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INVARIANCE OF FUZZY PROPERTIES

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Abstract. In this paper we study the invariance and the inverse invariance under fuzzy closed, fuzzy open, and various kinds of fuzzy perfect maps, of different properties of fuzzy topological spaces.

1. Introduction and definitions

In this paper we study the invariance and the inverse invariance under fuzzy closed, fuzzy open and various types of fuzzy perfect maps, of different properties of fuzzy topological space in the Chang's sense.

Next, we list the definitions which we will use in this paper.

Definition 1. [9] A fuzzy topological space (X, T) is weakly induced by a topological space (X, T_0) if:

(a) $T_0 = [T]$, where $[T] = \{ A \subset X \mid \chi_A \in T \}$ and

(b) every $\mu \in T$ is a lower semicontinuous function from (X, T_0) into $[0, 1]$.

Definition 2. [6] A fuzzy extension of a topological property of $(X, [T])$ is said to be good, when it is possessed by (X, T) if, and only if, the original property is possessed by $(X, [T])$.

Definition 3. [10] Let μ_1, μ_2 be two fuzzy sets in X. μ_1 is said to be quasi-coincident with μ_2, denoted by $\mu_1 \; q \; \mu_2$, if there exists $x \in X$ such that $\mu_1(x) > \mu_2'(x)$ where μ_2' is the fuzzy complement of μ_2.

Definition 4. [1] Let f be a map from X to Y. Let μ be a fuzzy set in Y, then the inverse of μ, written as $f^{-1}(\mu)$, is defined by $f^{-1}(\mu)(x) = \mu(f(x))$ for all x in X. Conversely, if μ is a fuzzy set in X, the image of μ, written as $f(\mu)$, is a fuzzy set in Y given by

$$f(\mu)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \{\mu(x)\}, & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

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Definition 5. [6] Let Top be the category of all the topological spaces and the continuous maps and let CFT be the category of the fuzzy topological spaces in the Chang's sense and the F-continuous maps. We will denote by

$$\omega\colon Top \to CFT$$

the functor which associates to any topological space (X, T) the fuzzy space $(X, \omega T)$ where ωT is the totality of all lower semicontinuous maps of (X, T) to the unit interval, and by

$$j\colon CFT \to Top$$

the functor which associates to a fuzzy topological space the topological space such that its open sets have their characteristic maps in the fuzzy topology.

Definition 6. [15] A map f from a fuzzy topological space (X, T) to a fuzzy topological space (Y, S) is fuzzy closed (resp. fuzzy open) if $f(\mu)$ is fuzzy closed (resp. fuzzy open) in (Y, S), for each fuzzy closed (resp. fuzzy open) set μ in X.

Definition 7. [1] A family of fuzzy sets $\{\mu_j \mid j \in J\}$ is a cover of a fuzzy set μ if $\mu \leqslant \bigvee\{\mu_j \mid j \in J\}$. A subcover of $\{\mu_j \mid j \in J\}$ is its subfamily which is also a cover. A fuzzy topological space is compact in the Chang's sense, if each open cover has a finite subcover.

Definition 8. [5] A fuzzy set μ in (X, T) is fuzzy compact in the Lowen's sense, if for each family of fuzy open sets $\{\mu_j \mid j \in J\}$ such that $\mu \leqslant \bigvee\{\mu_j \mid j \in J\}$ and for all $\varepsilon > 0$ there exists a finite subfamily $\{\mu_j \mid j \in J_0\}$ such that $\mu - \varepsilon \leqslant \bigvee\{\mu_j \mid j \in J_0\}$.

Definition 9. [4] A collection $\mathcal{U}$ of fuzzy sets in (X, T) is called a q-cover of a fuzzy set μ if for each $x \in \operatorname{supp}\mu$, $x_{\mu(x)}\ q \bigvee_{U\in\mathcal{U}} U$. If each member of $\mathcal{U}$ is fuzzy open, then $\mathcal{U}$ is called an open q-cover. A fuzzy set μ in a fuzzy topological space (X, T) is called q-compact if for every open q-cover $\mathcal{U}$ of μ, there exists a finite subcollection $\mathcal{U}_0$ of $\mathcal{U}$ such that $\sup\{(1 - \bigvee_{U\in\mathcal{U}_0} U)(x) \mid x \in \operatorname{supp}\mu\} < \mu(x)$ for every $x \in \operatorname{supp}\mu$.

