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CONTINUITY OF THE ESSENTIAL SPECTRUM IN THE CLASS OF QUASIHYPONORMAL OPERATORS

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Abstract. Let H be a separable Hilbert space. We write $\sigma(A)$ for the spectrum of $A \in B(H)$, $\sigma_w(A)$ for the Weyl spectrum and $\sigma_b(A)$ for the Browder spectrum. Operator $A \in B(H)$ is quasihyponormal if $A^*(A^*A - AA^*)A \geq 0$, i.e. $\|A^*Ax\| \leq \|A^2x\|$, for every $x \in H$.

1. Introduction

Let H be a complex infinite-dimensional separable Hilbert space and let $B(H)$ ($K(H)$) denote a Banach algebra of all bounded operators (the ideal of all compact operators) on H. If $A \in B(H)$, then $\sigma(A)$ denotes the spectrum of A and $\rho(A)$ denotes the resolvent set of A. The following sets are well-known semigroups of semi-Fredholm operators on H:

$$\Phi_+(H) = \{\, A \in B(H) : \mathcal{R}(A) \text{ is closed and } \dim \mathcal{N}(A) < \infty \,\}$$
$$\Phi_-(H) = \{\, A \in B(H) : \mathcal{R}(A) \text{ is closed and } \dim H/\mathcal{R}(A) < \infty \,\}.$$

The semigroup of Fredholm operators is $\Phi(H) = \Phi_+(H) \cap \Phi_-(H)$. If A is semi-Fredholm and $\alpha(A) = \dim \mathcal{N}(A)$ and $\beta(A) = \dim H/\mathcal{R}(A)$, then we may define an index: $i(A) = \alpha(A) - \beta(A)$. We also consider a class $\Phi_0(H) = \{\, A \in \Phi(H) : i(A) = 0 \,\}$ (Weyl operators). For $A \in B(H)$, the following familiar spectra are defined

$$\sigma_a(A) = \{\, \lambda \in \mathbf{C} : \inf_{x \in H\,,\, \|x\|=1} \|(A-\lambda)x\| = 0 \,\} \ - \text{ the approximate spectrum,}$$
$$\sigma_e(A) = \{\, \lambda \in \mathbf{C} : A - \lambda \notin \Phi(H) \,\} \ - \text{ the Fredholm spectrum,}$$
$$\sigma_w(A) = \{\lambda \in \mathbf{C} : A - \lambda \notin \Phi_0(H) \,\} \ - \text{ the Weyl spectrum, and}$$
$$\sigma_b(A) = \bigcap \{\, \sigma(A+K) : AK = KA, K \in K(H) \,\} \ - \text{ the Browder spectrum}.$$

We use $\sigma_{le}(A)$ ($\sigma_{re}(A)$) *left* (*right*) *essential spectrum* of A (that is left (right) spectrum of $\pi(A)$ in $B(H)/K(H)$), and $\sigma_{lre}(A) = \sigma_{le}(A) \cap \sigma_{re}(A)$.

Let $\pi_{00}(A)$ be the set of all $\lambda \in \mathbf{C}$ such that λ is an isolated point of $\sigma(A)$ and $0 < \dim \mathcal{N}(A-\lambda) < \infty$ and let $\pi_0(A)$ be the set of all normal eigenvalues

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of A, that is the set of all isolated points of $\sigma(A)$ for which the corresponding spectral projection has finite-dimensional range and let $\sigma^0(A) = \sigma_{lre}(A) \cup \pi_0(A)$. It is well-known that $\sigma_b(A) = \sigma(A) \setminus \pi_0(A)$ [2, 7].

We say that A obeys Weyl's theorem [7, 10], if

$$\sigma_w(A) = \sigma(A) \backslash \pi_{00}(A).$$

Let $\Gamma_{0e}(A)$ be the union of all trivial components of the set

$$(\sigma_e(A) \backslash [\rho_{s-F}^{\pm}(A)]^-) \cup (\bigcup_{-\infty<n<\infty} \{[\rho_{s-F}^{n}(A)]^- \backslash \rho_{s-F}^{n}(A)\}) ,$$

where $\rho_{s-F}^{\pm}(A) = \{ \lambda \in \mathbf{C} : i(A-\lambda) \neq 0 \}$ and $\rho_{s-F}^{n}(A) = \{ \lambda \in \mathbf{C} : i(A-\lambda) = n \}$.

