(a),\ k=1,\dots,m\ \wedge\ \alpha'_0(a)\overset{\text{def}}{=}f'(a)$, we get the other formulation of the Taylor's expansion:

$$f(a+\Delta x)-f(a)=\sum_{k=1}^{m}\alpha'_{k-1}(a)\,(\Delta x)^k+o\big((\Delta x)^m\big).$$

Hence, we get a geometric interpretation of the coefficient of the k^{th} order of the Taylor's expansion as the slope of the tangent function α_{k-1} at the point $A[a,0]$.

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