Definition 10. [4] A map f from a fuzzy topological space (X, T) to a fuzzy topological space (Y, S) is called fuzzy perfect, in the Ghosh's sense, if f is onto, fuzzy closed, F-continuous and $f^{-1}(y_\alpha)$ is q-compact, for each fuzzy point y_α in Y.

Definition 11. [12] A map f from a fuzzy topological space (X, T) to a fuzzy topological space (Y, S) is called fuzzy perfect in the Srivastava and Lal's sense, if f is onto, fuzzy closed, F-continuous and $f^{-1}(y_\alpha)$ is fuzzy compact in the Lowen's sense, for each fuzzy point y_α in Y.

Definition 12. [2] A map f from a fuzzy topological space (X, T) to a fuzzy topological space (Y, S) is called fuzzy perfect in the Christoph's sense, if f is onto, fuzzy closed, F-continuous and $f^{-1}(y_\alpha)$ is compact in the Chang's sense, for each fuzzy point y_α in Y.

2. Invariance theorems

Let (X,T) and (Y,S) be two T_2 weakly induced fuzzy topological spaces [9] and $f\colon (X,[T]) \to (Y,[S])$ an onto map. We denote $\omega(f)\colon (X,T) \to (Y,S)$ the map $\omega(f)(\mu)(y) = \sup_{t\in f^{-1}(y)}\{\mu(t)\}$ for each fuzzy set μ in X.

LEMMA 2.1. *If the map $\omega(f)\colon (X,T) \to (Y,S)$ is F-continuous, then the map $f\colon (X,[T]) \to (Y,[S])$ is continuous.*

Proof. For every $A \in [S]$, we have that $\chi_A \in S$, then $\omega(f)^{-1}(\chi_A) = \chi_{f^{-1}(A)} \in T$. Thus, $f^{-1}(A) \in [T]$. ∎

LEMMA 2.2. *If the map $\omega(f)\colon (X,T) \to (Y,S)$ is fuzzy closed (resp. fuzzy open), then the map $f\colon (X,[T]) \to (Y,[S])$ is closed (resp. open).*

Proof. If C is closed in $(X,[T])$, we have $X\setminus C \in [T]$, $\chi_{X\setminus C} \in T$ and χ_C fuzzy closed in (X,T). Then, if $\omega(f)$ is fuzzy closed, $\omega(f)(\chi_C) = \chi_{f(C)}$ is fuzzy closed in (Y,S). Thus $\chi'_{f(C)}$ is fuzzy open in (Y,S), $Y\setminus f(C) \in [S]$ and finally $f(C)$ is closed in $(Y,[S])$. If A is open in $(X,[T])$, we have $\chi_A \in T$. Then, if $\omega(f)$ is fuzzy open, $\omega(f)(\chi_A) = \chi_{f(A)}$ is fuzzy open in (Y,S) and $f(A) \in [S]$. ∎

THEOREM 2.3. *Let (X,T), (Y,S) be two weakly induced fuzzy topological spaces and $\omega(f)\colon (X,T) \to (Y,S)$ be F-continuous (resp. F-continuous and closed fuzzy, or F-continuous and open fuzzy). If:*

(a) (X,T) verifies the fuzzy vrersion of a property (P) of topological spaces,

(b) the property (P) is invariant under continuous (resp. continuous and closed, or continuous and open) maps,

(c) the fuzzy version of (P) is a good extension of (P),

then (Y,S) verifies the fuzzy version of (P).