If (τ_n) is a sequence of compact subsets of $\mathbf{C}$, then its limit inferior is

$$\liminf_{n\to\infty} \tau_n = \{ \lambda \in \mathbf{C} : \text{ there are } \lambda_n \in \tau_n \text{ with } \lambda_n \to \lambda \}$$

and its limit superior is

$$\limsup_{n\to\infty} \tau_n = \{ \lambda \in \mathbf{C} : \text{ there are } \lambda_{n_k} \in \tau_{n_k} \text{ with } \lambda_{n_k} \to \lambda \} .$$

If $\liminf_{n\to\infty} \tau_n = \limsup_{n\to\infty} \tau_n$, then $\lim_{n\to\infty} \tau_n$ is said to exist and is equal to this common limit. A mapping p, defined on $B(H)$, whose values are compact subset of $\mathbf{C}$ is said to be upper (lower) semi-continuous at A, provided that if $A_n \to A$ then $\limsup_{n\to\infty} p(A_n) \subset p(A)$ ($p(A) \subset \liminf_{n\to\infty} p(A_n)$). If p is both upper and lower semi-continuous at A, then it is said to be continuous at A and in this case $\lim_{n\to\infty} p(A_n) = p(A)$.

We say that $A \in B(H)$ is hyponormal provided that $\|A^*x\| \leq \|Ax\|$ for all $x \in H$ and A is quasihyponormal, if $\|A^*Ax\| \leq \|A^2x\|$ for all $x \in H$. Note that the Weyl's theorem is proved for hyponormal and quasihyponormal operators [6, 7, 10].

2. Results

THEOREM 2.1. *Let $A \in B(H)$ obeys Weyl's theorem. Then σ_w is continuous at A if and only if σ is continuous at A.*

Proof. Let σ_w is continuous at $A \in B(H)$ and let $\{A_n\}$ be a sequence in $B(H)$ such that $A_n \to A$. Since σ is upper semi-continuous [3, 4] we have to show that σ is lower semi-continuous at A, or $\sigma(A) \subset \liminf_{n\to\infty} \sigma(A_n)$. Let $\lambda \in \sigma(A)$. Then, if $\lambda \in \sigma_w(A) \subset \sigma(A)$, we have $\lambda \in \sigma_w(A) \subset \liminf_{n\to\infty} \sigma_w(A_n) \subset \liminf_{n\to\infty} \sigma(A_n)$. Suppose that $\lambda \in \sigma(A) \backslash \sigma_w(A)$. Since A obeys Weyl's theorem we have that $\lambda \in \pi_{00}(A)$, so λ is an isolated point of $\sigma(A)$. Now from [9, Theorem 3.26] it follows that $\lambda \in \liminf_{n\to\infty} \sigma(A_n)$.

Now, let σ be continuous at A and let A obeys Weyl's theorem. Since $\pi_0(A) \subset \pi_{00}(A)$, we have

$$\overline{\pi_0(A)} \cap \sigma_e(A) \subset \overline{\pi_{00}(A)} \cap \sigma_w(A) = \overline{\pi_{00}(A)} \cap (\sigma(A) \backslash \pi_{00}(A)) \subset \overline{\Gamma_{oe}(A)} ,$$

and so, by [1, Theorem 14.17] σ_w is continuous at A. ∎

Corollary 2.2. *Let $A \in B(H)$ obeys Weyl's theorem. If σ_a is continuous at A then σ_w is continuous at A.*

Proof. If σ_a is continuous at A, then by [4, Theorem 5.1.] we have that σ is continuous at A, too. Now, since A obeys Weyl's theorem, by Theorem 2.1 it follows that σ_w is continuous at A. ∎

Lemma 2.3. *Let $A \in B(H)$. If σ is continuous at A, then σ^0 is upper semi-continuous at A.*

Proof. Since σ is continuous at A, by [3, Corollary 3.2] it follows that $\operatorname{int} \rho^0_{s-F}(A) = \emptyset$. Now, by [5, Theorem 1.3] we have that σ^0 is upper semi-continuous. ∎

Theorem 2.4. *Let $A \in B(H)$. If σ and σ_w are continuous at A, then σ_b is continuous at A.*

Proof. Suppose that σ_b is not continuous at A. Since σ_b is upper semi-continuous at every $A \in B(H)$ [2, Lemma 2.1], then we have a sequence of operators $\{A_n\} \subset B(H)$ such that

$$\sigma_b(A) \not\subseteq \liminf_{n\to\infty} \sigma_b(A_n)\,,$$

i.e. there exsist $\lambda \in \sigma_b(A)$, $\epsilon > 0$ and nonnegative integer n_1 such that $B(\lambda, \epsilon) \cap \sigma_b(A_n) = \emptyset$, for every $n > n_1$. Since σ_w is continuous at A we have that $\lambda \in \sigma_b(A) \setminus \sigma_w(A)$.