Proof. The map $f\colon (X,[T]) \to (Y,[S])$ is continuous (resp. continuous and closed, or continuous and open) by Lemmas 2.1 and 2.2. Then, from the hypothesis, it follows that (Y,S) verifies the fuzzy version of (P). ∎

COROLLARIES 2.4. 1. *T_1 fuzzy topological spaces* [14] *are invariant by closed fuzzy maps.*

2. *S-paracompactness, S^*-paracompactness* [7]*, fuzzy paracompactness and $*$-fuzzy paracompactness* [3] *are invariant by closed fuzzy maps.*

3. *All the good extensions of compactness are invariant by F-continuous maps.*

4. *All the good extensions of connectedness are invariant by F-continuous maps.*

PROPOSITION 2.5. *If the map $\omega(f)\colon (X,T) \to (Y,S)$ verifies that $\omega(f)^{-1}(y_\alpha)$ is q-compact for every fuzzy point y_α in Y, then $f^{-1}(y)$ is compact for every $y \in Y$.*

Proof. For every open cover $\mathcal{U}$ of $f^{-1}(y)$, we have that $\mathcal{U}^* = \{\chi_U \mid U \in \mathcal{U}\}$ is open q-cover of $\omega(f)^{-1}(y_\alpha)$, where y_α is the fuzzy point in Y that takes the value α at its support y. Indeed: for each $x \in \operatorname{supp}\omega(f)^{-1}(y_\alpha)$, $\omega(f)^{-1}(y_\alpha)(x) =$

$y_\alpha(f(x)) = \alpha$, then $f(x) = y$, $x_{\omega(f)^{-1}(y_\alpha)(x)} + \chi_{\bigcup_{U\in\mathcal{U}} U}(x) = \alpha + 1 > 1$, thus $x_{\omega(f)^{-1}(y_\alpha)(x)}\, q\, \chi_{\bigcup_{U\in\mathcal{U}} U}$.

By the hypothesis, there exists a finite subcollection $\mathcal{U}_0^*$ of $\mathcal{U}^*$ such that $\sup\{(1 - \bigvee_{\chi_U\in\mathcal{U}_0^*} \chi_U)(x) \mid x \in \operatorname{supp}\omega(f)^{-1}(y_\alpha)\} < \omega(f)^{-1}(y_\alpha)(x)$, for every $x \in \operatorname{supp}\omega(f)^{-1}(y_\alpha)$ (or equivalently, $x \in f^{-1}(y)$). Then $(\bigvee_{\chi_U\in\mathcal{U}_0^*} \chi_U)(x) + \omega(f)^{-1}(y_\alpha)(x) > 1$ for every $x \in f^{-1}(y)$ [4, Corollary 2.18], $\chi_{\bigcup_{U\in\mathcal{U}_0} U}(x) + \alpha > 1$ for every $x \in f^{-1}(y)$ (where $\mathcal{U}_0 \subset \mathcal{U}$ is a finite subfamily), $x \in \bigcup_{U\in\mathcal{U}_0} U$ for all $x \in f^{-1}(y)$ and, finally, $\mathcal{U}_0$ covers $f^{-1}(y)$. Thus $f^{-1}(y)$ is compact. ■

Corollary 2.6. *If the map $\omega(f)\colon (X,T) \to (Y,S)$ is fuzzy perfect in the Ghosh's sense, then the map $f\colon (X,[T]) \to (Y,[S])$ is perfect.*

Proposition 2.7. *If the map $\omega(f)\colon (X,T) \to (Y,S)$ verifies that $\omega(f)^{-1}(y_\alpha)$ is fuzzy compact in the Lowen's sense for every fuzzy point y_α in Y, then $f^{-1}(y)$ is compact for every $y \in Y$.*

Proof. For every open cover $\mathcal{U}$ of $f^{-1}(y)$, we have that $\mathcal{U}^* = \{\chi_U \mid U \in \mathcal{U}\}$ is open L-cover of $\omega(f)^{-1}(y_\alpha)$, where y_α is the fuzzy point in Y that takes the value α at its support y. Indeed: for each $x \in X$,

$$\omega(f)^{-1}(y_\alpha)(x) = y_\alpha(f(x)) = \begin{cases} \alpha, & \text{if } x \in f^{-1}(y), \\ 0, & \text{if } x \notin f^{-1}(y), \end{cases}$$

then $\omega(f)^{-1}(y_\alpha) \leqslant \chi_{\bigcup_{U\in\mathcal{U}} U} = \bigvee_{U\in\mathcal{U}} \chi_U$.

By the hypothesis for all ξ, with $\alpha > \xi > 0$, there exists a finite subfamily $\mathcal{U}_\xi \subset \mathcal{U}$ such that $\{\chi_U \mid U \in \mathcal{U}\}$ is L- cover of $\omega(f)^{-1}(y_\alpha) - \xi$. Then for every $x \in f^{-1}(y)$, we have that $\chi_{\bigcup_{U\in\mathcal{U}} U}(x) \geqslant \omega(f)^{-1}(y_\alpha)(x) - \xi = y_\alpha(f(x)) - \xi = \alpha - \xi > 0$. Thus, $\mathcal{U}_\xi$ covers $f^{-1}(y)$. ■

Corollary 2.8. *If the map $\omega(f)\colon (X,T) \to (Y,S)$ is fuzzy perfect in the Srivastava and Lal's sense, then the map $f\colon (X,[T]) \to (Y,[S])$ is perfect.*

Proposition 2.9. *If the map $\omega(f)\colon (X,T) \to (Y,S)$ verifies that $\omega(f)^{-1}(y_\alpha)$ is compact, in the Chang's sense, for every fuzzy point y_α in Y, then $f^{-1}(y)$ is compact for every $y \in Y$.*

Proof. For every open cover $\mathcal{U}$ of $f^{-1}(y)$, we have that $\mathcal{U}^* = \{\chi_U \mid U \in \mathcal{U}\}$ is open fuzzy cover of $\omega(f)^{-1}(y_\alpha)$, where y_α is the fuzzy point in Y that takes the value α at y. By the hypothesis, there exists a finite subfamily $\mathcal{U}_0 \subset \mathcal{U}$ such that $\{\chi_U \mid U \in \mathcal{U}_0\}$ covers $\omega(f)^{-1}(y_\alpha)$. Thus, also $\mathcal{U}_0$ covers $f^{-1}(y)$. ■

Corollary 2.10. *If the map $\omega(f)\colon (X,T) \to (Y,S)$ is fuzzy perfect in the Christoph's sense, then the map $f\colon (X,[T]) \to (Y,[S])$ is perfect.*

Theorem 2.11. *Let (X,T), (Y,S) be two weakly induced fuzzy topological spaces and $\omega(f)\colon (X,T) \to (Y,S)$ be fuzzy perfect in the Ghosh's sense (resp. in the Srivastava and Lal's sense, or in the Christoph's sense). If:*

(a) (X,T) verifies the fuzzy version of a property (P) of topological spaces,

(b) the property (P) is invariant under perfect maps,

(c) the fuzzy version of (P) is a good extension of (P),

then (Y,S) verifies the fuzzy version of (P).

Proof. The map $f\colon (X,[T]) \to (Y,[S])$ is perfect by Corollaries 2.6, 2.8 and 2.10. Then the result is clear. ∎

Corollaries 2.12. 1. *Hausdorff fuzzy topological spaces* [13] *are invariant by fuzzy perfect maps in the various senses.*

2. *All the good extensions of paracompactness are invariant by fuzzy perfect maps in the various senses.*

Theorem 2.13. *Let (X,T), (Y,S) be two weakly induced fuzzy topological spaces and $\omega(f)\colon (X,T) \to (Y,S)$ be fuzzy perfect in the Ghosh's sense (resp. in the Srivastava and Lal's sense, or in the Christoph's sense). If:*

(a) (Y,S) verifies the fuzzy version of a property (P) of topological spaces,

(b) the property (P) is inverse invariant under perfect maps,

(c) the fuzzy version of (P) is a good extension of (P),

then (X,T) verifies the fuzzy version of (P).

Proof. The map $f\colon (X,[T]) \to (Y,[S])$ is perfect by Corollaries 2.6, 2.8 and 2.10. Then the result is clear. ∎

Corollaries 2.14. 1. *Hausdorff fuzzy topological spaces are inverse invariant by fuzzy perfect maps in the various senses.*

2. *All the good extensions of paracompactness are inverse invariant by fuzzy perfect maps in the various senses.*

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