Now, from continuity of σ at A we have

$$\lambda \in \sigma_b(A) \subset \sigma(A) \subset \liminf_{n\to\infty} \sigma(A_n)\,,$$

i.e. there exists a nonnegative integer n_2 such that $B(\lambda, \epsilon) \cap \sigma(A_n) \neq \emptyset$, for every $n > n_2$. There exists a $\lambda_n \in B(\lambda, \epsilon) \cap \sigma(A_n)$ such that $\lambda_n \in \sigma(A_n) \setminus \sigma_b(A_n) = \pi_0(A_n)$, i.e. $\lambda_n \in \pi_0(A_n) \cup \sigma_{lre}(A_n) = \sigma^0(A_n)$, for every $n > n_0 = \max\{n_1, n_2\}$.

Since σ is continuous at A, by Lemma 2.3. we have that σ^0 is upper semi-continuous at A. As $B(\lambda, \epsilon) \cap \sigma^0(A_n) \neq \emptyset$, $n > n_0$ it follows that

$$\lambda \in \limsup_{n\to\infty} \sigma^0(A_n) \subset \sigma^0(A) = \sigma_{lre}(A) \cup \pi_0(A)\,.$$

Since $\lambda \notin \sigma_w(A)$, we have that $\lambda \notin \sigma_{lre}(A)$, i.e. $\lambda \in \pi_0(A) = \sigma(A) \setminus \sigma_b(A)$. This contradiction concludes the proof. ∎

Theorem 2.5. *If A_n, A are quasihyponormal operators in $B(H)$ such that $A_n \to A$, then $\sigma_w(A_n) \to \sigma_w(A)$.*

Proof. As proved in [3, 7, 10], quasihyponormal operators obeys Weyl's theorem and so, by [8, Theorem 1] we have that $\sigma(A_n) \to \sigma(A)$. Now, by Theorem 2.1 we have that $\sigma_w(A_n) \to \sigma_w(A)$. ∎

Corollary 2.6. *Let A_n, A are quasihyponormal operators in $B(H)$ such that $A_n \to A$. Then $\sigma_b(A_n) \to \sigma_b(A)$.*

Proof. Since A_n, A are quasihyponormal operators, by [8, Theorem 2.] we have that $\lim_{n\to\infty} \sigma(A_n) = \sigma(A)$ and by Theorem 2.5 we have that $\lim_{n\to\infty} \sigma_w(A_n) = \sigma_w(A)$. Now by Theorem 2.4 it follows that $\lim_{n\to\infty} \sigma_b(A_n) = \sigma_b(A)$. ∎

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REFERENCES

[**1**] C. Apostol, L. A. Fialkow, D. A. Herrero and D. Voiculescu, *Aproximation of Hilbert space operators, Vol. II*, Reserch Notes in Mathematics **102**, Pitman, Boston, 1984.

[**2**] J. J. Buoni, *The variation of Browder's essential spectrum*, Proc. Amer. Math. Soc. **48** (1975), 140–144.

[**3**] J. B. Conway and B. B. Morrel, *Operators that are points of spectral continuity*, Integral Equations Operator Theory **2** (1979), 174–198.

[**4**] J. B. Conway and B. B. Morrel, *Operators that are points of spectral continuity II*, Integral Equations Operator Theory **4** (1981), 459–503.

[**5**] J. B. Conway and B. B. Morrel, *Operators that are points of spectral continuity III*, Integral Equations Operator Theory **6** (1983), 319–344.

[**6**] L. A. Coburn, *Weyl's theorem for nonnormal operators*, Michigan Math. J. **13** (1966), 285–288.

[**7**] В. А. Еровенко, *Теорема Вейля о существенном спектре для k-паранормальних операторов*, Весцi Академии наук БССР, Сер. Физ-Мат. **5** (1986), 30–35.

[**8**] В. А. Еровенко, *Непрерывност спектра некоторих классов операторов*, Докл. АН БССР **30** (1986), 681–684.

[**9**] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin 1966.

[**10**] S. Prasanna, *Weyl's theorem and thin spectra*, Proc. Indian Acad. Sci. Math. Sci. **91**, 1 (1982), 59–63.

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